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The Standard Model of Particle Physics - SoSe 2010 Assignment 3

(Due: May 6, 2010)

1 The gamma-matrices

a.) The gamma-matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{1}$$

The significance of the Clifford algebra is that it induces a representation of the Lorentz algebra as follows: Consider the set of matrices

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{2}$$

These satisfy the relation

$$[\sigma^{\mu\nu}, \sigma^{\alpha\beta}] = 2i\Big(g^{\mu\beta}\sigma^{\nu\alpha} + g^{\nu\alpha}\sigma^{\mu\beta} - g^{\mu\alpha}\sigma^{\nu\beta} - g^{\nu\beta}\sigma^{\mu\alpha}\Big) \tag{3}$$

as a consequence of the Clifford algebra and thus form a representation of the Lorentz algebra, as promised (cf. Assignment 1, Exercise 1).

Give the four-dimensional representation of the gamma-matrices introduced in the lecture and check explicitly that they satisfy (1) as well as

$$\gamma^0 = (\gamma^0)^\dagger, \qquad \gamma^i = -(\gamma^i)^\dagger. \tag{4}$$

b.) The matrix γ_5 is defined as $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Without using a concrete representation of the gamma-matrices, prove that

$$\gamma^5 = (\gamma^5)^\dagger, \qquad (\gamma^5)^2 = 1, \qquad \{\gamma^5, \gamma^\mu\} = 0. \tag{5}$$

2 The Dirac spinor

a.) A Dirac spinor is defined by its properties under Lorentz transformations. Give these properties in terms of the matrix

$$S(\Lambda) = \exp(-\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}). \tag{6}$$

What's the difference between the transformation of a vector and a Dirac spinor?

Optional: Show that

$$\frac{i}{4}[\sigma^{\mu\nu}, \gamma^{\rho}]\,\omega_{\mu\nu} = \omega^{\rho}{}_{\nu}\gamma^{\nu}. \tag{7}$$

Hint: Rewrite the commutators in terms of anti-commutators.

Argue that this is the infinitesimal version of the more general relation

$$S^{-1}(\Lambda)\gamma^{\mu}S^{-1}(\Lambda) = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}. \tag{8}$$

b.) The Dirac equation for a spinor field of mass m is

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0. \tag{9}$$

Transform this equation into a new frame $x' = \Lambda x$ and show that if $\psi(x)$ satisfies the Dirac equation in the original frame, then $\psi'(x')$ does so in the new frame.
Hint: Equation (8) is useful.

c.) The conjugate of a Dirac spinor is defined as

$$\bar{\psi}(x) = \psi^{\dagger}\gamma^{0}. \tag{10}$$

Show that the spinor bilinear

$$S(x) = \bar{\psi}(x)\psi(x) \tag{11}$$

transforms like a scalar under Lorentz transformations. Do so by showing that $\bar{\psi}(x)$ transforms like

$$\bar{\psi}'(x') = \bar{\psi}(\Lambda^{-1}x')S^{-1}(\Lambda) \tag{12}$$

under Lorentz transformations. Show similarly that

$$J^{\mu} = \bar{\psi}(x)\gamma^{\mu}\psi(x) \tag{13}$$

transforms like a vector.

d.) Recall the form of the plane wave solution $\psi(x) = u(p)\text{exp}(-ipx)$ and $\psi(x) = v(p)\text{exp}(ipx)$ as given in the lecture.

Show that

$$\begin{aligned} \gamma^0 \, (i\vec{\gamma}\cdot\vec{\partial} + m) \, u(p) \, \exp(-ipx) &= p^0 \, u(p)\text{exp}(-ipx), \\ \gamma^0 \, (i\vec{\gamma}\cdot\vec{\partial} + m) \, v(p) \, \exp(ipx) &= -p^0 \, u(p)\text{exp}(ipx). \end{aligned} \tag{14}$$

e.) The Hamiltonian associated with the Dirac Lagrangian $\mathcal{L} = \bar{\psi}(i\gamma\partial - m)\psi$ is

$$H = \int d^3x \Big(\Pi(0,\vec{x})\psi(0,\vec{x}) - \mathcal{L}|_{x=(0,\vec{x})} \Big), \qquad \Pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^\dagger. \tag{15}$$

Use (14) to show that the mode expansion of H is

$$H = \int d^3p \frac{1}{(2\pi)^3} \frac{1}{2E_p} \frac{1}{2} \sum_s \Big(\alpha_s^*(\vec{p})\alpha_s(\vec{p}) - \beta_s(\vec{p})\beta_s^*(\vec{p}) \Big), \tag{16}$$

where

$$\psi(x) = \int d^3x \frac{1}{(2\pi)^3} \frac{1}{2E_p} \sum_s \Big(e^{-ipx} u_s(p)\alpha_s(p) + e^{ipx} v_s(p)\beta_s^*(p) \Big). \tag{17}$$

Discuss the consequences of the minus sign for quantisation of the theory and compare to the case of a scalar field.

3 Spin and $SU(2)$

a.) Consider quantum mechanics in three dimensional space $\mathbb{R}^3$, setting, as always, $\hbar = 1$. The angular momentum operator is defined as

$$\vec{L} = (L)_i, \qquad\qquad L_i = -i\,\epsilon_{ijk}\,x_j\partial_k, \tag{18}$$

where we are not distinguishing between up and down indices as is appropriate in $\mathbb{R}^3$. Show that under a rotation $\vec{x} \to \vec{x}'$ by an angle θ around an axis in normalised direction $\vec{n}$, a wavefunction $\psi(\vec{x})$ transforms as

$$\psi(\vec{x}) \to \psi'(x') = e^{-i\,\theta\,\vec{n}\cdot\vec{L}}\psi(\vec{x}). \tag{19}$$

Do so by considering the infinitesimal version of this transformation.

b.) Show that the components of the angular momentum satisfy the commutation relations

$$[L_i, L_j] = i\epsilon_{ijk}L_k. \tag{20}$$

These are the commutation relations of the Lie algebra of the group $SU(2)$, which is the double cover of the group $SO(3)$ of rotations in $\mathbb{R}^3$. Explain the appearance of this group in view of the results of a.)

Show that the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy the relation

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k. \tag{21}$$

Conclude that the matrices $\frac{1}{2}\sigma^i$ form a representation of $SU(2)$ as in b.)

c.) Deduce that a spin $\frac{1}{2}$ particle is described by a 2-component spinor χ that transforms under spatial rotations as

$$\chi \to e^{-\frac{i}{2}\theta \vec{n}\cdot\vec{\sigma}}\chi. \tag{22}$$

In particular this shows that a spin $\frac{1}{2}$ only comes back to itself up to a minus sign under rotations by 2π.

d.) Consider now Dirac theory in 3+1 dimensions. Review carefully the logic in the lecture and argue that a Dirac spinor indeed describes a spin $\frac{1}{2}$ particle. **This derivation is examinable!**

4 Noether's Theorem

Consider a Lagrangian $\mathcal{L}(\phi, \partial\phi)$ and suppose the corresponding action is invariant under a *global* symmetry transformation parametrised by the *constant* parameter α whose infinitesimal form is

$$\phi(x) \to (1 + \delta_\alpha)\phi(x). \tag{23}$$

This symmetry induces a conserved current $J^\mu(x)$ with the property

$$\partial_\mu J^\mu = 0. \tag{24}$$

This can be seen by promoting the parameter α in the the transformation (23) to a spacetime-dependent quantity $\alpha(x)$. The action is now not invariant under (23) any more. Since it is invariant if α is constant, the variation can only depend on $\partial\alpha$. This defines the current J^μ via

$$\delta S = \int d^4x J^\mu(x)\, \partial_\mu \alpha(x). \tag{25}$$

Upon integration by parts and using that $\delta S = 0$ for α constant, one recovers (24).

a.) Show that the quantity

$$Q(t) = \int d^3x \, J^0(t, \vec{x}) \tag{26}$$

is conserved:

$$\frac{\partial}{\partial t} Q(t) = 0. \tag{27}$$

It is called the conserved charge associated with the symmetry (23). In particular, Q defines good, i.e. conserved, quantum numbers for the states of the theory.

b.) Consider now the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma\partial - m)\psi. \tag{28}$$

Argue that it is invariant under

$$\psi \rightarrow exp(i\alpha)\psi(x). \tag{29}$$

Deduce from this the existence of the conserved current

$$J^\mu(x) = -e\bar{\psi}(x)\gamma^\mu\psi(x). \tag{30}$$

Give the form of the associated conserved charge. Argue that Dirac theory describes both particles and anti-particles.

c.) Use the canonical anti-commutation relations to show that

$$Q\psi(x) = \psi(x)(Q - 1) \tag{31}$$

and thus

$$e^{-i\alpha Q}\psi(x)e^{+i\alpha Q} = e^{i\alpha}\psi(x). \tag{32}$$

This means that the conserved charge generates the continuous symmetry underlying its existence.