Heidelberg University
Summer term 2024

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Geometric Algebra for Physicists - Problem Set 1

to be handed in by **TBA1**

1 Rotors and matrices

If we want to handle rotations in two dimensions, we normally either use complex numbers or two-by-two rotation matrices. Geometric algebra allows us to link these two concepts together.
Let $\mathbf{v} = v_1 e_1 + v_2 e_2$ be some 2D vector. We want to rotate it by an angle θ. In 2D geometric algebra, we construct the even multivector

$$z = \exp(e_{12}\theta) = \cos(\theta) + e_{12}\sin(\theta) \tag{1.1}$$

We can apply it from the right to get the rotated vector.

$$\mathbf{v}' = \mathbf{v}z \tag{1.2}$$

1. Derive the 2×2 rotation matrix $\mathsf{R}(\theta)$ corresponding to this rotation. Use the expression for $\mathbf{v}z$.
2. We might have also multiplied z from the left, $\mathbf{v} \mapsto z\mathbf{v}$. Is the result any different?
3. We can convert the vector $\mathbf{v}$ to a "complex number" (even 2D multivector) by premultiplying it with e_1, i.e. $v = e_1\mathbf{v}$. What is the expression for v in terms of the components of $\mathbf{v}$? Can we also rotate it by multiplying z? Does it matter from which side?
4. Could we also have chosen another vector than e_1 to convert $\mathbf{v}$ to a complex number?

1.1

The rotation $\mathbf{v} \mapsto \mathbf{v}' = \mathbf{v}z$ is a linear transformation, so the corresponding matrix form is given by

$$\mathsf{R}(\theta)_{ij} = e_i \cdot (e_j z). \tag{1.3}$$

Therefore we have to expand the expression

$$\boxed{\mathsf{R}(\theta)_{ij} = e_i \cdot \left(e_j(\cos\theta + e_{12}\sin\theta)\right).} \tag{1.4}$$

The four components of the matrix are:

$$\begin{aligned}
\mathsf{R}(\theta)_{11} &= e_1 \cdot (e_1\cos\theta + e_2\sin\theta) = \cos\theta \\
\mathsf{R}(\theta)_{12} &= e_2 \cdot (e_1\cos\theta + e_2\sin\theta) = \sin\theta \\
\mathsf{R}(\theta)_{21} &= e_1 \cdot (e_2\cos\theta - e_1\sin\theta) = -\sin\theta \\
\mathsf{R}(\theta)_{22} &= e_2 \cdot (e_2\cos\theta - e_1\sin\theta) = \cos\theta
\end{aligned} \tag{1.5}$$

Putting together these components into a matrix, we write

$$\mathsf{R}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{1.6}$$

1.2

We take the product

$$z\mathbf{v} = (\cos\theta + e_{12}\sin\theta)\mathbf{v} \tag{1.7}$$

In the 2D geometric algebra Cl(2), vectors and bivectors anticommute, so pulling $\mathbf{v}$ towards the left yields

$$\boxed{\mathbf{v}(\cos(\theta) - e_{12}\sin(\theta))} \tag{1.8}$$

which is the same as taking

$$\mathbf{v}(\cos(-\theta) + e_{12}\sin(-\theta) = \mathbf{v}z'. \tag{1.9}$$

where

$$z' = \exp(-e_{12}\theta) \tag{1.10}$$

Multiplying a vector with the z from the left performs the opposite of the intended rotation.

1.3

We obtain the even multivector

$$v = e_1\mathbf{v} = e_1(v_1e_1 + v_2e_2) = v_1 + v_2e_{12}. \tag{1.11}$$

We can see that the e_1-axis part of $\mathbf{v}$ is converted into the scalar/"real" part, and the e_2-axis part is converted into a bivector/"imaginary" part. We could have also chosen a vector different than e_1 to perform the conversion - then, the split into real and imaginary parts would have been different.
Both v and z are even multivectors, so they behave exactly like complex numbers. It does not matter whether we multiply z from the left or from the right.

2 Pythagorean theorem

Given two orthogonal vectors $\mathbf{a}$ and $\mathbf{b}$, expand the geometric product $(\mathbf{a}+\mathbf{b})^2 = (\mathbf{a}+\mathbf{b})(\mathbf{a}+\mathbf{b})$.

$$\begin{aligned}
(\mathbf{a}+\mathbf{b})^2 &= (\mathbf{a}+\mathbf{b})(\mathbf{a}+\mathbf{b}) && (2.1)\\
&= \mathbf{aa}+\mathbf{ab}+\mathbf{ba}+\mathbf{bb} && (2.2)\\
&= |\mathbf{a}|^2+|\mathbf{b}|^2+\mathbf{ab}-\mathbf{ab} && (2.3)\\
&= |\mathbf{a}|^2+|\mathbf{b}|^2 && (2.4)
\end{aligned}$$

3 Multivector expressions

Simplify the following multivector-valued expressions. Use $e_i^2 = 1$ for all i.

1. e_{1234}^2
2. $e_{12}^{-3} \wedge e_{34}$
3. $e_{41} \cdot e_{1234}$
4. $e_{12} \cdot (e_{23} \cdot e_2)$
5. $e_1 \wedge 1$
6. $(e_{12} + e_{23})^2$
7. $e_{123} \cdot (e_1 \wedge e_{12})$
8. $\langle (e_1 \wedge e_4)e_{13}e_2^{-1}\rangle_3$
9. $-\langle (1 + e_{12})^5\rangle_2$
10. $(\cos\gamma + e_{12}\sin\gamma)^2$

1. 1
2. e_{1234}
3. e_{23}
4. 0
5. $\langle e_1 \rangle_1 = e_1$
6. -2
7. 0
8. $-e_{234}$
9. $14e_{12}$
10. $\cos 2\gamma + e_{12} \sin 2\gamma$

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