6-1 Some group theory

a) A Lie Algebra

$$[T^a, T^b] = i f^{abc} T^c \qquad ※\ f^{abc} : \text{structure constants}$$

and the Jacobi identity

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0$$

imply that

$$0 = [T^a, i f^{bcd} T^d] + [T^b, i f^{cad} T^d] + [T^c, i f^{abd} T^d]$$
$$= -\underbrace{(f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd})}_{=0} T^e$$

b)-2.

- irreducible rep. : if it is not reducible ...
- Simple Lie algebra : a non-Abelian Lie algebra which cannot be devided into two mutually commuting sets of generators.
- semi-simple Lie algebra : a direct sum of simple Lie algebra.

b)-3.

$$[T_R^a, T_R^b] = i f^{abd} T_R^d$$

$$\longrightarrow \mathrm{tr}\{[T_R^a, T_R^b] T_R^c\} = i f^{abd} \underbrace{\mathrm{tr}[T_R^d T_R^c]}_{= C(R)\,\delta^{dc}}$$

$$\Longrightarrow f^{abc} = -\frac{i}{C(R)} \mathrm{tr}\{[T_R^a, T_R^b] T^c\}$$

Because of the cyclic property of the trace

$$\mathrm{tr}\{[T^a, T^b] T^c\} = \mathrm{tr}\{T^a T^b T^c - T^b T^a T^c\}$$
$$= \mathrm{tr}\{T^b T^c T^a - T^c T^b T^a\}$$
$$= \mathrm{tr}\{[T^b, T^c] T^a\}$$

$$\Longrightarrow f^{abc} = f^{bca}$$

Taken together with

$$[T^a, T^b] = i f^{abc} T^c \quad \Longrightarrow \quad f^{abc} = -f^{bac},$$

f^{abc} is totally antisymmetric in all three indices :

$$f^{abc} = f^{bca} = f^{cab} = -f^{bac} = -f^{acb} = -f^{cba}.$$

6-2 Yang-Mills theory

The gauge transformation :

$$\psi(x) \to U(x)\psi(x) \quad , \quad U(x) = \exp[\, i\theta^a(x) T^a]$$

The covariant derivative :

$$D_\mu(x) = \partial_\mu + i g T^a A^a_\mu(x) \qquad \begin{cases} g : \text{the gauge coupling} \\ T^a : \text{the generators} \quad [T^a, T^b] = i f^{abc} T^c \end{cases}$$

a) The gauge bosons transform such that the covariant derivative transforms covariantly under the gauge transformation :

$$\begin{aligned}
D_\mu(x) \to\ & U(x) D_\mu(x) U^{-1}(x) \\
&= U(x)[\partial_\mu + i g T^a A^a_\mu(x)] U^{-1}(x) \\
&= U(x)(\partial_\mu U^{-1}(x)) + \underbrace{U(x) U^{-1}(x)}_{=\mathbb{1}} \partial_\mu + i g\, U(x) T^a A^a_\mu(x) U^{-1}(x) \\
&= \partial_\mu + i g \Big[U(x) T^a A^a_\mu U^{-1}(x) \underbrace{- \tfrac{i}{g} U(x)(\partial_\mu U^{-1}(x))}_{= +\frac{i}{g}(\partial_\mu U(x)) U^{-1}(x)} \Big] \\
&= \partial_\mu + i g T^a A^{a\prime}_\mu(x)
\end{aligned}$$

Let's check the gauge transformation of $D_\mu(x)\psi(x)$:

$$\begin{aligned}
D_\mu(x)\psi(x) \to\ & [\partial_\mu + i g T^a A^{a\prime}_\mu(x)] U(x)\psi(x) \\
&= [\partial_\mu + i g U(x) T^a A^a_\mu(x) U^{-1}(x) + U(x)(\partial_\mu U^{-1}(x))] U(x)\psi(x) \\
&= (\partial_\mu U(x))\psi(x) + U(x)\partial_\mu\psi(x) + i g U(x) T^a A^a_\mu(x)\psi(x) \\
&\qquad + U(x)(\partial_\mu U^{-1}(x)) U(x)\psi(x) \\
&= U(x)[\partial_\mu + i g T^a A^a_\mu(x)]\psi(x) \\
&\qquad + [\cancel{\underbrace{(\partial_\mu U(x))}_{= iT^a(\partial_\mu\theta^a(x))U(x)} + U(x)\underbrace{(\partial_\mu U^{-1}(x))}_{= -iT^a(\partial_\mu\theta^a(x))U^{-1}(x)} U(x)}]\psi(x) \\
&= U(x) D_\mu(x)\psi(x)
\end{aligned}$$

b)

$$\begin{aligned}
[D_\mu, D_\nu] &= [\partial_\mu + i g T^b A^b_\mu \,,\ \partial_\nu + i g T^c A^c_\nu] \\
&= i g T^a (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + (i g)^2 \underbrace{[T^b, T^c]}_{= i f^{abc} T^a} A^b_\mu A^c_\nu \\
&= i g T^a [\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu] \\
&\equiv i g T^a F^a_{\mu\nu} \qquad (F^a_{\mu\nu} : \text{the Yang-Mills field strength})
\end{aligned}$$

$$\begin{aligned}
T^a F^a_{\mu\nu} \to\ & \frac{-i}{g}[D'_\mu, D'_\nu] \\
&= \frac{-i}{g}[U D_\mu U^{-1}, U D_\nu U^{-1}] \\
&= \frac{-i}{g} U [D_\mu, D_\nu] U^{-1} \\
&= U T^a F^a_{\mu\nu} U^{-1}
\end{aligned}$$

Let us show that the Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - M)\psi$$

(※ The mass term of gauge fields $\frac{1}{2} m^2 A^a_\mu A^{a\mu}$ violates gauge invariance.)

is gauge invariant.

The gauge part is rewritten as a trace:

$$\mathrm{tr}\{[D_\mu, D_\nu][D^\mu, D^\nu]\} = \mathrm{tr}\{ig T^a F^a_{\mu\nu} \cdot ig T^b F^{b,\mu\nu}\}$$
$$= (ig)^2 \underbrace{\mathrm{tr}\{T^a T^b\}}_{= C(R)\delta^{ab}} F^a_{\mu\nu} F^{b,\mu\nu}$$
$$= (ig)^2 C(R) F^a_{\mu\nu} F^{a,\mu\nu}$$

$$\Rightarrow F^a_{\mu\nu} F^{a,\mu\nu} = \frac{1}{(ig)^2 C(R)} \mathrm{tr}\{[D_\mu, D_\nu][D^\mu, D^\nu]\}$$

The gauge transformation of the gauge part:

$$F^a_{\mu\nu} F^{a,\mu\nu} \to \sim \mathrm{tr}\{U[D_\mu, D_\nu][D^\mu, D^\nu]U^{-1}\}$$
$$= \mathrm{tr}\{[D_\mu, D_\nu][D^\mu, D^\nu]\} \quad (\because \text{the cyclic property of trace})$$

The gauge transformation of the fermion part:

$$\bar{\psi}(i\gamma^\mu D_\mu - M)\psi \to \bar{\psi} U^{-1}(i\gamma^\mu U D_\mu U^{-1} - M) U\psi$$
$$= \bar{\psi}(i\gamma^\mu D_\mu - M)\psi$$

c) $$\mathcal{L}^{gauge}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu}$$
$$= -\frac{1}{4}[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu][\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} - g f^{ade} A^{d\mu} A^{e\nu}]$$

$$\longrightarrow + \frac{g}{2}(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) f^{abc} A^b_\mu A^c_\nu \quad \cdots \text{cubic}$$
$$- \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} \quad \cdots \text{quartic}$$

} self-interactions for the gauge fields

d) The Lagrangian of the QED: $G = U(1) \to f^{abc} = 0$ (No self-interaction)

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - M)\psi,$$

where
$$\begin{cases} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ D_\mu = \partial_\mu + ieQA_\mu \end{cases}$$

$\left(\begin{array}{l} e = \sqrt{4\pi\alpha} \\ Q = -1 \ (e, \mu, \tau) \quad Q = 2/3 \ (u, c, t) \quad Q = -1/3 \ (d, s, b) \end{array} \right)$

The gauge transformation:

$$\begin{cases} \psi(x) \to U(x)\psi(x) = e^{i\theta(x)Q}\psi(x) \\ A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \theta(x) \end{cases}$$

6-3 Propagator, gauge fixing and massive U(1)s

a)
$$\begin{aligned}
S_{(x)} &= -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} \\
&= -\frac{1}{4}\int d^4x\, (\partial_\mu A_\nu - \partial_\nu A_\mu)F^{\mu\nu} \\
&= -\frac{1}{4}\int d^4x\, [-2A_\nu \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)] \\
&= \frac{1}{2}\int d^4x\, A_\nu[\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu]A_\mu \\
&= \frac{1}{2}\int d^4x\, A^\mu(x)\,[\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu]\,A^\nu(x) \\
&= \frac{1}{2}\int d^4x \int \frac{d^4p'}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\,\tilde{A}^\mu(p')\,e^{-ip'x}\,\underbrace{[\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu]}_{=-p^2 g_{\mu\nu}+p_\mu p_\nu}\,\tilde{A}^\nu(p)\,e^{-ipx} \\
&= \frac{1}{2}\int \frac{d^4p'}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\,\tilde{A}^\mu(p')\,[-p^2 g_{\mu\nu} + p_\mu p_\nu]\,\tilde{A}^\nu(p)\cdot(2\pi)^4\delta^{(4)}(p'+p) \\
&= \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\,\tilde{A}^\mu(-p)\,\underbrace{\left[-\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)p^2\right]}_{\equiv\, \mathcal{O}_{\mu\nu}(p)}\,\tilde{A}^\nu(p)
\end{aligned}$$

(※ Fourier trans.
$$f(x) = \int \frac{d^4p}{(2\pi)^4}\tilde{f}(p)\,e^{-ip\cdot x}$$
$$\tilde{f}(p) = \int d^4x\, f(x)\,e^{ip\cdot x}$$)

(※ $\int d^4x\, e^{ipx} = (2\pi)^4\delta^{(4)}(p)$)

b)
- $P^T_{\mu\nu}P^{T\nu\lambda} = \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\left(g^{\nu\lambda} - \frac{p^\nu p^\lambda}{p^2}\right) = g_\mu{}^\lambda - \frac{p_\mu p^\lambda}{p^2} - \cancel{\frac{p_\mu p^\lambda}{p^2}} + \cancel{\frac{p^2 p_\mu p^\lambda}{p^4}} = P^T_\mu{}^\lambda$
- $P^L_{\mu\nu}P^{L\nu\lambda} = \frac{p_\mu p_\nu}{p^2}\frac{p^\nu p^\lambda}{p^2} = P^L_\mu{}^\lambda$
- $P^T_{\mu\nu}P^{L\nu\lambda} = \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{p^\nu p^\lambda}{p^2} = \frac{p_\mu p^\lambda}{p^2} - \frac{p^2 p_\mu p^\lambda}{p^4} = 0 = P^L_{\mu\nu}P^{T\nu\lambda}$
- $P^T_{\mu\nu} + P^L_{\mu\nu} = g_{\mu\nu}$

A conserved current $J^\mu(x) \Rightarrow \partial_\mu J^\mu(x) = 0 \longrightarrow -ip_\mu \tilde{J}^\mu(p) = 0$.

Therefore, $P^L_{\mu\nu} = \frac{p_\mu p_\nu}{p^2}$ does not couple to a conserved current.

c)
$$\begin{aligned}
S_{fix} &= -\int d^4x\, \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \\
&= \frac{1}{2\xi}\int d^4x\, A^\mu(x)\,\partial_\mu\partial_\nu A^\nu(x)
\end{aligned}$$

$$\Rightarrow \mathcal{O}_{\mu\nu}(p,\xi) = -\left[g_{\mu\nu} - \left(1-\frac{1}{\xi}\right)\frac{p_\mu p_\nu}{p^2}\right]p^2 = -\left[P^T_{\mu\nu} + \frac{1}{\xi}P^L_{\mu\nu}\right]p^2$$

The propagator $\tilde{\Delta}_{\mu\nu}(p,\xi)$ is defined as

$$\mathcal{O}_{\mu\nu}(p,\xi)\,\tilde{\Delta}^{\nu\lambda}(p,\xi) = i\delta_\mu{}^\lambda$$

$$\longrightarrow -\left(P^T_{\mu\nu} + \frac{1}{\xi}P^L_{\mu\nu}\right)p^2\cdot\left(A P^{T\nu\lambda} + B P^{L\nu\lambda}\right) = -p^2 A\, P^T_\mu{}^\lambda - \frac{p^2}{\xi}B\, P^L_\mu{}^\lambda = i\delta_\mu{}^\lambda$$

$$\Rightarrow -p^2 A = -\frac{p^2}{\xi}B = i$$

Therefore,

$$\begin{aligned}
\tilde{\Delta}_{\mu\nu}(p,\xi) &= \left(P^T_{\mu\nu} + \xi P^L_{\mu\nu}\right)\frac{-i}{p^2+i\varepsilon} \\
&= \left(g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)\frac{-i}{p^2+i\varepsilon}
\end{aligned}$$

$$\begin{cases} \xi = 0 : & \text{Landau gauge} \\ \xi = 1 : & \text{Feynman gauge} \end{cases}$$

d) $$S_{mass} = \int d^4x \, \frac{1}{2} M_A^2 A_\mu A^\mu$$
$$= \frac{1}{2} \int d^4x \, A^\mu(x) \, M_A^2 \, g_{\mu\nu} \, A^\nu(x)$$

$$\Rightarrow \quad O_{\mu\nu}(P, M_A) = (-p^2 + M_A^2)\, g_{\mu\nu} + P_\mu P_\nu = -(p^2 - M_A^2)(P^T_{\mu\nu} + P^L_{\mu\nu}) + P^2 P^L_{\mu\nu}$$
$$= -(P^2 - M_A^2) P^T_{\mu\nu} + M_A^2 P^L_{\mu\nu}$$

$$O_{\mu\nu}(P, M_A)\, \tilde{\Delta}^{\nu\lambda}(P, M_A) = i \delta_\mu^{\ \lambda}$$
$$\rightarrow \left(-(P^2 - M_A^2) P^T_{\mu\nu} + M_A^2 P^L_{\mu\nu}\right)\left(A\, P^{T\nu\lambda} + B\, P^{L\nu\lambda}\right)$$
$$= -(P^2 - M_A^2) A\, P^{T\ \lambda}_{\mu} + M_A^2 B\, P^{L\ \lambda}_{\mu} = i \delta_\mu^{\ \lambda}$$
$$\Rightarrow \quad -(P^2 - M_A^2) A = M_A^2 B = i$$

Therefore,

$$\tilde{\Delta}_{\mu\nu}(P, M_A) = \frac{-i}{P^2 - M_A^2 + i\varepsilon} P^T_{\mu\nu} + \frac{i}{M_A^2} P^L_{\mu\nu}$$
$$= \frac{-i}{P^2 - M_A^2 + i\varepsilon}\left(g_{\mu\nu} - \frac{P_\mu P_\nu}{P^2}\right) + \frac{i}{M_A^2}\frac{P_\mu P_\nu}{P^2}$$
$$= \frac{-i}{P^2 - M_A^2 + i\varepsilon}\left[g_{\mu\nu} - \left(1 + \frac{P^2 - M_A^2}{M_A^2}\right)\frac{P_\mu P_\nu}{P^2}\right]$$
$$= \frac{-i}{P^2 - M_A^2 + i\varepsilon}\left[g_{\mu\nu} - \frac{P_\mu P_\nu}{M_A^2}\right]$$

e) As we saw in b), a term proportional to $P_\mu P_\nu$ in the propagator does not couple to a conserved current. Therefore, the relevant part of the propagator falls off as $1/p^2$, which ensures the renormalizability of the theory.