Introduction to the Standard Model of Particle Physics

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Chapter 1

Pre-requisits

1.1 Special Relativity

The Minkowski space is a four dimensional space with the following metric:

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{1.1}$$

One point in the Minkowski space is a contravariant vector:

$$(x^\mu) = \begin{pmatrix} c \cdot t \\ \vec{x} \end{pmatrix}$$

with $\mu = 0, 1, 2, 3$ and $c = \hbar = 1$

$$(x^\mu) = \begin{pmatrix} t \\ \vec{x} \end{pmatrix} .$$

The scalar product of space-time differences

$$\begin{aligned} (x-y)^2 &= (x-y)^\mu \cdot g_{\mu\nu} \cdot (x-y)^\nu \\ &= (x-y)^\mu \cdot (x-y)_\mu \\ &= (x-y)_0^2 - (\vec{x}-\vec{y})^2 \end{aligned} \tag{1.2}$$

with $x_\mu = g_{\mu\nu} \cdot x^\nu$ is a covariant vector.

Symmetry transformation? What leaves $(x-y)^\mu (x-y)_\mu$ invariant?
Poincaré transformations (Λ, a):

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu \, x^\nu + a^\mu \tag{1.3}$$

Composition:

$$(\Lambda_1, a_1) \circ (\Lambda_2, a_2) = (\Lambda_1 \cdot \Lambda_2, \Lambda_1 a_2 + a_1) \tag{1.4}$$

Invariance:

$$\Lambda^T \; g \; \Lambda = g \tag{1.5}$$

In components:

$$\Lambda^\rho{}_\mu \; g_{\rho\sigma} \; \Lambda^\sigma{}_\nu = g_{\mu\nu}$$

or

$$\begin{aligned} \Lambda_\sigma{}^\mu \; \Lambda^\sigma{}_\nu &= g^\mu{}_\nu \\ &= \delta^\mu{}_\nu \end{aligned}$$

$$\begin{aligned} \det \Lambda &= \pm 1 \\ \Rightarrow (\Lambda^{-1})^\mu{}_\sigma &= \Lambda_\sigma{}^\mu \end{aligned} \tag{1.6}$$

Lorentz group $(\Lambda, 0)$:
Component of unity (orthochron):

$$\det \Lambda = 1, \;\; \Lambda^0{}_0 > 0 \tag{1.7}$$

Parity P: $\vec{x} \to -\vec{x}$

$$\Lambda_P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{1.8}$$

Time reversal T: $x_0 \to -x_0$

$$\Lambda_T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{1.9}$$

Generators of Lorentz group:
Boost along x-axis:

$$\begin{aligned} \Lambda^\mu{}_\nu &= \begin{pmatrix} \gamma & -\gamma \cdot \frac{v}{c} & & \\ -\gamma \cdot \frac{v}{c} & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \; \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \\ &= \left(e^{\omega K_1}\right)^\mu{}_\nu \end{aligned} \tag{1.10}$$

with rapidity ω

$$\omega = \operatorname{arctanh}\left(\frac{v}{c}\right) \tag{1.11}$$

and generator K_1

$$K_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{1.12}$$

$\Rightarrow$ Lorentz algebra
Boosts K_i, rotations J_i:

$$\begin{aligned} [J_i, J_j] &= i\,\varepsilon_{ijk}\,J_k \\ [K_i, K_j] &= -i\,\varepsilon_{ijk}\,J_k \\ [J_i, K_j] &= i\,\varepsilon_{ijk}\,K_k \end{aligned} \tag{1.13}$$

1.2 Transition rates

Perturbation theory for computing decay rates and cross sections

$$H_0\phi_n = E_n\phi_n \tag{1.14}$$

with

$$\int_V \phi_m^* \phi_n \mathrm{d}^3x = \delta_{mn} .$$

Perturbation H':

$$(H_0 + H')\psi = i\,\frac{\partial\psi}{\partial t} \tag{1.15}$$

What is the *transition rate* from ϕ_i at $-\frac{T}{2}$ to ϕ_f at $\frac{T}{2}$?

$$\phi(x) = \sum_n c_n(t) \cdot \phi_n(\vec{x}) \cdot e^{-iE_nt}$$

$$c_n\left(-\frac{T}{2}\right) = \delta_{ni} \tag{1.16}$$

From equation (1.15):

$$\frac{\partial\phi}{\partial t} = -i(H_0 + H')\phi$$

$$\begin{aligned} \frac{\mathrm{d}c_f(t)}{\mathrm{d}t} &= -i \sum_n c_n(t) \underbrace{\int \mathrm{d}^3r\;\phi_f^* H' \phi_n}_{\langle f|H'|n\rangle} e^{i(E_f-E_n)t} \\ &\cong -i\,\langle f|H'|i\rangle\, e^{i(E_f-E_i)t} + O(H'^2) \\ c_i(t) = 1 + O(H'^2) & \\ \Rightarrow c_f(t) &\cong -i \int_{-\frac{T}{2}}^{t} \mathrm{d}t'\,\langle f|H'|i\rangle\, e^{i(E_f-E_i)t'} \end{aligned} \tag{1.17}$$

$\Rightarrow$ Transition amplitude A_{fi}:

$$
\begin{aligned}
A_{fi} &= c_f\left(\frac{T}{2}\right) \\
A_{fi} &= -i \int\limits_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{d}t \, \langle f|H'|i\rangle \, e^{i(E_f-E_i)t} \qquad (1.18)
\end{aligned}
$$

or

$$
A_{fi}(T\to\infty) = -i\int \phi_f^*(x)H'\phi_i(x)\;\mathrm{d}^4x \qquad (1.19)
$$

with

$$
\phi_n(x) = \phi_n(\vec{x})e^{-iE_nt}
$$

Transition probability: H' time-independent

$$
\begin{aligned}
\lim_{T\to\infty}|A_{fi}|^2 &= |\,\langle f|H'|i\rangle\,|^2\cdot\int\limits_{-\frac{T}{2}}^{\frac{T}{2}}\mathrm{d}t\; e^{i(E_f-E_i)t}\cdot\int\limits_{-\frac{T}{2}}^{\frac{T}{2}}\mathrm{d}t'\; e^{i(E_f-E_i)t'} \\
\text{Fermi}'\text{ s trick} &= T\cdot 2\pi\,|\,\langle f|H'|i\rangle\,|^2\;\delta(E_f-E_i) \qquad (1.20)
\end{aligned}
$$

$\Rightarrow$ Transition rate Γ:

$$
\Gamma(i\to f) = \lim_{T\to\infty}\frac{|A_{fi}|^2}{T} = 2\pi\,|\,\langle f|H'|i\rangle\,|^2\;\delta(E_f-E_i) \qquad (1.21)
$$

Integrating over final states (ρ: phase space density):

$$
\Gamma\,[i\to f] = \int \mathrm{d}E_f\;\rho(E_f)\cdot 2\pi\,|\,\langle f|\,H'\,|i\rangle\,|^2\;\delta(E_f-E_i) \qquad (1.22)
$$

$$
\boxed{\Rightarrow \Gamma\,[i\to f] = 2\pi\,|\,\langle f|H'|i\rangle\,|^2\;\rho(E_i)} \qquad (1.23)
$$

In general:

$$
\Gamma\,[i\to f] = 2\pi\;|T_{fi}|^2\;\rho(E_i)
$$

with

$$
T_{fi} = \langle f|\,H'\,|i\rangle + \sum \frac{\langle f|\,H'\,|n\rangle\,\langle n|\,H\,|i\rangle}{1}\;.
$$

Chapter 2

Quantum Field Theory

2.1 The free scalar field

Spin 0, neutral particles, e.g. Π_0, described by a real scalar field φ:

$$\varphi^*(x) = \varphi(x) \tag{2.1}$$

Property under Lorentz transformations:

$$\varphi'(x') = \varphi(x) \quad \text{scalar} \tag{2.2}$$

The equation of motion, free, up to second order in derivatives (unique if local) is called *Klein-Gordon equation*:

$$(\partial_\mu \partial^\mu + m^2)\varphi(x) = 0 \tag{2.3}$$

with

$$\begin{aligned} \partial_\mu \partial^\mu &= \partial_t^2 - \vec{\nabla}^2 \\ &= \partial_t^2 - \triangle \\ &= \Box \\ &= -p_\mu^2 \,. \end{aligned}$$

$\Box$: d′Alembert operator, m^2: mass of the scalar particle
The Klein-Gordon equation can be derived out of Boosts from the rest frame equation of motion:

$$(E^2 - m^2)\varphi(x) = 0 \quad \text{unique} \tag{2.4}$$

The most fruitful approach to Elementary Particle Physics is via the *action principle*.
Lagrange density of a free scalar field:

$$\mathcal{L}(x) = \frac{1}{2}[\partial_\mu \varphi(x) \partial^\mu \varphi(x) - m^2 \varphi^2(x)] \,. \tag{2.5}$$

Action S:

$$\begin{aligned} S[\varphi] &= \int \mathrm{d}^4x \, \mathcal{L}(x) \\ &= \frac{1}{2}\int \mathrm{d}^4x \big(\partial_\mu\varphi(x)\partial^\mu\varphi(x) - m^2\varphi^2(x)\big) \end{aligned} \tag{2.6}$$

Action principle:

$$\delta S[\varphi] = 0$$

or

$$\frac{\delta S}{\delta\varphi(x)} = 0$$

with

$$\frac{\delta\varphi(y)}{\delta\varphi(x)} = \delta^{(4)}(x-y)$$

and

$$\frac{\delta\partial_\mu\varphi(y)}{\delta\varphi(x)} = \partial_\mu^y \, \delta^{(4)}(x-y) \tag{2.7}$$

results in the following equation

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = 0 \tag{2.8}$$

which equals (2.3).

Classical solutions of the Klein-Gordon equation, take e.g. $\varphi = \varphi(x^1)$:

$$(-\partial_1^2 + m^2)\varphi(x_1) = 0 \, .$$

$\Rightarrow$ Solutions are plane waves:

$$\varphi(x) = e^{\pm ikx} \tag{2.9}$$

with

$$kx = k^\mu x_\mu$$

with

$$k^2 = m^2 \, , \; k^0 = \pm\omega = \pm\sqrt{\vec{k}^2 + m^2} \, .$$

General solution: linear superposition of plane waves.

$$\varphi(x) = \underbrace{\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\omega}}_{\int \frac{\mathrm{d}^4k}{(2\pi)^4}\delta(k^2-m^2)} \big(e^{ikx}\alpha^*(\vec{k}) + e^{-ikx}\alpha(\vec{k})\big) \tag{2.10}$$

with

$$k = \begin{pmatrix} \omega \\ \vec{k} \end{pmatrix} \, .$$

QFT:

$$\varphi(x) \to \phi(x) \quad \text{operator}$$

The expectation value $\langle \phi(x) \rangle$ is a classical field. ϕ obeys canonical commutation relations:

$$[\phi(\vec{x}, t), \dot{\phi}(\vec{y}, t)] = i\, \delta^{(3)}(\vec{x} - \vec{y})\ . \tag{2.11}$$

$\dot{\phi}(\vec{y}, t)$ is the canonical conjugated momentum.

Let a be the lattice parameter of a crystal.

$$\underbrace{\circ\quad\circ}_{a}\quad\circ\quad\circ$$

$$\text{Classical mechanics} \quad \xrightarrow{a \to 0} \quad \text{field theory}$$

$$\begin{array}{ccc} \text{QM} & \xrightarrow{a \to 0} & \text{QFT} \\ [x, p] = i(\hbar) & & [\phi, \Pi_0] = i \end{array}$$

$\phi(x)$ still obeys the Klein-Gordon equation $(\Box + m^2)\phi = 0$.

$$\Rightarrow \phi(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2\omega} \left[e^{ikx} a^\dagger(\vec{k}) + e^{-ikx} a(\vec{k}) \right] \tag{2.12}$$

Inserting (2.12) into (2.11) results in

$$\begin{aligned} \left[a(\vec{k}), a^\dagger(\vec{k}') \right] &= (2\pi)^3 \cdot 2\omega\, \delta^{(3)}(\vec{k} - \vec{k}') \\ \left[a(\vec{k}), a(\vec{k}') \right] &= 0 = \left[a^\dagger(\vec{k}), a^\dagger(\vec{k}') \right] \end{aligned} \tag{2.13}$$

Fock space
$|0\rangle$: normalised vacuum state: $\langle 0|0\rangle = 1$ with

$$a(\vec{k})\, |0\rangle = 0\ .$$

$|0\rangle$ is the lowest energy state! a annihilates the vacuum.
Heisenberg picture:

$$\partial_t\, |0\rangle = 0 \tag{2.14}$$

All states are generated by applying a, $a^\dagger$ on $|0\rangle$. a, $a^\dagger$ are annihilation and creation operators, respectively.
One particle states:

$$|k\rangle = a^\dagger(\vec{k})\, |0\rangle\ . \tag{2.15}$$

The states $|k\rangle$ are orthogonal:

$$\begin{aligned} \langle k'|k\rangle &= \langle 0|\, a(\vec{k}') a^\dagger(\vec{k})\, |0\rangle \\ &= \langle 0|\, [a(\vec{k}'), a^\dagger(\vec{k})]\, |0\rangle \\ &= (2\pi)^3 \cdot 2\omega\, \delta^{(3)}(\vec{k} - \vec{k}') \end{aligned} \tag{2.16}$$

⇒ General one-particle state:

$$|f\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3 \cdot 2\omega} f(\vec{k})\ a^\dagger(\vec{k})\ |0\rangle \tag{2.17}$$

Example: two state system (spin)
With

$$\begin{aligned} aa^\dagger + a^\dagger a &= 1 \\ a^\dagger\,|0\rangle &= |1\rangle \\ a\,|0\rangle &= 0 \\ a^\dagger\,|1\rangle &= 0 \end{aligned}$$

Realisation:

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad a^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ .$$

Realised in spin system.
Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ ,\ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ ,\ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ .$$

$$[\sigma^i, \sigma^j] = 2i\ \varepsilon^{ijk}\sigma^k$$

Note that:

$$\begin{aligned} \{\sigma^i, \sigma^j\} &= 2\ \delta^{jk} \\ \sigma_\pm &= \frac{1}{2}(\sigma^1 \pm i\sigma^2) \\ \sigma_+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ , \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\ . \end{aligned} \tag{2.18}$$

N-Particle states

$$\begin{aligned} &\text{2 particles:} \quad a^\dagger(k_2)\ a^\dagger(k_1)\,|0\rangle \\ &\text{3 particles:} \quad a^\dagger(k_3)\ a^\dagger(k_2)\ a^\dagger(k_1)\,|0\rangle \\ &\qquad\vdots \end{aligned} \tag{2.19}$$

Have Bose symmetry, as

$$a^\dagger(k_2)\ a^\dagger(k_1) = a^\dagger(k_1)\ a^\dagger(k_2) \tag{2.20}$$

Energy-momentum is additive. Take some state $|\beta\rangle$, then $a^\dagger(k)\,|\beta\rangle$ is a state with *one* additional particle with momentum k.

Annihilation:
$a(\vec{k})\,|\beta\rangle$ is a state, where a particle with momentum k is removed from the state $|\beta\rangle$.
Example with a general particle state $|f\rangle$ (see equation (2.17)):

$$\begin{aligned} a(\vec{k})|f> &= a(\vec{k}) \int \frac{\mathrm{d}^3k'}{(2\pi)^3} \frac{1}{\sqrt{2\omega'}} f(\vec{k}') \cdot a^\dagger(\vec{k}')\,|0\rangle \\ &= \int \frac{\mathrm{d}^3k'}{(2\pi)^3} \frac{1}{\sqrt{2\omega'}} f(\vec{k}') \cdot \underbrace{[a(\vec{k}),\ a^\dagger(\vec{k}')]}_{(2.14)(2\pi)^3\cdot 2\omega\delta^{(3)}(\vec{k}-\vec{k}')} |0\rangle \\ &= \sqrt{2\omega} f(\vec{k})\,|0\rangle \end{aligned} \tag{2.21}$$

Interpretation of $\phi(x)$:

- states with defined particle number n have vanishing expectation values of ϕ, as ϕ creates and annihilates a particle, see equation (2.12).
  This follows from

$$\begin{aligned} \langle 0|\,a^\dagger\,|0\rangle &= \langle 0|\,a\,|0\rangle = 0 \\ \langle k|\,a^\dagger\,|k'\rangle &= \langle k|\,a\,|k'\rangle = 0 \\ &\vdots \\ \Rightarrow\ \langle 0|\,\phi(x)\,|0\rangle &= 0\ \ldots \end{aligned} \tag{2.22}$$

- coherent states: $\langle\phi\rangle$ behaves like a classical wave.

$$|\alpha\rangle = \frac{1}{\mathcal{N}} \underbrace{\exp\left\{\int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{2\omega} \alpha(\vec{k}) a^\dagger(\vec{k})\right\} |0\rangle}_{|\alpha_0\rangle} \tag{2.23}$$

  with

$$\mathcal{N} = \exp\left\{\frac{1}{2} \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{2\omega} |\alpha(\vec{k})|^2\right\} = \langle\alpha_0|\alpha_0\rangle$$

  and $\alpha(\vec{k})$ coefficient function.

$$\langle\alpha|\alpha\rangle = 1$$

  via

$$\langle 0|\,\alpha(\vec{k_1'})\cdots\alpha(\vec{k_m'})\cdot\alpha^\dagger(\vec{k_n'})\ldots\alpha^\dagger(\vec{k_1'})\,|0\rangle\ \sim \delta_{nm}\left[2\omega(2\pi)^3\right]^n$$

$$\Rightarrow\ \langle\alpha|\,\phi(x)\,|\alpha\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{2\omega} \left\{e^{ikx}\alpha^*(\vec{k}) + e^{-ikx}\alpha(\vec{k})\right\} \tag{2.24}$$

Using

$$
\begin{aligned}
& \frac{1}{n!} a(k) \left[\int \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{1}{2\omega'} \alpha(\vec{k}') a^\dagger(\vec{k}') \right]^n |0\rangle = \\
= \quad & \frac{1}{n!}\ n\ \alpha(\vec{k}') \left[\int \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{1}{2\omega'} \alpha(\vec{k}') a^\dagger(\vec{k}') \right]^{n-1} |0\rangle \quad \text{(see equation (2.16))} \\
= \quad & \alpha(\vec{k}')\ \frac{1}{(n-1)!} \left[\int \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{1}{2\omega'} \alpha(\vec{k}') a^\dagger(\vec{k}') \right]^{n-1} |0\rangle
\end{aligned}
\tag{2.25}
$$

and similarily

$$
\frac{1}{n!} \left[\int \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{1}{2\omega'} \alpha^*(\vec{k}') a(\vec{k}') \right]^n a^\dagger(k) = \frac{1}{(n-1)!} \left[\int \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{1}{2\omega'} \alpha^*(\vec{k}') a(\vec{k}') \right]^{n-1} \alpha^*(\vec{k}') \tag{2.26}
$$

Symmetries:
By partial integration of equation (2.5) one gets:

$$
S[\phi] = \int \mathrm{d}^4 x\ \mathcal{L}(x) = \frac{1}{2} \int \mathrm{d}^4 x\ \phi(x)[-\partial_\mu \partial^\mu - m^2]\phi(x)\ . \tag{2.27}
$$

1. Invariance of S[ϕ] under orthochronous Poincaré transformations

$$
\begin{aligned}
x'^\mu &= \Lambda^\mu{}_\nu x^\nu + a^\mu \quad \text{(see equation (1.3))} \\
\phi'(x') &= \phi(x) \\
\Rightarrow \partial_\mu{}' \partial^{\mu\prime} &= \partial_\mu \partial^\mu
\end{aligned}
\tag{2.28}
$$

with

$$
\Lambda^T g \Lambda = g
$$

and

$$
(\Box' + m^2)\phi'(x') = (\Box + m^2)\phi(x) = 0
$$

Unitary Representation: $U(\Lambda, a)$

$$
\phi(x) = \phi'(x') = U^\dagger(\Lambda, a)\ \phi(x')\ U(\Lambda, a)
$$

$$
\begin{aligned}
U(\Lambda, a)\ \phi(x)\ U^\dagger(\Lambda, a) &= \phi(x') \\
&= \phi(\Lambda x + a)
\end{aligned}
\tag{2.29}
$$

On Fockspace:

$$
\begin{aligned}
U(\Lambda, a)\,|0\rangle &= |0\rangle \\
U(\Lambda, a)\ a^\dagger(\vec{k})\ U^\dagger(\Lambda, a) &= e^{ik'a} a^\dagger(\vec{k}')
\end{aligned}
$$

with

$$
k'^\mu = \Lambda^\mu{}_\nu k^\nu
$$

2. Invariance of S[ϕ] under Parity transformations

$$x'^\mu = \Lambda_{P\ \nu}^{\ \mu}\, x^\nu \tag{2.30}$$

with

$$\Lambda_P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} .$$

Unitary Representation

$$\begin{aligned} U(P)\, \phi(x)\, U^\dagger(P) &= \eta_P\, \phi(x') \\ \Rightarrow U(P)\, \phi(\vec{x},t)\, U^\dagger(P) &= \eta_P\, \phi(-\vec{x},t) \end{aligned} \tag{2.31}$$

with intrinsic parity $\eta_P = \pm 1$.
On Fockspace:

$$\begin{aligned} U(P)\, |0\rangle &= |0\rangle \\ U(P)\, a^\dagger(\vec{k})\, U^\dagger(P) &= \eta_P\, a^\dagger(-\vec{k}) \end{aligned} \tag{2.32}$$

Parity reverses 3-momentum of particle:

Scalar fields:	$\eta_P = +1$
Pseudo scalar fields:	$\eta_P = -1$

e.g. Π_0
Parity:

$$\begin{aligned} \vec{x} &\to -\vec{x} \\ \vec{p} &\to -\vec{p} \end{aligned}$$

What about Parity transformations of pseudovectors like e.g. the angular momentum $\vec{L}$: $\vec{L} = \vec{x} \times \vec{p}$?

$$\vec{L} \to \vec{x} \times \vec{p} \;\; \text{pseudo vector}$$

So what about e.g. $\vec{x} \cdot \vec{L}$ or $\vec{p} \cdot \vec{L}$?

$$\begin{aligned} \vec{x} \cdot \vec{L} &\to -\vec{x} \cdot \vec{L} \\ \vec{p} \cdot \vec{L} &\to -\vec{p} \cdot \vec{L} \;\; \text{pseudoscalars} \end{aligned}$$

3. Invariance of S[ϕ] under time reversal

$$x'^\mu = \Lambda_{T\ \nu}^{\ \mu}\, x^\nu \tag{2.33}$$

with

$$\Lambda_T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} .$$

Anti-unitary transformation V with

(a)
$$V(c_1 \cdot |a\rangle + c_2 \cdot |b\rangle) = c_1^* \cdot V\,|a\rangle + c_2^* \cdot V\,|b\rangle \tag{2.34}$$
(b)
$$V^\dagger V = VV^\dagger = \mathbb{1} \tag{2.35}$$
(c)
$$\langle a|\, V^\dagger\, |b\rangle \;=\; \langle b|\, V\, |a\rangle \tag{2.36}$$
We have
$$\left[V(\Lambda_T)\;\phi(\vec{x},t)\;V^\dagger(\Lambda_T)\right]^\dagger = \phi(\vec{x}-t) \tag{2.37}$$
On Fockspace:
$$\begin{aligned} V(\Lambda_T)\,|0\rangle &= |0\rangle \\ V(\Lambda_T)\;a^\dagger(\vec{k})\;V^\dagger(\Lambda_T) &= a^\dagger(-\vec{k}) \end{aligned} \tag{2.38}$$
Note that
$$\begin{aligned} |a'\rangle &= V(\Lambda_T)\,|a\rangle \\ |b'\rangle &= V(\Lambda_T)\,|b\rangle \end{aligned} \tag{2.39}$$
and
$$\langle a'|b'\rangle \;=\; \langle b|a\rangle \;=\; \langle a|b\rangle^* \tag{2.40}$$

4. Charge conjugation: complex
   2., 3., 4. CPT-invariance is required for any *local*, *relativistic* QFT.
   But CP, P, T violation is permitted and realised.

2.2 The interacting scalar field

In this chapter some basic concepts on scattering/perturbation theory are introduced. Interaction of a real scalar field with a *static potential* $V(\vec{x})$, e.g. a localised potential produced by a nucleus.
Langrange density ($H = H_0 + H'$):
$$\begin{aligned} \mathcal{L}(x) &= \mathcal{L}_0(x) + \mathcal{L}'(x) \\ &= \underbrace{\frac{1}{2}\varphi(x)\left(-\partial_\mu\partial^\mu - m^2\right)\varphi(x)}_{\mathcal{L}_0(x)} \underbrace{-\frac{1}{2}V(\vec{x})\varphi^2(x)}_{\mathcal{L}'(x)} \\ \mathcal{L}'(x) &= -\frac{1}{2}V(\vec{x})\varphi^2(x) \end{aligned} \tag{2.41}$$

$\mathcal{L}_0(x)$ is the Lagrange density of a free scalar field. $\mathcal{L}'(x)$ is the Lagrange interaction density.

QM revisited: interaction picture
$$i\,\frac{\partial}{\partial t}\,|t\rangle = H'(t)\,|t\rangle \tag{2.42}$$

H' is the *interaction Hamiltonian* (see (1.15)).

$$\begin{aligned} \left| t < -\frac{T}{2} \right\rangle &= |i\rangle \quad \text{adiabatic} \\ \left| t > \frac{T}{2} \right\rangle &= |f\rangle \\ t_0 = -\frac{T}{2} & \end{aligned}$$

with the solution

$$|t\rangle = U(t, t_0)\,|t_0\rangle \tag{2.43}$$

where $U(t, t_0)$ describes a unitary time evolution:

$$\begin{aligned} U(t, t_0) &= \mathbb{1} + \underbrace{(-i)\int_{t_0}^{t} \mathrm{d}t' H'(t')}_{\text{first order term, see page 5}} + (-i)^2 \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t'' \cdot H'(t') \cdot H'(t'') + \ \ldots \\ &= T \exp\{-i \int_{t_0}^{t} \mathrm{d}t' H'(t')\} \end{aligned} \tag{2.44}$$

so that the time is ordered.

We have

$$\boxed{i\,\frac{\partial}{\partial t} U(t, t_0) = H'(t) U(t, t_0)} \tag{2.45}$$

Iterate (2.42) in its infinitesimal form:

$$\begin{aligned} |t + \Delta t\rangle &= |t\rangle - i\ \Delta t\ H'(t)\,|t\rangle \\ &= (1 - i\ \Delta t\ H'(t))\,|t\rangle \end{aligned}$$

This defines the S-Matrix:

$$S = \lim_{\substack{t_0 \to -\infty \\ t \to +\infty}} U(t, t_0)\ . \tag{2.46}$$

Back to field theory:

$$\begin{aligned} H'(t) &= -\int \mathrm{d}^3x\ \mathcal{L}(x, t) \\ &= \int \mathrm{d}^3x\ \frac{1}{2} V(\vec{x}) :\!\phi(\vec{x}, t)\phi(\vec{x}, t)\!: \end{aligned} \tag{2.47}$$

where $\phi(\vec{x}, t)$ is a operator, which includes annihilation and creation of particles and $:\ :$ denotes normal ordering:

$$:\!a(\vec{k})\ a^\dagger(\vec{k}')\!: = +a^\dagger(\vec{k}')\ a(\vec{k})\ . \tag{2.48}$$

Example: transition amplitude for transition

$$\begin{aligned} \text{from} \quad |i\rangle &= |\vec{k}\rangle = a^\dagger(\vec{k})\,|0\rangle \ \text{ at } t_0 \to -\infty \\ \text{to} \quad |f\rangle &= |\vec{k}'\rangle = a^\dagger(\vec{k}')\,|0\rangle \ . \end{aligned} \tag{2.49}$$

We have

$$\begin{aligned} A_{fi} &= \langle f|\,S\,|i\rangle = \langle \vec{k}'|\,S\,|\vec{k}\rangle \\ &= \langle \vec{k}'|\,\mathbb{1} - i\int_{\mathbb{R}} \mathrm{d}t\; H'(t) + \ldots |\vec{k}\rangle \\ &= \langle \vec{k}'|\,\mathbb{1} + i\int \mathrm{d}^4x\; \mathcal{L}'(x) + \ldots |\vec{k}\rangle \end{aligned} \tag{2.50}$$

Consider weak interactions:

$$V^2(\vec{x}) \sim 0 \ .$$

Then

$$\langle f|\,S\,|i\rangle = \delta_{fi} - i\int \mathrm{d}^4x\; \frac{1}{2}\cdot V(\vec{x})\cdot 2\cdot \langle \vec{k}'|\,\phi(x)\,|0\rangle\,\langle 0|\,\phi(x)\,|\vec{k}\rangle \ . \tag{2.51}$$

The factor 2 in (2.51) stands for the two permutations of $a^\dagger$ and a, included in $\phi(\vec{x})$, which contribute. They are: $a^\dagger a$ and $aa^\dagger$, because there is neither an overlap between three particles and one particle nor between one and 0, the annihilated vacuum state $|0\rangle$.
Furthermore

$$\begin{aligned} \delta_{fi} &= \langle f|\,\mathbb{1}\,|i\rangle = \langle \vec{k}'|\vec{k}\rangle \\ &= \langle 0|\,a(\vec{k}')\;a^\dagger(\vec{k})\,|0\rangle \\ &= \langle 0|\,[a(\vec{k}'), a^\dagger(\vec{k})]\,|0\rangle \\ &= (2\pi)^3\cdot 2\omega\;\delta^{(3)}(\vec{k}-\vec{k}') \end{aligned} \tag{2.52}$$

The last equation follows from equation (2.14) on page 9.

Interpretation:

1. δ_{fi}: no interaction $\Rightarrow \vec{k} = \vec{k}'$.
2. state $\vec{k}$ scatters *once* at $V(\vec{x})$ into state $\vec{k}'$.

$$\langle 0|\,\phi(x)\,|\vec{k}\rangle = \langle 0|\int \frac{\mathrm{d}^3k'}{(2\pi)^3}\,\frac{1}{2\omega'}\left\{e^{ik'x}a^\dagger(\vec{k}') + e^{-ik'x}a(\vec{k}')\right\}a^\dagger(\vec{k})\,|0\rangle$$

with

$$\langle 0|\,a^\dagger = (a\,|0\rangle)^* = 0$$

follows

$$\begin{aligned}
\langle 0|\,\phi(x)\,|\vec{k}\rangle &= \langle 0|\int \frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{1}{2\omega'}e^{-ik'x}\,\langle 0|\,[a(\vec{k}'),a^\dagger(\vec{k})]\,|0\rangle \\
&= \int \frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{1}{2\omega'}e^{-ik'x}\cdot 2\omega\cdot(2\pi)^3\;\delta^{(3)}(\vec{k}-\vec{k}') \\
&= e^{-ikx} \qquad (2.53)
\end{aligned}$$

and similarily

$$\langle\vec{k}'|\,\phi(x)\,|0\rangle = e^{ik'x}\;. \qquad (2.54)$$

Interpretation:
Amplitudes for annihilating/ creating particles with momentum $\vec{k}/\,\vec{k}'$ at space-time point x.

We infer:

$$\begin{aligned}
A_{fi} &= \langle f|\,S\,|i\rangle = \delta_{fi} - i\int \mathrm{d}^4x\;V(\vec{x})e^{ik'x}e^{-ikx} \\
&= \delta_{fi} + i\int \mathrm{d}t\left[\int \mathrm{d}^3x\;V(\vec{x})e^{-i(\vec{k}'-\vec{k})\vec{x}}\right]e^{i(k'^0-k^0)x^0} \\
&= \delta_{fi} - 2\,i\;\delta(k'^0-k^0)\,\tilde{V}(\vec{q}) \qquad (2.55)
\end{aligned}$$

with

$$\begin{aligned}
\tilde{V}(\vec{q}) &= \int \mathrm{d}^3x\;V(\vec{x})e^{i\vec{q}\cdot\vec{x}} \\
\vec{q} &= \vec{k}-\vec{k}'\;.\;\;\text{3-momentum transfer} \qquad (2.56)
\end{aligned}$$

Interpretation revisited:

1. State $\vec{k}$ scatters at $V(\vec{x})$ with 'strength' $\tilde{V}(\vec{q})$ into state $\vec{k}'$ where $\vec{q}=\vec{k}-\vec{k}'$.
2. Energy is conserved as $k_0 = k'_0$.

Final remark:
Relation between the scattering amplitudes in momentum space and the form/ range of potential in space(-time):

Example:

$$\begin{aligned}
V(\vec{x}) &= V_0\,\frac{1}{(2\pi)^{3/2}}\,\frac{1}{l^3}\;\exp\left\{-\frac{1}{2}\cdot\frac{\vec{x}^2}{l^2}\right\} \qquad (2.57) \\
\to \tilde{V}(\vec{q}) &= V_0\;\exp\left\{-\frac{1}{2}\cdot l^2\cdot\vec{q}^{\,2}\right\} \qquad (2.58)
\end{aligned}$$

The potential $V(\vec{x})$ and its Fourier transformation $V(\vec{q})$ are plotted in figures 2.1 and 2.2 for one V_0-l-combination.

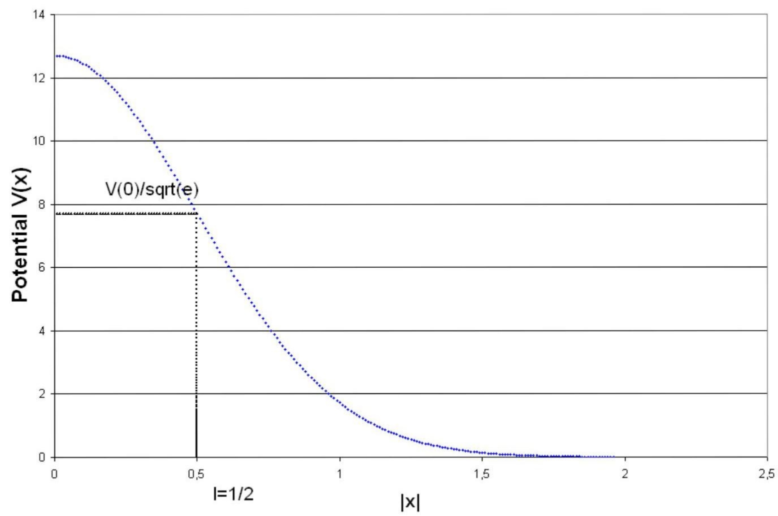

Figure 2.1: Potential $V(\vec{x})$ for V_0=1000, l=0.5

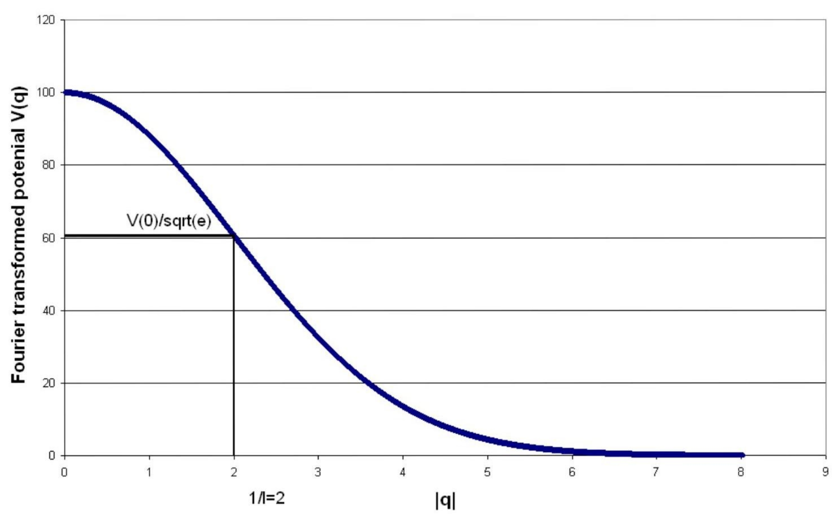

Figure 2.2: Fourier transformed potential $V(\vec{q})$ for V_0=1000, l=0.5

Remark on self-interaction (and Feynman rules):

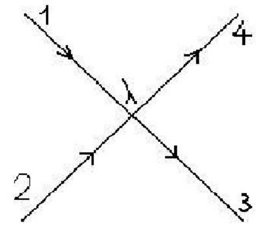

Figure 2.3: Feynman diagram for self-interaction.

$$\begin{aligned} & V(\vec{x}) \ :\phi(\vec{x},t)\phi(\vec{x},t): \\ \rightarrow & \frac{1}{4!}\lambda \ :\phi(\vec{x},t)\phi(\vec{x},t)\phi(\vec{x},t)\phi(\vec{x},t): \end{aligned} \tag{2.59}$$

$$\begin{aligned} & \langle 4,3| \, \frac{1}{4!} \ : \quad : \ |1,2\rangle \\ \sim \quad & \langle 4,3| \, \frac{1}{4!} \ a^\dagger \ a^\dagger \ a \ a \ |1,2\rangle \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ \sim \quad & \lambda \cdot \left\langle a a^\dagger \right\rangle^4 \end{aligned}$$

Remark on complex fields:

$$\phi = \phi_1 + i\,\phi_2 \tag{2.60}$$

$$\Rightarrow \mathcal{L}_0 = \frac{1}{2}\,\phi^*(x)(-\partial_\mu\partial^\mu - m^2)\phi(x) \tag{2.61}$$

$$\mathcal{L}' = -\frac{1}{2}\,V(\vec{x})\phi\phi^* \tag{2.62}$$

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}'$$

In general:

$$\mathcal{L}' = \mathcal{L}'[\phi\phi^*]$$

It follows that $\mathcal{L}$ is invariant under global U(1)-transformation of ϕ:

$$\phi(x) \rightarrow e^{i\alpha}\phi(x) \tag{2.63}$$

with

$$\partial_\mu \alpha = 0\ .$$

$$\begin{aligned} &\Rightarrow \quad \phi^*(x) \to \phi^*(x)\, e^{-i\alpha} \\ &\Rightarrow \quad \mathcal{L}[\phi] \to \mathcal{L}[\phi\, e^{i\alpha}] = \mathcal{L}[\phi] \end{aligned} \tag{2.64}$$

Noether theorem:

$$\partial_\mu\, j^\mu = 0 \;\; \text{equation of motion} \tag{2.65}$$

with

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \phi} \delta\phi \\ Q &= \int \mathrm{d}^3 x\; j^0 \\ \dot{Q} &= 0 \end{aligned} \tag{2.66}$$

Noether theorem (for internal Symmetry):

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \phi} = 0 \;\; \text{equation of motion}$$

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi}\, \delta\phi + \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \phi}\, \partial_\mu \delta\phi \\ &= \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \phi} \right) \delta\phi + \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \phi}\, \partial_\mu \delta\phi \\ &= \partial_\mu \Bigg(\underbrace{\frac{\partial \mathcal{L}}{\partial\, \partial_\mu \phi}\, \delta\phi}_{j^\mu} \Bigg) = 0 \\ \Rightarrow \partial_\mu\, j^\mu &= 0 \end{aligned}$$

and

$$\dot{Q} = \int \mathrm{d}^3 x \dot{j}^0 = + \int \mathrm{d}^3 x\; \partial^i j^i = 0$$

No boundary terms.

2.3 Spin $\frac{1}{2}$ Fields

Motivation: Algebra of Lorentz group, see (1.14) on page 5.
Boosts K_i, Rotations J_i

$$\begin{aligned} N_i &= \frac{1}{2}(J_i + i\, K_i) \\ N_i^\dagger &= \frac{1}{2}(J_i - i\, K_i) \end{aligned} \tag{2.67}$$

with SU(2)-algebra

$$\left[N_i^{(\dagger)}, N_j^{(\dagger)}\right] = i\,\varepsilon_{ijk}\,N_k^{(\dagger)} \tag{2.68}$$

$\Rightarrow$ We have 2-dim. representations of the Lorentz group, the spin $\frac{1}{2}$ representations.

Example:

$$\begin{aligned} \Lambda_L &= \exp\left\{\frac{i}{2}\sigma_i(\omega^i - iv^i)\right\} \text{ left-handed} \\ \Lambda_R &= \exp\left\{\frac{i}{2}\sigma_i(\omega^i + iv^i)\right\} \text{ right-handed} \end{aligned} \tag{2.69}$$

$\omega:$ rotation, $v:$ boost, $\sigma_i:$ Pauli matrices
The left-handed spin $\frac{1}{2}$ representation Λ_L can be mapped to the right-handed spin $\frac{1}{2}$ representation Λ_R by parity transformation.

Dirac equation:

$$\begin{aligned} \psi(x) &= \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_4(x) \end{pmatrix} \\ (i\gamma^\mu \partial_\mu - m)\psi(x) &= 0 \end{aligned} \tag{2.70}$$

with γ^μ are 4×4 matrices with

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\} &= \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \\ &= 2g^{\mu\nu} \cdot \mathbb{1} \quad \textit{Clifford algebra} \end{aligned} \tag{2.71}$$

Standard representation:

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \\ \gamma^i &= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \end{aligned} \tag{2.72}$$

with Pauli matrices σ^i (see (2.18) on page 10).

Remarks:

1. $\psi(x)$ consists of a two-component left-handed and a two-component right-handed spinor.
   Chiral representation:

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \\ \gamma_i &= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \end{aligned} \tag{2.73}$$

2. γ^μ transforms as a vector under Lorentztransformations.

Equation of motion (2.70) from Lagrange density:

$$\mathcal{L}_D = \bar{\psi}\,(i\,\gamma^\mu\,\partial_\mu - m)\,\psi(x) \tag{2.74}$$

with the Dirac conjugate $\bar{\psi} = \psi^\dagger\,\gamma^0$.

$$\frac{\partial \mathcal{L}_D}{\partial \bar{\psi}} - \partial_\mu \frac{\partial \mathcal{L}_D}{\partial\, \partial_\mu \bar{\psi}} = \frac{\partial \mathcal{L}_D}{\partial \bar{\psi}} = 0 \tag{2.75}$$

For the result of equation (2.75) see equation (2.70).
Also

$$\begin{array}{llll} & \frac{\partial \mathcal{L}_D}{\partial \psi} & -\partial_\mu \frac{\partial \mathcal{L}_D}{\partial\, \partial_\mu \psi} & = 0 \\ & \downarrow & \downarrow & \\ \Rightarrow & -m\bar{\psi} & -i\,\partial_\mu\,\bar{\psi}\,\gamma^\mu & = 0 \end{array} \tag{2.76}$$

Classical solution:

$$\begin{aligned} (-i\,\gamma^\mu\,\partial_\mu - m)\cdot(i\,\gamma^\nu\,\partial_\nu - m)\,\psi(x) &= \left[\frac{1}{2}\underbrace{\{\gamma^\mu,\gamma^\nu\}}_{2\cdot g^{\mu\nu}}\underbrace{\partial_\mu\,\partial_\nu}_{\partial_\nu\,\partial_\mu} + m^2\right]\psi(x) \\ &= \left[g^{\mu\nu}\partial_\nu\,\partial_\mu + m^2\right]\psi(x) \\ &= \left[\Box + m^2\right]\,\psi(x) \end{aligned} \tag{2.77}$$

$$\Rightarrow \psi(x) \sim e^{\pm ipx} \text{ plane wave} \tag{2.78}$$

We have

$$(i\,\gamma^\mu\,\partial_\mu - m)\,e^{\pm ipx} = (\mp\tilde{p} - m)\,e^{\pm ipx} \tag{2.79}$$

with

$$\tilde{p} := \gamma^\mu\,p_\mu = \gamma^0\,p_0 - \gamma^1 p^1 - \gamma^2 p^2 - \gamma^3 p^3.$$

A solution to the Dirac equation reads, $s = \pm\frac{1}{2}$

$$\begin{aligned} \psi(x) &\sim u_s(p)\,e^{-ipx} \\ \psi(x) &\sim v_s(p)\,e^{ipx} \end{aligned} \tag{2.80}$$

with

$$\begin{aligned} (\tilde{p} - m)\,u_s(p) &= 0 \\ &= (\tilde{p} + m)\,v_s(p). \end{aligned} \tag{2.81}$$

Equation (2.81) is satisfied with

$$\begin{aligned} u_s &= \sqrt{p^0 + m}\begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{p^0+m}\,\chi_s \end{pmatrix} \\ v_s &= -\sqrt{p^0 + m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{p^0+m}\,\varepsilon\,\chi_s \\ \varepsilon\,\chi_s \end{pmatrix} \end{aligned} \tag{2.82}$$

with

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$p_0 = +\sqrt{\vec{p}^{\,2} + m^2},\ \sigma :\ \text{Pauli matrices}$$

ε describes the metric in spin $\frac{1}{2}$ space.
Additional identity:

$$\begin{aligned} \sum_{s=\pm\frac{1}{2}} u_s(p)\, \bar{u}_s(p) &= \not{p} + m \\ \sum_{s=\pm\frac{1}{2}} v_s(p)\, \bar{v}_s(p) &= \not{p} - m \end{aligned} \tag{2.83}$$

As for the scalar field the general solution is given by the Fourier integral:

$$\psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2p_0} \sum_{s=\pm\frac{1}{2}} \left\{ e^{ipx}\ v_s(p)\ \beta_s^*(\vec{p}) + e^{-ipx}\ u_s(p)\ \alpha_s(\vec{p}) \right\} \tag{2.84}$$

β_s^* and α_s are independent of each other.
One gets the Hamiltonian density $\mathcal{H}_D$ via a Legendre transformation of $\mathcal{L}_D$:

$$\mathcal{H}_D = \Pi\, \dot{\psi} - \mathcal{L}_D$$

with

$$\begin{aligned} \Pi = \frac{\partial \mathcal{L}_D}{\partial \dot{\psi}},\ \bar{\Pi} = \frac{\partial \mathcal{L}_D}{\partial \dot{\bar{\psi}}} = 0 \\ \Pi = \bar{\psi}\ i\ \gamma^0 = i\ \psi^\dagger. \end{aligned} \tag{2.85}$$

It follows

$$\begin{aligned} \mathcal{H}_D &= \bar{\psi}\ i\ \gamma^0\ \dot{\psi} - \bar{\psi}\ (i\ \gamma^\mu\ \partial_\mu - m)\ \psi \\ &= \bar{\psi}\ (i\ \vec{\gamma}\ \vec{\partial} + m)\ \psi \end{aligned} \tag{2.86}$$

Hamiltonian:

$$\begin{aligned} H_D &= \int \mathrm{d}^3 x\ \bar{\psi}\ (i\ \vec{\gamma}\ \vec{\partial} + m)\ \psi \\ &= \int \mathrm{d}^3 x\ \psi^\dagger\ (i\ \gamma^0 \vec{\gamma}\ \vec{\partial} + \gamma^0\ m)\ \psi \end{aligned} \tag{2.87}$$

with

$$\vec{\partial} = \vec{\nabla}.$$

Inserting (2.84) into (2.87) leads to

$$H_D = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2p_0} \sum_{s=\pm\frac{1}{2}} \left(\alpha_s^*(\vec{p})\ \alpha_s(\vec{p})\ p_0 - \beta_s(\vec{p})\ \beta_s^*(\vec{p})\ p_0 \right) \tag{2.88}$$

from

$$\begin{aligned} \gamma^0\ (i\,\vec\gamma\,\vec\partial + m)\ u_s(p) &= p^0\ u_s(p) \\ \gamma^0\ (i\,\vec\gamma\,\vec\partial + m)\ v_s(p) &= -p^0\ v_s(p) \end{aligned} \tag{2.89}$$

$$\Rightarrow H_D = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\ \frac{1}{2} \sum_{s=\pm\frac{1}{2}} \left(\alpha_s^*(\vec p)\ \alpha_s(\vec p) - \beta_s(\vec p)\ \beta_s^*(\vec p)\right) \tag{2.90}$$

$\Rightarrow$ Negative energy states lead to unbounded Hamiltonian, no classical interpretation!

Quantisation:

$$\psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\ \frac{1}{2p^0} \sum_{s=\pm\frac{1}{2}} \left\{ e^{ipx}\ v_s(p)\ b_s^\dagger(\vec p) + e^{-ipx}\ u_s(p)\ a_s(\vec p) \right\} \tag{2.91}$$

with *anti*-commutation relations

$$\begin{aligned} \{a_r(\vec p), a_s^\dagger(\vec p\,')\} &= \delta_{rs}\ (2\pi)^3\ 2p_0\ \delta^{(3)}(\vec p - \vec p\,') \\ \{b_r(\vec p), b_s^\dagger(\vec p\,')\} &= \delta_{rs}\ (2\pi)^3\ 2p_0\ \delta^{(3)}(\vec p - \vec p\,') \end{aligned} \tag{2.92}$$

and

$$\{a^{(\dagger)}, a^{(\dagger)}\} = \{b^{(\dagger)}, b^{(\dagger)}\} = \{a^{(\dagger)}, b^{(\dagger)}\} = \{a, b^\dagger\} = 0 \tag{2.93}$$

Remarks:

1. The anti-commutation relations (ACR) are a manifestation of the Spin-statics theorem:

   Spin $\frac{2n+1}{2}$ particles have fermi-statistics (ACR, Pauli principle), spin n particles have Bose-statistics.

2. Electric charge (Noether): $J^\mu = -e\ \bar\psi\ \gamma^\mu\ \psi$

$$\begin{aligned} Q &= \int \mathrm{d}^3 x\ J^0 \\ &= -e \int \mathrm{d}^3 x\ \psi^\dagger\ \psi \\ &= -e \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\ \frac{1}{2p_0} \sum_{s=\pm\frac{1}{2}} \left(a_s^\dagger(\vec p)\ a_s(\vec p) + b_s(\vec p)\ b_s^\dagger(\vec p)\right) \end{aligned} \tag{2.94}$$

Please notice the positive sign in equation (2.94), that turns into a minus sign again!

Fockspace:
Construction as for scalar field, but ACR $\rightarrow$ anti-symmetric states

$|0\rangle$: normalised vacuum state, $\langle 0|0\rangle = 1$

$$\begin{aligned} a_s(\vec{p})\,|0\rangle &= 0 \\ b_s(\vec{p})\,|0\rangle &= 0 \end{aligned}$$

One-particle states:

$$\begin{aligned} \left|e^-(\vec{p},s)\right\rangle &= a_s^\dagger(\vec{p})\,|0\rangle \\ \left|e^+(\vec{p},s)\right\rangle &= b_s^\dagger(\vec{p})\,|0\rangle \end{aligned}$$

$|e^-(\vec{p},s)\rangle$ / $|e^+(\vec{p},s)\rangle$ describe electron/ positron with momentum p and spin $s = \pm\frac{1}{2}$ (s_z in rest-frame).

Remark:
Prediction of e^+, e^- with identical mass is triumpf of the Dirac theory.

Orthogonality:

$$\begin{aligned} \left\langle e^-(\vec{p}\,', s')|e^-(\vec{p},s)\right\rangle &= \langle 0|\, a_{s'}(\vec{p}\,')\; a_s^\dagger(\vec{p})\,|0\rangle \\ &= \langle 0|\,\{a_{s'}(\vec{p}\,'), a_s^\dagger(\vec{p})\}\,|0\rangle \\ &= (2\pi)^3 \cdot 2p_0\; \delta_{s's}\; \delta^{(3)}(\vec{p}-\vec{p}\,') \end{aligned}$$

Two-particle states:

$$\left|e^-(\vec{p}_1,s_1)\; e^-(\vec{p}_2,s_2)\right\rangle = a_{s_1}^\dagger(\vec{p}_1)\; a_{s_2}^\dagger(\vec{p}_2)\,|0\rangle$$

Pauli principle

$$\begin{aligned} \left|e^-(\vec{p}_1,s_1)\; e^-(\vec{p}_2,s_2)\right\rangle &= a_{s_1}^\dagger(\vec{p}_1)\; a_{s_2}^\dagger(\vec{p}_2)\,|0\rangle \\ &= -a_{s_2}^\dagger(\vec{p}_2)\; a_{s_1}^\dagger(\vec{p}_1)\,|0\rangle \\ &= -\left|e^-(\vec{p}_2,s_2)\; e^-(\vec{p}_1,s_1)\right\rangle \end{aligned} \tag{2.95}$$

N-particle states:

$$a_{s_1}^\dagger(\vec{p}_1)\;\ldots\; a_{s_n}^\dagger(\vec{p}_n)\; b_{r_1}^\dagger(\vec{q}_1)\;\ldots\; b_{r_m}^\dagger(\vec{q}_m)\,|0\rangle$$

Finally, with

$$\psi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{1}{2k^0} \sum_{r=\pm\frac{1}{2}} \left\{ e^{ikx}\; v_r(k)\; b_r^\dagger(\vec{k}) + e^{-ikx}\; u_r(k)\; a_r(\vec{k}) \right\}$$

$$\begin{aligned} \langle 0|\,\psi(x)\left|e^-(\vec{p},s)\right\rangle &= u_s(p)\; e^{-ipx} \\ \left\langle e^+(\vec{p},s)\right|\psi(x)\,|0\rangle &= v_s(p)\; e^{ipx} \end{aligned}$$

ψ: Annihilation of an electron/ creation of a positron at x.

$$\begin{aligned} \langle 0|\,\bar{\psi}(x)\left|e^+(\vec{p},s)\right\rangle &= \bar{v}_s(p)\; e^{-ipx} \\ \left\langle e^-(\vec{p},s)\right|\bar{\psi}(x)\,|0\rangle &= \bar{u}_s(p)\; e^{ipx} \end{aligned}$$

$\bar{\psi}$: Annihilation of a positron/ creation of an electron at x.

Symmetries:

$$S_D[\psi, \bar{\psi}] = \int \mathrm{d}^4x\; \bar{\psi}\; (i\, \gamma^\mu\, \partial_\mu - m)\; \psi \tag{2.96}$$

1. Invariance of $S_D[\psi, \bar{\psi}]$ under orthochronous Poincaré transformations:

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu + a^\mu \quad \text{see (1.3)} \tag{2.97}$$

$$U(\Lambda, a)\; \psi(x)\; U^\dagger(\Lambda, a) = S^{-1}(\Lambda)\; \psi(\Lambda x + a)$$

   where S satisfies

$$S^{-1}(\Lambda)\; \gamma^\mu\; S(\Lambda) = \Lambda^\mu{}_\rho\; \gamma^\rho \tag{2.98}$$

   and U is unitary.

   Dirac adjoint spinor:

$$U(\Lambda, a)\; \bar{\psi}(x)\; U^\dagger(\Lambda, a) = \bar{\psi}(\Lambda x + a)\; S(\Lambda) \tag{2.99}$$

   The invariance of S is to show:

$$\begin{aligned}
& \int \mathrm{d}^4x\; \bar{\psi}\; (i\, \gamma^\mu\, \partial_\mu - m)\; \psi(x) \\
\rightarrow\; & \int \mathrm{d}^4x\; \bar{\psi}(\Lambda x + a)\; S\; (i\, \gamma^\mu\, \partial_\mu - m)\; \psi(\Lambda x + a) \\
=\; & \int \mathrm{d}^4x\; \bar{\psi}(x)\; S\; (i\, \gamma^\mu\, \partial_\nu (\Lambda^{-1})_\mu{}^\nu - m)\; S^{-1}\; \psi(x) \\
=\; & \int \mathrm{d}^4x\; \bar{\psi}(x)\; S\; (i\, \Lambda^\nu{}_\mu\; \gamma^\mu\; \partial_\nu - m)\; S^{-1}\; \psi(x) \\
=\; & \int \mathrm{d}^4x\; \bar{\psi}(x)\; S\; (i\, S^{-1}\; \gamma^\nu\; S\; \partial_\nu - m)\; S^{-1}\; \psi(x) \\
=\; & \int \mathrm{d}^4x\; \bar{\psi}\; (i\, \gamma^\mu\, \partial_\mu - m)\; \psi(x)
\end{aligned} \tag{2.100}$$

   General bilinears:

   (a) $\bar{\psi}\; \psi$ scalar: $m\; \bar{\psi}\; \psi$
       pseudo scalar later

   (b) $\bar{\psi}\; \gamma^\mu\; \psi$ vector
       pseudo vector later

   (c) $\bar{\psi}\; \sigma^{\mu\nu}\; \psi$ tensor, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

2. Invariance of $S_D[\psi, \bar{\psi}]$ under Parity

$$\Lambda_P = \begin{pmatrix} 1 & \\ & -\mathbb{1}_3 \end{pmatrix} \quad \text{see equation (1.8)} \tag{2.101}$$

Unitary representation:

$$\begin{aligned} U(P)\ \psi(\vec{x},t)\ U^\dagger(P) &= \gamma^0\ \psi(-\vec{x},t) \\ U(P)\left|e^-(\vec{p},s)\right\rangle &= \left|e^-(-\vec{p},s)\right\rangle \\ U(P)\left|e^+(\vec{p},s)\right\rangle &= -\left|e^+(-\vec{p},s)\right\rangle \end{aligned} \tag{2.102}$$

e^+, e^- are parity 'eigen states'.
Relative intrinsic parity can be measured:

3. Invariance of $S_D[\psi,\bar{\psi}]$ under time reversal

$$\Lambda_T = \begin{pmatrix} -1 & \\ & 1\!\!1_3 \end{pmatrix} \quad \text{see equation (1.9)} \tag{2.103}$$

Anti-unitary transformation V:

$$\left(V(T)\ \psi(\vec{x},t)\ V^{-1}(T)\right)^\dagger = S(T)\ \bar{\psi}^T(\vec{x},t) \tag{2.104}$$

with

$$S(T) = i\ \gamma^2\ \gamma_5 \tag{2.105}$$

and

$$\begin{aligned} \gamma_5 &= \frac{i}{4!}\ \varepsilon_{\mu\nu\rho\sigma}\ \gamma^\mu\ \gamma^\nu\ \gamma^\rho\ \gamma^\sigma \\ \gamma^5 &= i\ \gamma^0\ \gamma^1\ \gamma^2\ \gamma^3 \\ \varepsilon_{0123} &= 1 \\ \{\gamma_5,\gamma^\mu\} &= 0 \end{aligned}$$

We have

$$\begin{aligned} V(T)\left|e^-(\vec{p},s)\right\rangle &= (-1)^{s-\frac{1}{2}}\left|e^-(-\vec{p},-s)\right\rangle \\ V(T)\left|e^+(\vec{p},s)\right\rangle &= (-1)^{s-\frac{1}{2}}\left|e^+(-\vec{p},-s)\right\rangle \end{aligned} \tag{2.106}$$

4. Charge conjugation C

$$\mathcal{C} : e^+ \rightleftarrows e^-$$

$$U(C)\ \psi(x)\ U^{-1}(C) = S(C)\ \bar{\psi}^T(x) \tag{2.107}$$

with

$$S(C) = i\ \gamma^2\ \gamma^0 = \begin{pmatrix} 0 & -\varepsilon \\ -\varepsilon & 0 \end{pmatrix} \tag{2.108}$$

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\begin{aligned} U(C)\left|e^-(\vec{p},s)\right\rangle &= \left|e^+(\vec{p},s)\right\rangle \\ U(C)\left|e^+(\vec{p},s)\right\rangle &= \left|e^-(\vec{p},s)\right\rangle \end{aligned} \tag{2.109}$$

Bilinears:

(1) scalar:	$\bar\psi\,\psi(x)$	1 generator	
pseudo-scalar:	$i\,\bar\psi\,\gamma_5\,\psi$	1 generator	
(2) vector:	$\bar\psi\,\gamma^\mu\,\psi$	4 generators	with
pseudo-vector:	$\bar\psi\,\gamma_5\,\gamma^\mu\,\psi$	4 generators	
(3) tensor:	$\bar\psi\,\sigma^{\mu\nu}\,\psi$	6 generators	

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{2.110}$$

$\Rightarrow$ 16 generators of the Lorentzgroup

Remark:

$$(\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\varepsilon_{\mu\nu\rho\sigma}) \cdot \gamma^\sigma\,\gamma^\sigma \sim \gamma_5\,\gamma^\sigma$$

2.4 The interacting fermionic field (a first glimps of QED)

Classical: Langrangian density

$$\begin{aligned} \mathcal{L}(x) &= \mathcal{L}_D(x) + \mathcal{L}'(x) \qquad (2.111)\\ &= \underbrace{\bar\psi(x)(i\,\gamma^\mu\,\partial_\mu - m)\psi(x)}_{\mathcal{L}_D(x)} + \underbrace{e\,A_\mu(x)\,\bar\psi(x)\,\gamma^\mu\,\psi(x)}_{\mathcal{L}'(x)} \end{aligned}$$

with

$$\mathcal{L}'(x) = e\,A_\mu(x)\,\bar\psi(x)\,\gamma^\mu\,\psi(x) \tag{2.112}$$

Since $\bar\psi(x)\gamma^\mu\psi(x)$ transforms as a vector under Lorentz transformations, $A_\mu(x)$ has to transform as a vector:

$$A^\mu(x) \to \Lambda^\mu{}_\nu\,A^\nu(x) \tag{2.113}$$

A_μ is a vector field.

Remark:
$\mathcal{L}(x)$ is invariant under

$$\begin{aligned} \psi(x) &\to e^{ie\,\alpha(x)}\psi \\ \bar\psi(x) &\to \bar\psi e^{-ie\,\alpha(x)} \\ A_\mu(x) &\to A_\mu(x) + \partial_\mu\alpha(x) \\ \to j^\mu &= -e\,\bar\psi\,\gamma^\mu\,\psi \qquad (2.114) \end{aligned}$$

Quantisation in interaction picture

$$i\frac{\partial}{\partial t}\,|t\rangle = H'(t)\,|t\rangle \tag{2.115}$$

with

$$H'(t) = -\int \mathrm{d}^3x\, \mathcal{L}'_{op}(x) \tag{2.116}$$
$$= -e\int \mathrm{d}^3x\, A_\mu(x)\ :\bar{\psi}(x)\gamma^\mu\psi(x):$$

A_μ can be either a background field (classical) or a quantum field $A_{\mu_{op}}$: it is bosonic (as a vector spin 1) and commutes with $\psi, \bar{\psi}$, also: $:A_\mu: = A_\mu$ with creation/ annihilation operators $a^\dagger_\mu, a_\mu$.
Subtleties concerning the quantisation of A_μ later, physical state $|\gamma\rangle \sim a^\dagger |0\rangle$
Equation (2.115) is solved by

$$|t\rangle = U(t,t_0)\,|t_0\rangle \tag{2.117}$$

with

$$\boxed{U(t,t_0) = T\exp\left\{-i\int_{t_0}^{t} H'(t')\,\mathrm{d}t'\right\} = T\exp\left\{ie\int_{t_0}^{t}\mathrm{d}^4x\, A_\mu\ :\bar{\psi}\gamma^\mu\psi:\right\}} \tag{2.118}$$

$\mathcal{L}'$ couples $e^\pm$ to the electromagnetic field A_μ, the photon.
Similarily we couple $\mu^\pm$ and $\tau^\pm$ to A_μ:

$$\psi(x) = \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix} \tag{2.119}$$

with

$$\mathcal{L}_D = \bar{\psi}_e(i\gamma^\mu\partial_\mu - m_e)\psi_e + \bar{\psi}_\mu(i\gamma^\mu\partial_\mu - m_\mu)\psi_\mu + \bar{\psi}_\tau(i\gamma^\mu\partial_\mu - m_\tau)\psi_\tau \tag{2.120}$$

and

$$m_e = 0{,}511\,\mathrm{MeV}$$
$$m_\mu = 105{,}7\,\mathrm{MeV}$$
$$m_\tau = 1784\,\mathrm{MeV}$$

$$\mathcal{L}'(x) = e\big[\bar{\psi}_e\, A\!\!\!/\, \psi_e + \bar{\psi}_\mu\, A\!\!\!/\, \psi_\mu + \bar{\psi}_\tau\, A\!\!\!/\, \psi_\tau\big] \tag{2.121}$$

Computation of transition amplitude
Initial state at $t_0 \to -\infty$:

$$|t_0\rangle = |i\rangle = \big|e^-(p_1)\dots e^-(p_n)e^+(q_1)\dots e^+(q_m)\gamma(k_1)\dots\gamma(k_l)\big\rangle \tag{2.122}$$

and $\mu's$, $\tau's$.
Final state at $t \to +\infty$:

$$|t\rangle = |f\rangle = \big|e^-(p'_1)\dots e^-(p'_n)e^+(q'_1)\dots e^+(q'_m)\gamma(k'_1)\dots\gamma(k'_l)\big\rangle \tag{2.123}$$

e.g.:

$$\left|\mu^-(p_1')\dots\mu^-(p_n')\mu^+(q_1')\dots\mu^+(q_m')\gamma(k_1')\dots\gamma(k_l')\right\rangle$$

This is related to the S matrix element.

$$e^-(p_1)+\ldots+\gamma(k_l)\to e^-(p_1')+\ldots+\gamma(k_l')$$

Here, we are interested in

$$e^+e^-\to\mu^+\mu^-.$$

$$e^+(k)+e^-(k')\to\mu^-(p)+\mu^+(p')$$

In general:

$$|t\rangle = U(t,t_0)\,|i\rangle$$

$$\begin{aligned}
S_{fi} &= \langle f|t=\infty\rangle \\
&= \lim_{\substack{t_0\to-\infty\\ t\to+\infty}} \langle f|\,U(t,t_0)\,|i\rangle \\
&= \langle f|\,T\exp\left\{ie\int \mathrm{d}^4x A_\mu\ :\bar\psi\gamma^\mu\psi:\right\}|i\rangle
\end{aligned} \tag{2.124}$$

Expanding $T\exp\{i\int \mathrm{d}^4x\,\mathcal{L}'\}$ leads to

$$\mathcal{L}'(x) = eA_\mu\ :\bar\psi\gamma^\mu\psi:.$$

$$\begin{aligned}
S_{fi} = & \underbrace{\langle f|i\rangle}_{\delta_{fi}} + ie\,\langle f|\,T\int \mathrm{d}^4x A_\mu\ :\bar\psi\gamma^\mu\psi:|i\rangle + \\
& + \frac{(ie)^2}{2!}\langle f|\int \mathrm{d}^4x_1\int \mathrm{d}^4x_2 T\mathcal{L}'(x_1)\mathcal{L}'(x_2)\,|i\rangle + \\
& + \ldots + \\
& + \frac{(ie)^n}{n!}\langle f|\int \mathrm{d}^4x_1\ldots\int \mathrm{d}^4x_n T\mathcal{L}'(x_1)\ldots\mathcal{L}'(x_n)\,|i\rangle + \\
& + \ldots
\end{aligned} \tag{2.125}$$

First order:

$$\int \mathrm{d}^4x\,\langle f|\,A_\mu(x)\ :\bar\psi(x)\gamma^\mu\psi(x):|i\rangle \tag{2.126}$$

$$\int \mathrm{d}^4x\,\langle f|\,(\ldots a_\mu^\dagger+\ldots a_\mu)(\ldots b_s^\dagger b_r+\ldots a_s^\dagger b_r^\dagger+\ldots b_s a_r+\ldots a_s^\dagger a_r)\,|i\rangle$$

We have the following processes:
Time ordered diagrams

1. Scattering of e^+ with emission/absorption of γ.
2. Creation of e^+e^- pairs with emission/absorption of γ.

3. Annihilation of e^+e^- pairs with emission/absorption of γ.
4. Scattering of e^- with emission/absorption of γ.

Higher order processes are composed out of first order processes, e.g. second order with

$$\begin{aligned} |i\rangle &\sim e^+e^- \\ |f\rangle &\sim \mu^+\mu^- \end{aligned}$$

or $|i\rangle \sim \gamma$, $|f\rangle \sim \gamma$
Problem:

- Convergence of expansion in

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137}.$$

  Series is an asymptotic series: does not converge.

- All orders are infinite $\Rightarrow$ renormalisation.

Programme:

- Write down all diagrams for a given order in α for matrix element $\langle f|\, S\, |i\rangle$.
- Sort out combinatorics (normal ordering), compute the remaining integrals.

$\Rightarrow$ Feynman rules/ Loop integrals

Reminder: differential cross section (page 6)

$$\begin{aligned} \mathrm{d}\sigma &= \frac{\mathrm{d}\Gamma[i \to f]}{\Phi} \\ &= \frac{(2\pi)^4}{V^4}\, \delta^{(4)}(p_A + p_B - p_C - p_D)\, |A_{fi}|^2\, \frac{\mathrm{d}\rho_f}{\Phi} \end{aligned} \tag{2.127}$$

for two particle scattering.
Φ: particle flux, normalisation of the states $|i\rangle$, $|f\rangle$ and
$|A_{fi}|^2 = |M_{fi}|^2 = |\langle f|\, H\, |i\rangle|^2$.

$$\mathrm{d}\rho_f : \frac{V \cdot \mathrm{d}^3 p_C}{(2\pi)^3 \cdot 2p_C^0} \cdot \frac{V \cdot \mathrm{d}^3 p_D}{(2\pi)^3 \cdot 2p_D^0} \tag{2.128}$$

Example:

$$e^+e^- \to \mu^-\mu^+$$

Cross section $\mathrm{d}\sigma$:

$$\mathrm{d}\sigma = \frac{1}{T} \underbrace{\frac{V \cdot \mathrm{d}^3 p_3}{(2\pi)^3 \cdot 2p_3^0} \frac{V \cdot \mathrm{d}^3 p_4}{(2\pi)^3 \cdot 2p_4^0}}_{\text{phasespace density}} \frac{1}{F} \cdot \sum_{\text{spins}} |\underbrace{\left\langle \mu^+(p_4)\mu^-(p_3) \right| S \left| e^+(p_2)e^-(p_1) \right\rangle}_{\text{S-Matrix element}}|^2$$

F is the incident particle flux.

Differential cross-section $d\sigma$ per unit volume V for $e^- e^+ \to \mu^- \mu^+$:

$$
\begin{aligned}
d\sigma &= \frac{1}{T\,F}\frac{d^3p_3}{(2\pi)^3}\frac{1}{2p_3^0}\frac{d^3p_4}{(2\pi)^3}\frac{1}{2p_4^0} \\
&\times \sum_{\text{spins}} |\langle \mu^+(p_4)\,\mu^-(p_3)|\,S\,|e^+(p_2)\,e^-(p_1)\rangle|^2
\end{aligned}
\qquad (2.129)
$$

with incident particle flux F, and unit volume $V = 1$.

Consider the term between the absolut value bars in (2.129) first:

S-Matrix:

$$
S = T\, e^{ie\int d^4x\, A_\nu(x) :\bar\psi\gamma^\nu\psi(x):} \qquad (2.130)
$$

with

$$
\bar\psi\,\gamma^\nu\,\psi(x) = \bar\psi_e\,\gamma^\nu\,\psi_e(x) + \bar\psi_\mu\,\gamma^\nu\,\psi_\mu(x).
$$

Expansion of S-matrix element for $e^- e^+ \to \mu^- \mu^+$:

$$
\begin{aligned}
&\langle \mu^+(p_4)\,\mu^-(p_3)|\,S\,|e^+(p_2)\,e^-(p_1)\rangle = \\
=\;& \frac{(ie)^2}{2}\langle \mu^+(p_4)\,\mu^-(p_3)|\,T\int d^4x\,d^4x' A_\nu(x)\,A_\mu(x')\,\times \\
&\times :\bar\psi\,\gamma^\nu\,\psi(x)::\bar\psi\,\gamma^\mu\,\psi(x'):\;|e^+(p_2)\,e^-(p_1)\rangle + O(e^4)
\end{aligned}
\qquad (2.131)
$$

Consider the states in (2.131):

$$
|e^+(p_2)\,e^-(p_1)\rangle = b_e^\dagger(\vec p_2)\,a_e^\dagger(\vec p_1)|0\rangle
$$

Fermionic field operator

$$
\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\sum_{s=\pm 1/2}\left\{e^{ipx}\,v_s(p)\,b^\dagger(\vec p) + e^{-ipx}\,u_s(p)\,a(\vec p)\right\} \qquad (2.132)
$$

A_μ commutes with $\psi, \bar\psi$:

$$
\begin{aligned}
&\langle \mu^+(p_4)\,\mu^-(p_3)|\,S\,|e^+(p_2)\,e^-(p_1)\rangle = \\
=\;& \frac{(ie)^2}{2}\int d^4x\,d^4x' \langle 0\,|T\,A_\nu(x)\,A_\mu(x')\,|0\rangle\,\times \\
&\times \langle \mu^+(p_4)\,\mu^-(p_3)|\,T\,:\bar\psi\,\gamma^\nu\,\psi(x)::\bar\psi\,\gamma^\mu\,\psi(x'):\;|e^+(p_2)\,e^-(p_1)\rangle + \\
&+ O(e^4) \\
=\;& \frac{(ie)^2}{2}\int d^4x\,d^4x' \langle 0\,|T\,A_\nu(x)\,A_\mu(x')\,|0\rangle \\
&\times \langle \mu^+(p_4)\,\mu^-(p_3)|\;:\bar\psi\,\gamma^\nu\,\psi(x)::\bar\psi\,\gamma^\mu\,\psi(x'):\;|e^+(p_2)\,e^-(p_1)\rangle + \\
&+ O(e^4) \\
=\;& (ie)^2\int d^4x\,d^4x' \langle 0\,|T\,A_\nu(x)\,A_\mu(x')\,|0\rangle\,\times \\
&\times \langle \mu^+(p_4)\,\mu^-(p_3)|\;:\bar\psi_\mu\gamma^\nu\psi_\mu(x)::\bar\psi_e\gamma^\mu\psi_e(x'):\;|e^+(p_2)\,e^-(p_1)\rangle + \\
&+ O(e^4)
\end{aligned}
\qquad (2.133)
$$

Counting annihilation/creation operators: $a^{(\dagger)}$, $b^{(\dagger)}$:

$$\begin{aligned}
&\langle \mu^+(p_4)\mu^-(p_3)|\ :\bar{\psi}_\mu\gamma^\nu\psi_\mu(x)::\bar{\psi}_e\gamma^\mu\psi_e(x'):\ |e^+(p_2)\,e^-(p_1)\rangle = \\
&= \langle \mu^+(p_4)\,\mu^-(p_3)|\ :\bar{\psi}_\mu\,\gamma^\nu\,\psi_\mu(x):\ |0\rangle\ \times \\
&\times\langle 0|\ :\bar{\psi}_e\,\gamma^\mu\,\psi_e(x'):\ |e^+(p_2)\,e^-(p_1)\rangle
\end{aligned} \tag{2.134}$$

Further reduction of the last part of (2.134):

$$\langle 0|\ :\bar{\psi}_e\gamma^\mu\psi_e(x'):\ |e^+(p_2)\,e^-(p_1)\rangle = \langle 0|\ :\bar{\psi}_e\,\gamma^\mu\,\psi_e(x'):\ b_e^\dagger(\vec{p}_2)\,a_e^\dagger(\vec{p}_1)|0\rangle \tag{2.135}$$

For the fermionic field operator $\psi(x)$ see (2.132).
Further reduction of the right side of (2.135) leads to

$$\langle 0|\ :\bar{\psi}_e\,\gamma^\mu\,\psi_e(x'):\ b_e^\dagger(\vec{p}_2)\,a_e^\dagger(\vec{p}_1)|0\rangle = -\langle 0|\,\bar{\psi}_e(x')\,b_e^\dagger(\vec{p}_2)|0\rangle\gamma^\mu\langle 0|\psi_e(x')\,a_e^\dagger(\vec{p}_1)|0\rangle \tag{2.136}$$

Expectation value $\langle 0|\psi|e^-\rangle$

$$\langle 0|\psi_e(x')\,a_e^\dagger(\vec{p}_1)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\,e^{-ipx'}\sum_{s=\pm 1/2} u_s(p)\langle 0|\,a(\vec{p})\,a_e^\dagger(\vec{p}_1)|0\rangle \tag{2.137}$$

with the commutator trick:

$$\langle 0|\psi_e(x')\,a_e^\dagger(\vec{p}_1)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\,e^{-ipx'}\sum_{s=\pm 1/2} u_s(p)\langle 0|\,\{a_e(\vec{p})\,,\,a_e^\dagger(\vec{p}_1)\}|0\rangle. \tag{2.138}$$

For more information about the result of the anti-commutation relation above

$$\{a_s(\vec{p})\,,\,a_r^\dagger(\vec{p}_1)\} = (2\pi)^3 2p^0\delta_{rs}\delta(\vec{p}-\vec{p}_1)$$

see (2.92) on page 24.
So one gets for $\langle 0|\psi|e^-\rangle$:

$$\langle 0|\psi_e(x')a_e^\dagger(\vec{p}_1)|0\rangle = e^{-ip_1x'}\,u_e(p_1) \tag{2.139}$$

Analogous one can calculate the expectation value $\langle 0|\bar{\psi}|e^+\rangle$.
In summary the expectation values $\langle 0|\psi|e^-\rangle, \langle 0|\bar{\psi}|e^+\rangle$:

$$\langle 0|\psi_e(x')\,a_e^\dagger(\vec{p}_1)|0\rangle = e^{-ip_1x'}\,u_e(p_1) \tag{2.140}$$
$$\langle 0|\bar{\psi}_e(x')\,b_e^\dagger(\vec{p}_2)|0\rangle = e^{-ip_2x'}\,\bar{v}_e(p_2) \tag{2.141}$$
$$\tag{2.142}$$

It follows a further simplification of (2.135):

$$\langle 0|\ :\bar{\psi}_e(x')\,\gamma^\mu\,\psi_e(x'):\ b_e^\dagger(\vec{p}_2)\,a_e^\dagger(\vec{p}_1)\,|0\rangle = -\bar{v}_e(p_2)\,\gamma^\mu\,u_e(p_1)\,e^{-i(p_1+p_2)x'} \tag{2.143}$$

Similarily for the muon:

$$\langle 0|\,b_\mu(\vec{p}_4)\,a_\mu(\vec{p}_3)\ :\bar{\psi}_\mu(x)\,\gamma^\nu\,\psi_\mu(x):\ |0\rangle = \bar{u}_\mu(p_3)\,\gamma^\nu\,v_\mu(p_4)\,e^{i(p_3+p_4)x} \tag{2.144}$$

Plug the results in (2.131):

$$\begin{aligned}
&\langle\mu^+(p_4)\,\mu^-(p_3)|\,S\,|e^+(p_2)\,e^-(p_1)\rangle \simeq\\
\simeq\quad & -(ie)^2\int d^4x\,d^4x'\langle 0\,|T\,A_\nu(x)\,A_\mu(x')\,|0\rangle e^{i(p_3+p_4)x}\,e^{-i(p_1+p_2)x'}\\
&\times\bar{u}_\mu(p_3)\,\gamma^\nu\,v_\mu(p_4)\,\bar{v}_e(p_2)\,\gamma^\mu\,u_e(p_1) \qquad (2.145)
\end{aligned}$$

Now consider the photon propagator:

$$\langle 0\,|T\,A_\nu(x)\,A_\mu(x')\,|0\rangle = -ig_{\mu\nu}\lim_{\epsilon\to 0_+}\int\frac{d^4k}{(2\pi)^4}\,\frac{e^{-ik(x-x')}}{k^2+i\epsilon}$$

Momentum conservation

$$\int d^4x\,d^4x'\int\frac{d^4k}{(2\pi)^4}\,\frac{e^{-i(k-p_1-p_2)x}\,e^{i(k-p_3-p_4)x'}}{k^2+i\epsilon} = \frac{1}{s}(2\pi)^4\,\delta(p_1+p_2-p_3-p_4) \qquad (2.146)$$

with the square of the total energy $s=(p_1+p_2)^2$ leads to

$$\begin{aligned}
&\langle\mu^+(p_4)\mu^-(p_3)|\,S\,|e^+(p_2)\,e^-(p_1)\rangle \simeq\\
\simeq\quad & \frac{ig_{\mu\nu}}{s}(ie)^2(2\pi)^4\,\delta(p_1+p_2-p_3-p_4)\bar{u}_\mu(p_3)\,\gamma^\nu\,v_\mu(p_4)\,\bar{v}_e(p_2)\,\gamma^\mu\,u_e(p_1)\\
\simeq\quad & \frac{i}{s}(ie)^2(2\pi)^4\,\delta(p_1+p_2-p_3-p_4)\bar{u}_\mu(p_3)\,\gamma^\nu\,v_\mu(p_4)\,\bar{v}_e(p_2)\,\gamma_\nu\,u_e(p_1) \qquad (2.147)
\end{aligned}$$

Now the term between the absolut value bars in (2.129) is calculated. The next step is the averaging over the spins in the initial and the final state (see (2.129)):

$$\begin{aligned}
&\frac{1}{4}\sum_{s,s',r,r'}|\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\;\bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2 =\\
=\quad & \frac{1}{2}\sum_{s,s'}\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\bar{v}_{\mu,s'}(p_4)\gamma^\rho u_{\mu,s}(p_3)\\
&\times\frac{1}{2}\sum_{r,r'}\bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)\bar{u}_{e,r'}(p_1)\gamma_\rho v_{e,r}(p_2) \qquad (2.148)
\end{aligned}$$

with

$$\begin{aligned}
\Big[\bar{v}_r(p)\,\gamma_\nu\,u_{r'}(q)\Big]^* &= u^\dagger_{r'}(q)\,\gamma^\dagger_\nu\,\gamma^{0\dagger}\,v_r(p)\\
&= \bar{u}_{r'}(q)\,\gamma_\nu\,v_r(p).
\end{aligned}$$

Consider the first sum in (2.148) first:

$$\frac{1}{2}\sum_{s,s'}\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\bar{v}_{\mu,s'}(p_4)\gamma^\rho u_{\mu,s}(p_3) = \frac{1}{2}\mathrm{Tr}(\not{p}_3+m_\mu)\gamma^\nu(\not{p}_4-m_\mu)\gamma^\rho \quad \text{see 2.83}$$

Similarily one gets for the second sum in (2.148):

$$\frac{1}{2}\sum_{s,s'} \bar{v}_{e,s}(p_2)\gamma_\nu u_{e,s'}(p_1)\bar{u}_{e,s'}(p_1)\gamma_\rho v_{e,s}(p_2) = \frac{1}{2}\mathrm{Tr}(\not{p}_2 - m_e)\gamma_\nu(\not{p}_1 + m_e)\gamma_\rho$$

In summary one gets as an intermediate result for the average over the spins in the initial and the final state (see (2.148)):

$$\begin{aligned} &\frac{1}{4}\sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2 = \\ = \quad &\frac{1}{4}\mathrm{Tr}\Big[(\not{p}_3 + m_\mu)\gamma^\nu(\not{p}_4 - m_\mu)\gamma^\rho\Big]\ \mathrm{Tr}\Big[(\not{p}_2 - m_e)\gamma_\nu(\not{p}_1 + m_e)\gamma_\rho\Big] \end{aligned} \tag{2.149}$$

In high energy limit

$$s \gg m_\mu^2, m_e^2$$

one can drop m_e, m_μ in the traces of (2.149).
So (2.149) turns into

$$\frac{1}{4}\sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2 = \frac{1}{4}\mathrm{Tr}\Big[\not{p}_3\gamma^\nu\not{p}_4\gamma^\rho\Big]\ \mathrm{Tr}\Big[\not{p}_2\gamma_\nu\not{p}_1\gamma_\rho\Big] \tag{2.150}$$

With the traces

$$\mathrm{Tr}\ \gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\rho = 4\left(g^{\alpha\nu}g^{\beta\rho} + g^{\rho\alpha}g^{\nu\beta} - g^{\alpha\beta}g^{\nu\rho}\right)$$

one gets for (2.148):

$$\frac{1}{4}\sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2 = 4\Big[(p_1p_4)(p_2p_3) + (p_2p_4)(p_1p_3)\Big] \tag{2.151}$$

High energy limit revisited

$$p_1p_3 = p_2p_4 = \frac{s}{4}(1 - \cos\vartheta)\,, \qquad p_1p_4 = p_2p_3 = \frac{s}{4}(1 + \cos\vartheta)$$

with scattering angle

$$\cos\vartheta = \frac{\vec{p}_1\vec{p}_3}{|\vec{p}_1|\,|\vec{p}_3|}$$

one gets the final result for (2.148):

$$\frac{1}{4}\sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2 = \frac{s^2}{2}\left(1 + \cos^2\vartheta\right) \tag{2.152}$$

Back to the differential cross-section $d\sigma$ per unit volume (see (2.129)). When one plugs in all results calculated above one gets for the differential cross-section $d\sigma$ per unit volume

$$d\sigma = \frac{1}{T\,F}\frac{d^3p_3}{(2\pi)^3}\frac{1}{2p_3^0}\,\frac{d^3p_4}{(2\pi)^3}\frac{1}{2p_4^0} \times \left[(2\pi)^4\delta(p_1 + p_2 - p_3 - p_4)\right]^2 \frac{e^4}{s^2}\,\frac{s^2}{2}(1 + \cos^2\vartheta) \tag{2.153}$$

With Fermi's trick

$$\begin{aligned}
& [(2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4)]^2 \\
= \;& (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \int_{VT} d^4x \, e^{ix((p_1+p_2-p_3-p_4)} \\
= \;& VT(2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4)
\end{aligned}$$

and $\alpha = \dfrac{e^2}{4\pi}$ one gets for the differential cross-section $d\sigma$ per unit volume

$$d\sigma = 2\alpha^2 \frac{1}{F} \frac{d^3p_3}{2p_3^0} \frac{d^3p_4}{2p_4^0} \delta(p_1 + p_2 - p_3 - p_4) \, (1 + \cos^2 \vartheta) \tag{2.154}$$

In high energy limit and the CMS-system, i.e.

$$\vec{p}_1 + \vec{p}_2 = 0, \qquad p_1^0 + p_2^0 \simeq \sqrt{s}$$

one gets for the differential cross-section $\dfrac{d\sigma}{d\Omega_3}$

$$\frac{d\sigma}{d\Omega_3} = \alpha^2 \frac{1}{2F} \int_0^\infty d|\vec{p}_3| \, |\vec{p}_3| \int \frac{d^3p_4}{|\vec{p}_4|} \delta(\sqrt{s} - |\vec{p}_3| - |\vec{p}_4|) \delta(\vec{p}_3 + \vec{p}_4) \, (1 + \cos^2 \vartheta). \tag{2.155}$$

With the flux F

$$F = 2p_1^0 \, 2p_2^0 \frac{|\vec{p}_1|}{p_1^0} \quad (= |\vec{v}_A| \, 2E_A \, 2E_B)$$

the differential cross-section $\dfrac{d\sigma}{d\Omega_3}$

$$\frac{d\sigma}{d\Omega_3} = \frac{\alpha^2}{4} (1 + \cos^2 \vartheta) \tag{2.156}$$

and the total cross-section $\sigma = \displaystyle\int d\Omega_3 \frac{d\sigma}{d\Omega_3}$

$$\sigma_{\text{total}}(e^- e^+ \to \mu^- \mu^+) = \frac{4\pi\alpha^2}{3} \tag{2.157}$$

for $e^- e^+ \to \mu^- \mu^+$ are calculated.

Chapter 3

Quantenelectrodynamics (QED)

3.1 The electromagnetic field

Maxwell's equations:

$$\partial_\mu F^{\mu\nu}(x) = j^\nu(x) \tag{3.1}$$

$$\varepsilon^{\mu\nu\rho\sigma}\,\partial_\nu F_{\rho\sigma}(x) = 0 \tag{3.2}$$

with field strength $F_{\mu\nu}$

$$F_{\mu\nu}(x) = \partial_\mu\, A_\nu(x) - \partial_\nu\, A_\mu(x) \tag{3.3}$$

and 4-vector potential $A_\mu(x)$.
(3.3) trivially satisfies (3.2):

$$2\,\varepsilon^{\mu\nu\rho\sigma}\,\partial_\nu\partial_\rho A_\sigma = 0$$

$j^\nu(x)$ is the 4-vector current density:

$$j^\nu(x) = \begin{pmatrix} \rho(x) \\ \vec{j}(x) \end{pmatrix} \tag{3.4}$$

and

$$F^{\mu\nu}(x) = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{3.5}$$

We use Heaviside units (rational), that is removing factors of $\sqrt{4\pi}$ from the equation. Maxwell's equations (see (3.2)) reads with (3.4) and (3.5):

$$\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \rho \\ \vec{\nabla}\times\vec{B} &= \frac{\partial\vec{E}}{\partial t}+\vec{j} \qquad (3.6)\\ \vec{\nabla}\cdot\vec{B} &= 0 \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \qquad (3.7) \end{aligned}$$

Relation to cgs units:

$$\begin{aligned} \alpha &= \frac{e_H^2}{4\pi}\frac{1}{\hbar c}\simeq\frac{1}{137} \\ e_H &= \sqrt{4\pi}\,e_{cgs} \\ \Rightarrow e_{cgs}^2 &= \frac{e_H^2}{4\pi}=\frac{e_{SI}^2}{4\pi\varepsilon_0} \qquad (3.8)\\ \vec{E}_H &= \frac{\vec{E}_{cgs}}{\sqrt{4\pi}} \\ \vec{B}_H &= \frac{\vec{B}_{cgs}}{\sqrt{4\pi}} \end{aligned}$$

(3.9)

The most important values in the cgs-system:

$$\begin{aligned} e_{cgs}^2 &= (4.8\cdot10^{-10})^2\,\mathrm{g}\cdot\mathrm{cm}^3/\mathrm{s}^2, \quad c=3\cdot10^{10}\mathrm{cm/s} \\ \hbar_{cgs} &= 1.05\cdot10^{-27}\,\mathrm{erg}\cdot\mathrm{s}, \quad \mathrm{erg}=\mathrm{g}\cdot\mathrm{cm}^2/\mathrm{s}^2=10^{-7}\mathrm{J} \end{aligned}$$

Remarks:

1. Inhomogeneous Maxwell equation (3.1):

$$\partial_\nu\,\partial_\mu\,F^{\mu\nu}=\partial_\nu\,j^\nu=0 \qquad (3.10)$$

   $\Rightarrow$ conserved current!

2. A^μ carries a redundancy, the gauge degrees of freedom: $F^{\mu\nu}$ is invariant under

$$\begin{aligned} A^\mu(x) &\to A^\mu(x)+\partial^\mu\,\alpha(x) \\ \Rightarrow F^{\mu\nu} &\to F^{\mu\nu}+[\partial^\mu,\partial^\nu]\,\alpha=F^{\mu\nu} \qquad (3.11) \end{aligned}$$

   This redundancy can be removed by imposing a constraint on A_μ (gauge fixing condition):
   Lorentz gauge (Landau):

$$\begin{aligned} \partial_\mu\,A^\mu(x) &= 0 \\ \Rightarrow \partial_\mu F^{\mu\nu} &= \Box A^\nu=j^\nu \qquad (3.12) \end{aligned}$$

   consistent with (3.10).
   For $j^\nu=0$, each component A^ν satisfies the Klein-Gordon equation.

Lagrangian density:

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu}(x)\, F^{\mu\nu}(x) - A_\mu(x)\, j^\mu(x) \tag{3.13}$$

Quantisation of free field:

$$A_\mu(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{2k_0} \left[e^{ikx}\, a_\mu^\dagger(\vec{k}) + e^{-ikx}\, a_\mu(\vec{k}) \right] \tag{3.14}$$

with $k_0 = |\vec{k}|$, and the commutators

$$[a_\nu(\vec{k}'), a_\mu^\dagger(\vec{k})] = -g_{\mu\nu}\, (2\pi)^3 \cdot 2k_0\, \delta(\vec{k} - \vec{k}')$$

$$\begin{aligned} [a_\nu(\vec{k}), a_\mu(\vec{k}')] &= 0 \\ &= [a_\nu^\dagger(\vec{k}), a_\mu^\dagger(\vec{k}')] \end{aligned} \tag{3.15}$$

Remarks:

1. We have the Klein-Gordon equation:

$$\Box A_\mu(x) = 0 \tag{3.16}$$

   but

$$\partial_\mu\, A_\mu(x) \simeq \int ik_\mu\, a_\mu^\dagger \ldots \neq 0.$$

   $(k_\mu [a_\nu(\vec{k}'), a_\mu^\dagger(\vec{k})] = -k_\nu\, (2\pi)^3\, 2k_0\, \delta(\vec{k} - \vec{k}'))$

$$\partial_\mu\, F^{\mu\nu}[A] \neq 0 \tag{3.17}$$

   Can we do better than (3.14) ?
   No, it was not possible to construct A_{op} with $\partial_\mu\, A_{op_\mu} = 0 +$ covariant.

2. Fockspace:

   - vacuum $|0\rangle$ with $\langle 0|0\rangle = 1$

$$a_\mu(\vec{k})\, |0\rangle = 0 \tag{3.18}$$

   - one particle states

$$a_\mu^\dagger(\vec{k})\, |0\rangle$$

     with norm

$$\begin{aligned} \langle 0|\, a_\nu(\vec{k}')\, a_\mu^\dagger(\vec{k})\, |0\rangle &= \langle 0|\, [a_\nu(\vec{k}'),\, a_\mu^\dagger(\vec{k})]\, |0\rangle \\ &= -g_{\mu\nu}\, (2\pi)^3\, 2k_0\, \delta(\vec{k} - \vec{k}') \end{aligned} \tag{3.19}$$

$$\Rightarrow \quad \begin{aligned} \mu = \nu = i: &\quad \text{positive norm states} \\ \mu = \nu = 0: &\quad \textit{negative} \text{ norm states} \end{aligned}$$

     $\Rightarrow$ No physical Hilbertspace (no probability interpretation).

Remedy:
Fockspace contains physical subspace F_{phys} with

$$\partial_\mu F^{\mu\nu}[A_\mu] \,|\text{ physical states }\rangle \stackrel{!}{=} 0$$

$$\Rightarrow \boxed{k^\mu \, a_\mu(\vec{k}) \,|\text{ physical states }\rangle = 0} \tag{3.20}$$

or

$$\partial_\mu A^\mu \,|\text{ physical states }\rangle = 0$$

Evidently $|0\rangle \in F_{phys}$.
Construction of F_{phys}:

$$\begin{aligned} \alpha_0^\dagger(\vec{k}) &= \frac{1}{\sqrt{2}} \frac{1}{|\vec{k}|} k^\mu \, a_\mu^\dagger(\vec{k}) \\ &= \frac{1}{\sqrt{2}} \left(a_0^\dagger(\vec{k}) - \hat{k} \, \vec{a}^\dagger(\vec{k}) \right) \end{aligned}$$

with $\hat{k} = \frac{\vec{k}}{|\vec{k}|}$.

$$\begin{aligned} \alpha_1^\dagger(\vec{k}) &= \hat{e}_1 \, \vec{a}^\dagger(\vec{k}) \\ \alpha_2^\dagger(\vec{k}) &= \hat{e}_2 \, \vec{a}^\dagger(\vec{k}) \\ \alpha_3^\dagger(\vec{k}) &= \frac{1}{\sqrt{2}} \left(\vec{a}_0^\dagger(\vec{k}) + \hat{k} \, \vec{a}^\dagger(\vec{k}) \right) \end{aligned} \tag{3.21}$$

$$\begin{aligned} \hat{e}_i \cdot \hat{k} &= 0 \\ \hat{e}_i \hat{e}_j &= \delta_{ij} \end{aligned}$$

Commutators:

$$\begin{aligned} [\alpha_0, \alpha_0^\dagger] = [\alpha_3, \alpha_3^\dagger] = 0 \\ [\alpha_0, \alpha_i^{(\dagger)}] = [\alpha_3, \alpha_i^{(\dagger)}] = 0 \\ [\alpha_0, \alpha_2^\dagger] = -(2\pi)^3 \cdot 2k^0 \, \delta \\ [\alpha_i, \alpha_i^\dagger] = (2\pi)^3 \cdot 2k^0 \, \delta \end{aligned} \tag{3.22}$$

Physical states:

$$\alpha_0(\vec{k}) \,|\text{physical state}\rangle = 0 \tag{3.23}$$

One particle states: $\alpha_0^\dagger(\vec{k}) \,|0\rangle$, $\alpha_1^\dagger(\vec{k}) \,|0\rangle$, $\alpha_2^\dagger(\vec{k}) \,|0\rangle$
but zero-norm states

$$\langle 0| \, \alpha_0(\vec{k}) \, \alpha_0^\dagger(\vec{k}) \,|0\rangle = 0 \tag{3.24}$$

$\Rightarrow$ Physical Hilbertspace $\mathcal{H}$:

$$|1\rangle \sim |2\rangle \text{ for } ||\,|1\rangle - |2\rangle\,|| = 0$$

$$\mathcal{H} = \frac{F_{phys.}}{\sim} \tag{3.25}$$

$\Rightarrow$ We have two one particle states in $\mathcal{H}$:

$$\left|\vec{k}, \varepsilon_1\right\rangle = \alpha_1^\dagger(\vec{k})\,|0\rangle$$

with Polarisation $\hat{e}_1$, $\hat{e}_2$, **g** and

$$\varepsilon_i = \begin{pmatrix} 0 \\ \hat{e}_i \end{pmatrix} \quad i = 1, 2 \tag{3.26}$$

General: $\left|\vec{k}, \varepsilon\right\rangle$ with $\varepsilon^0 = 0$ and $\vec{\varepsilon} \cdot \vec{k} = 0$, $\vec{\varepsilon} \in \mathbb{C}^3$

$\vec{E}$- and $\vec{B}$-field operators:

$$E^i(x) = -F^{0i} = -(\partial^0 A^i - \partial^i A^0)$$

$$\begin{aligned}
\Rightarrow \vec{E}(x) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2k_0} ik^0 \left\{(\vec{a} - \hat{k}\,a^0)\,e^{-ikx} - (\vec{a}^\dagger - \hat{k}\,a^{0\,\dagger})\,e^{ikx}\right\} \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2k_0} ik^0 \{(\hat{e}_1\,\vec{\alpha}_1(\vec{k}) + \hat{e}_2\,\alpha_2(\vec{k}))\,e^{-ikx} - \\
&\quad - (\hat{e}_1\,\alpha_1^\dagger(\vec{k}) + \hat{e}_2\,\alpha_2^\dagger(\vec{k}))\,e^{ikx}\} \\
&\quad - \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2k_0} ik^0 \left\{\hat{k}\,\alpha_0(\vec{k})\,e^{-ikx} - \hat{k}\,\alpha_0^\dagger(\vec{k})\,e^{ikx}\right\} \\
\vec{B}(x) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2k_0} i\vec{k}x \left\{\hat{e}_i\,\alpha_i(\vec{k})\,e^{-ikx} - \hat{e}_i\,\alpha_i^\dagger(\vec{k})\,e^{ikx}\right\}
\end{aligned} \tag{3.27}$$

$\Rightarrow$ Hamiltonian depends on α_i, $\alpha_i^\dagger$.

From

$$\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}\,F^{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) \tag{3.28}$$

follows the canonical momentum

$$\Pi^i = \frac{\partial \mathcal{L}}{\partial\,\partial_0 A_i} = -F^{0i} = E^i \tag{3.29}$$

Hamiltonian density

$$\begin{aligned}
\mathcal{H} &= \vec{\Pi}\,\partial_0\vec{A} + \frac{1}{4}F_{\mu\nu}\,F^{\mu\nu} \\
&= \vec{E}^2 - \frac{1}{2}(\vec{E}^2 - \vec{B}^2) + \vec{E}\nabla A_0 \\
&= \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \vec{\nabla}(\vec{E}A_0)
\end{aligned} \tag{3.30}$$

The last equation follows from $\vec{\nabla}\vec{E} = 0$ for $\rho = 0$.

$$\Rightarrow H = \int \mathrm{d}^3x\, \mathcal{H} = \frac{1}{2}\int \mathrm{d}^3x\, (\vec{E}^2 + \vec{B}^2) \tag{3.31}$$

Use field operators

$$\begin{aligned} H &\simeq -\frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\int \frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{1}{2k^0}\frac{1}{2k^0}(ik^0)^2\left[2\,\left(\alpha_i(\vec{k})\,\alpha_i^\dagger(\vec{k}) + \alpha_i^\dagger(\vec{k})\,\alpha_i(\vec{k})\right)\right]\cdot(2\pi)^3\,\delta(k-k') \\ &= \frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2k^0}\,k^0\left[\alpha_i(\vec{k})\,\alpha_i^\dagger(\vec{k}) + \alpha_i^\dagger(\vec{k})\,\alpha_i(\vec{k})\right] \end{aligned} \tag{3.32}$$

Normal ordering causes $\alpha_i(\vec{k})\,\alpha_i^\dagger(\vec{k}) + \alpha_i^\dagger(\vec{k})\,\alpha_i(\vec{k})$ to become $2\cdot\alpha_i^\dagger(\vec{k})\,\alpha_i(\vec{k})$.
Inserting the result of normal ordering in equation (3.32) leads to:

$$H \simeq \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2k^0}\,k^0\alpha_i^\dagger(\vec{k})\,\alpha_i(\vec{k}) \tag{3.33}$$

but ...

Interlude: The Casimir effect
Experiment: Lamoreaux et al 1997 Solution for $\vec{E}$, $\vec{B}$: plane waves

$$\vec{E} \simeq \vec{\varepsilon}\,e^{\pm ikx}, \quad \vec{B} \simeq \hat{k}\times\vec{E} \tag{3.34}$$

Boundary conditions:

$$\begin{aligned} \hat{n}\times\vec{E}|_{x=0,\,L} &= 0 \\ \hat{n}\cdot\vec{B}|_{x=0,\,L} &= 0 \end{aligned} \tag{3.35}$$

$$\Rightarrow \vec{E} \simeq \vec{\varepsilon}\,\sin(k_x x)\,e^{i(k_y y + k_z z - k^0 t)} \tag{3.36}$$

with the polarisation $\vec{\varepsilon}$ and

$$\begin{aligned} k^0 &= \sqrt{\vec{k}^2} \\ k_x &= \frac{n\pi}{L}, \quad n = 1, 2, \ldots \end{aligned}$$

Casimir energy:

$$\begin{aligned} \langle 0|\,H\,|0\rangle_L &= \frac{1}{2}\,\not{\Sigma}\!\!\!\!\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{1}{2}\,\langle 0|\,\alpha_i(\vec{k})\,\alpha_i^\dagger(\vec{k})\,|0\rangle\cdot 2 \\ &= \frac{1}{2}\frac{1}{L}\int \frac{\mathrm{d}^2k_{||}}{(2\pi)^2}\sum_{n=1}^{\infty} 2\sqrt{\vec{k}_{||}^2 + \left(\frac{n\pi}{L}\right)^2}\cdot\delta(0) \end{aligned} \tag{3.37}$$

with

$$\vec{k}_{||} = (0, k_y, k_z).$$

Equation () is divergent for large momenta (UV). Assume for the moment that the $\sum\!\!\!\!\int$ is cut off regularly for high energies/momenta:

$$\langle 0|\, H\, |0\rangle_L = \underbrace{\frac{V}{L}}_{\text{Area}} \int \frac{\mathrm{d}^2 k_{||}}{(2\pi)^2} \sum_{n=1}^{\infty} \sqrt{\vec{k}_{||}^2 + \left(\frac{n\pi}{L}\right)^2} \cdot r_\Lambda \left(\vec{k}_{||}^2 + \left(\frac{n\pi}{L}\right)^2 \right) \tag{3.38}$$

where $r_\Lambda(x \gg \Lambda^2) \to 0$, $r_\Lambda(x \ll \Lambda^2) \to 1$.

$$\Rightarrow E_L = \langle 0|\, H\, |0\rangle_L = \frac{V}{2\pi \cdot L} \sum_{n=1}^{\infty} R_\Lambda(n) \tag{3.39}$$

with

$$R_\Lambda(n) = \int_0^\infty \mathrm{d}k_{||}\, k_{||} \sqrt{\vec{k}_{||}^2 + \left(\frac{n\pi}{L}\right)^2}\; r_\Lambda \left(\vec{k}_{||}^2 + \left(\frac{n\pi}{L}\right)^2 \right)$$

$\Delta E_L = E_L - E_\infty$:

$$\Delta E_L = \frac{V}{2\pi \cdot L} \left[\sum_{n=1}^{\infty} R_\Lambda(n) - \int_0^\infty \mathrm{d}n\, R_\Lambda(n) \right] + \frac{1}{2} R_\Lambda(0) \tag{3.40}$$

Euler-McLaurin:

$$\int_0^\infty \mathrm{d}n\, R_\Lambda(n) = \sum_{n=1}^{\infty} \left[R_\Lambda(n) + \frac{1}{(2n)!} B_{2n}\, R_\Lambda^{(2n-1)}(0) \right] + \frac{1}{2} R_\Lambda(0) \tag{3.41}$$

with Bernoulli numbers

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad \dots \tag{3.42}$$

$$\Rightarrow \Delta E = \frac{V}{2\pi \cdot L} \left[-\frac{1}{12}\, R_\Lambda^{(1)}(0) + \frac{1}{720}\, R_\Lambda^{(3)}(0) + \dots \right] \tag{3.43}$$

Since

$$\begin{aligned} R_\Lambda(n) &= \int_0^\infty \mathrm{d}k_{||}\, k_{||}\, \underbrace{\sqrt{\vec{k}_{||}^2 + \left(\frac{n\pi}{L}\right)^2}}_{k} \\ &= \int_{\frac{n\pi}{L}}^\infty \mathrm{d}k\, k^2\, r_\Lambda(k^2) \cdot r_\Lambda(k^2) \end{aligned} \tag{3.44}$$

$$\begin{aligned} \Rightarrow R_\Lambda^{(1)}(n) &= -\frac{\pi}{L} \left(\frac{n\pi}{L}\right)^2 r_\Lambda \left(\frac{n\pi}{L}\right) \\ R_\Lambda^{(1)}(0) &= 0 \\ R_\Lambda^{(3)}(0) &= -2 \left(\frac{\pi}{L}\right)^3 \end{aligned} \tag{3.45}$$

$R_\Lambda^{(i>3)}(0)$ depends on r_Λ: $R_\Lambda^{(i>3)}(0) \sim (\frac{1}{\Lambda L})^{i-3}$.
Finally:

$$\Delta E = -\frac{\pi^2}{720}\frac{V/L}{L^3} \quad \text{for } \Lambda \to \infty$$

or

$$\Delta\varepsilon = \frac{\Delta E}{V/L} = -\frac{\pi^2}{720}\frac{1}{L^3} \tag{3.46}$$

Force/Area:

$$F = -\frac{\mathrm{d}\Delta\varepsilon}{\mathrm{d}L} = -\frac{\pi^2}{240}\frac{1}{L^4}(\hbar c) \simeq -\frac{1.3\cdot 10^{-27}}{L[m]^4}\,\text{pa m}^4$$

Summary:

$$\begin{aligned} R_\Lambda^{(1)}(n) &= -\frac{\pi}{L}\left(\frac{n\pi}{L}\right)^2 r_\Lambda\left(\frac{n\pi}{L}\right) \\ \Delta E &= -\frac{\pi^2}{720}\frac{V/L}{L^3} + O(R_\Lambda^{(4)}(0)) \\ &= -\frac{\pi^2}{720}\frac{V/L}{L^3\cdot L} \quad \text{for } \Lambda \to \infty \end{aligned} \tag{3.47}$$

with

$$R_\Lambda^{(i>3)}(0) \sim \left(\frac{1}{\Lambda L}\right)^{i-3}$$

Force/Area:

$$F = -\frac{\mathrm{d}(\frac{\Delta E}{V/L})}{\mathrm{d}L} = -\frac{\pi^2}{240}\frac{1}{L^4}(\hbar c) \simeq -\frac{1.3\cdot 10^{-27}}{L[m]^4}\,\text{pa m}^4$$

Idea: with plates, without plates
For high frequencies the difference between 'plates' and 'no plates' becomes smaller.
$\Delta E = \langle 0|\, H\, |0\rangle_L - \langle 0|\, H\, |0\rangle_{L=\infty}$ is finite. Divergent parts cancel. Computation is performed with regularisation

$$\Lambda : r_\Lambda\left(k_{||}^2 + \left(\frac{n\pi}{L}\right)^2\right)$$

3.2 Lagrangian of QED

In equation (3.13) on page 39 the Langrangian density of the electromagnetic field coupled to an external current $j^\mu(x)$ was presented. In QED, j^μ describes the coupling to the electron-, muon-, tau-current with

$$j^\mu(x) = -e\bar{\psi}\,\gamma^\mu\,\psi(x) \tag{3.48}$$

where ψ is given by

$$\psi = \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix} \tag{3.49}$$

Together with the Dirac term we get

$$\boxed{\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu}(x)\, F^{\mu\nu}(x) + \bar{\psi}\,(i\,\gamma^\mu\,\mathcal{D}_\mu - m)\,\psi} \tag{3.50}$$

where $\mathcal{D}_\mu$ is the covariant derivative:

$$\mathcal{D}_\mu\,\psi(x) = \big(\partial_\mu - ieA_\mu(x)\big)\,\psi(x) \tag{3.51}$$

and

$$m = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}$$

Gauge invariance: $g(x) = e^{i\,\alpha(x)} \in U(1)$

$$\begin{aligned}
\psi(x) &\;\rightarrow\; g(x)\,\psi(x) = e^{i\alpha(x)}\psi(x) = \psi^g \\
\bar{\psi}(x) &\;\rightarrow\; \bar{\psi}(x)\,g^\dagger(x) = \bar{\psi}(x)\,e^{-ie\alpha(x)} = \bar{\psi}^g \qquad (3.52)\\
A_\mu(x) &\;\rightarrow\; \frac{i}{e}\,g\,\mathcal{D}_\mu\,g^\dagger = A_\mu(x) + \partial_\mu\alpha(x) = A^g \\
\Rightarrow \mathcal{D}_\mu(x) &\;\rightarrow\; g\,\mathcal{D}_\mu\,g^\dagger(x) \\
-\frac{1}{4}F_{\mu\nu}\,F^{\mu\nu} &\;\rightarrow\; -\frac{1}{4}F_{\mu\nu}\,F^{\mu\nu} \qquad (3.53)\\
\bar{\psi}\,(i\not{D} - m)\,\psi &\;\rightarrow\; \bar{\psi}\,g^\dagger\,(ig\not{D}g^\dagger - m)\,g\psi = \bar{\psi}\,(i\not{D} - m)\,\psi \qquad (3.54)
\end{aligned}$$

with

$$\not{D} = \gamma^\mu\mathcal{D}_\mu$$

and

$$gg^\dagger = 1.$$

Remark: $\partial_\mu \rightarrow \partial_\mu - ieA_\mu$ is called 'minimal coupling'.
A_μ is a connection (Zusammenhang).
Consequences of gauge invariance:
Classical action of QED:

$$S_{QED}[A,\psi,\bar{\psi}] = \int \mathrm{d}^4x\,\mathcal{L}_{QED}(x)$$

with $\mathcal{L}_{QED}(x)$ from equation (3.50).
Gauge invariance:

$$S[A^g,\psi^g,\bar{\psi}^g] = S[A,\psi,\bar{\psi}]$$

Infinitesimal

$$
\begin{aligned}
& S[A+\partial_\mu\alpha, (1+i\alpha e)\,\psi, \bar\psi\,(1-i\alpha e)] - S[A,\psi,\bar\psi] = 0 \\
= \quad & \int \mathrm{d}^4x \left(\partial_\mu\alpha(x)\frac{\delta}{\delta A_\mu(x)} + ie\alpha(x)\,\psi(x)\frac{\delta}{\delta\psi(x)} - \bar\psi(x)\frac{\delta}{\delta\bar\psi(x)}\right) S
\end{aligned}
$$

e.g.: $\psi = \bar\psi = 0$:

$$
\begin{aligned}
\int \mathrm{d}^4x\,\partial_\mu\alpha(x)\frac{\delta}{\delta A_\mu(x)}\,S[A,0,0] &= 0 \\
\Rightarrow -\int \mathrm{d}^4x\,\alpha(x)\,\partial_\mu\frac{\delta S[A,0,0]}{\delta A_\mu(x)} &= 0
\end{aligned}
$$

$\alpha(x)$ arbitrary:

$$
\partial_\mu\frac{\delta S[A]}{\delta A_\mu(x)} = 0
$$

also

$$
\partial_\mu^*\,\frac{\delta^2 S}{\delta A_\mu(x)\delta A_\nu(y)}|_{A=0} = 0
$$

$$
\Rightarrow \partial_\mu^*\,(\partial_\rho\partial^\rho\,g^{\mu\nu} - \partial^\mu\partial^\nu)\,\delta(x-y) = 0
$$

Completion of Feynman rules:
Physical 1p states:

$$
\left|\vec k, \varepsilon_i\right\rangle = \alpha_i^\dagger(\vec k)\,|0\rangle = \hat e_i\,\vec a^\dagger(\vec k)\,|0\rangle \quad i = 1,2 \tag{3.55}
$$

with the 4-vector $\varepsilon_i = \begin{pmatrix} 0 \\ \hat e_i \end{pmatrix}$
General states: $|\varepsilon_\mu\varepsilon^\mu| = 1$

$$
\left|\vec k, \varepsilon\right\rangle = -\varepsilon^\mu\,\alpha_\mu^\dagger(\vec k)\,|0\rangle = \vec\varepsilon\,\vec a^\dagger(\vec k)\,|0\rangle \tag{3.56}
$$

with $\varepsilon^0 = 0$, $\varepsilon_\mu k^\mu = 0$, $\vec\varepsilon\vec k = 0$.
Norm

$$
\left\langle\vec k', \varepsilon'|\varepsilon, \vec k\right\rangle = \vec\varepsilon'\,^*\,\vec\varepsilon\,(2\pi)^3\,2k^0\,\delta(\vec k - \vec k') \tag{3.57}
$$

$\Rightarrow$ initial state ε^μ final state $\varepsilon^{\mu *}$

3.3 Magnetic moment of electron

Consider

$$
\begin{aligned}
\mathcal{D}_\mu &= \partial_\mu - ieA_\mu \\
\mathcal{L}_\mathcal{D} &= \bar\psi\,(i\,\gamma^\mu\,\mathcal{D}_\mu - m)\,\psi \\
\mathcal{H}_\mathcal{D} &= \psi^\dagger\,\underbrace{(-i\,\gamma^0\,\gamma^i\,\mathcal{D}_i + \gamma^0\,m)}_{H}\,\psi
\end{aligned} \tag{3.58}
$$

Evaluate non-relativistic limit of H^2:
First

$$\begin{aligned} H^2 &= \gamma^0 (i\vec{\gamma}\vec{\mathcal{D}} + m)\,\gamma^0 (i\vec{\gamma}\vec{\mathcal{D}} + m) \\ &= (\vec{\gamma}\vec{\mathcal{D}} + m)\,(i\vec{\gamma}\vec{\mathcal{D}} + m) \\ &= \gamma_i\gamma_j \mathcal{D}^i\mathcal{D}^j + m^2 \\ &= -\vec{\mathcal{D}}^{\,2} + \frac{1}{2}[\gamma_i, \gamma_j]\,\mathcal{D}^i\mathcal{D}^j + m^2 \\ &= -\vec{\mathcal{D}}^2 - i\,e\,\frac{1}{4}[\gamma_i, \gamma_j]\,F^{ij} + m^2 \end{aligned}$$

with

$$[\gamma^i, \gamma^j] = -2i\varepsilon_{ijk}\,\Sigma^k$$

and $\Sigma^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}$

$$\begin{aligned} H^2 &= -\vec{\mathcal{D}}^2 - \frac{e}{2}\,\varepsilon_{ijk}\,F^{ij} + m^2 \\ &= -\vec{\mathcal{D}}^2 + e\,\vec{\Sigma}\cdot\vec{B} + m^2\,. \end{aligned} \tag{3.59}$$

Then

$$\sqrt{H^2} = m + \frac{(\vec{p} - e\vec{A})^2}{2m} + \frac{e}{2m}\,\vec{\Sigma}\cdot\vec{B} + O(m^{-2}) \tag{3.60}$$

Remember $\vec{\Sigma} = 2\vec{S}$

$$\rightarrow i\frac{\partial}{\partial t}\psi = \frac{(\vec{p} - e\vec{A})^2}{2m} + m + \frac{e}{2m}\,2\,\vec{S}\cdot\vec{B} \tag{3.61}$$

Spin coupling:

$$g_e \cdot \frac{e}{2m}\,\vec{S}\cdot\vec{B} = \vec{\mu}_e \cdot \vec{B}$$

with gyromagnetic factor $g_e = 2$.
Experiment: $g_e = 2.0023193\ldots$
$(\frac{|e|}{2m}\,g_e = 1.760859770(44)\cdot 10^{11} s^{-1} T^{-1})$

Anomalous magnetic moment: $\vec{j}_{op}(x) = -e\ :\bar{\psi}(x)\,\gamma^\mu\,\psi(x):$

$$\vec{\mu}_{op} = \frac{1}{2}\int \mathrm{d}^3x\,\vec{x}\times\vec{j}_{op}(\vec{x}, t) \tag{3.62}$$

Expectation value: $\left|e(\vec{k}, s)\right\rangle = a_s^\dagger(\vec{k})\,|0\rangle$.
Non-interacting:

$$\left\langle e(\vec{k}', r)\right|\vec{\mu}\left|e(\vec{k}, s)\right\rangle = -\frac{e}{2}\int \mathrm{d}^3x\,\vec{x}\times\left\langle e(\vec{k}', r)\right|\ :\bar{\psi}\,\vec{\gamma}\,\psi:\left|e(\vec{k}, s)\right\rangle \tag{3.63}$$

When one proceeds as with scattering, one gets $g_e = 2$.

Magnetic moment of an electron in general:

$$\vec{\mu}(t) = \frac{1}{2} \int \mathrm{d}^3x\, \vec{x} \times \vec{j}(x) \tag{3.64}$$

Electron: $t = 0$.

$$\begin{aligned}
\langle e(\vec{p}\,', r)|\, \vec{\mu}\, |e(\vec{p}, s)\rangle &= -\frac{e}{2} \int \mathrm{d}^3x\, \vec{x} \times \langle e(\vec{p}\,', r)|\, :\bar{\psi}(\vec{x}, 0)\, \vec{\gamma}\, \psi(\vec{x}, 0) :\, |e(\vec{p}, s)\rangle \\
&= -\frac{e}{2} \int \mathrm{d}^3x\, \vec{x} \times \langle 0|\, a_r(\vec{p}\,')\, :\bar{\psi}(\vec{x}, 0)\, \vec{\gamma}\, \psi(\vec{x}, 0) :\, a_s^\dagger(\vec{p})\, |0\rangle \\
&= -\frac{e}{2} \int \mathrm{d}^3x\, \vec{x} \times \langle 0|\, a_r(\vec{p}\,')\bar{\psi}(\vec{x}, 0)\, |0\rangle\, \langle 0|\, \psi(\vec{x}, 0) a_s^\dagger(\vec{p})\, |0\rangle \\
&= -\frac{e}{2} \int \mathrm{d}^3x\, e^{i(p-p')x}\, \vec{x} \times \bar{u}_r(p')\, \vec{\gamma}\, u_s(p) \\
&= \ldots
\end{aligned} \tag{3.65}$$

Anomalous magnetic moment
Classically the interaction is given by

$$-e\, \bar{\psi}\, \gamma^\mu\, A_\mu\, \psi$$

$\Rightarrow$ Looking for quantum corrections to the Vertex:
Lowest order in α and A:

$$e\, \Gamma_\mu(q, q+p)\, A^\mu(p)$$

with

$$\begin{aligned}
\Gamma_\mu &\sim -\frac{i}{2m}\, \sigma_{\mu\nu}\, p^\nu\, \frac{\alpha}{2\pi} \\
\sigma^{\mu\nu} &= \frac{i}{2}[\gamma^\mu, \gamma^\nu].
\end{aligned} \tag{3.66}$$

For $\sigma^{\mu\nu}$ see page 28.

$$\begin{aligned}
\Rightarrow (\Gamma_\mu \cdot A^\mu)(x) &\simeq -\frac{ie}{2m}\, \sigma_{\mu\nu}\, \frac{\alpha}{2\pi}\, \partial^\nu\, A^\mu(x) \\
&= -\frac{ie}{4m}\, \frac{\alpha}{2\pi}\, \sigma_{\mu\nu}\, F^{\nu\mu}(x) \\
&= \frac{e}{2m}\, \frac{\alpha}{2\pi}\, \vec{\Sigma} \cdot \vec{B}
\end{aligned} \tag{3.67}$$

One gets the last equation, because $\vec{E} = 0$.

$$\Rightarrow \mu \to \mu + \frac{e}{2m}\, \frac{\alpha}{2\pi} \vec{\Sigma} = 2\, \frac{e}{2m}\, \left(1 + \frac{\alpha}{2\pi}\right)\, \underbrace{\frac{\vec{\Sigma}}{2}}_{\vec{s}} \tag{3.68}$$

3.4 Renormalisation

$\sim \alpha$
higher order term $\sim \alpha^2$ vacuum polarisation
Computation from Feynman rules:

$$\Pi^{\mu\nu}(p) = -(ie)^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \mathrm{Tr} \frac{i}{\not{q} - m} \gamma^\mu \frac{i}{\not{p} + \not{q} - m} \gamma^\nu \tag{3.69}$$

$$\left[v - (ie)^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \mathrm{Tr} \frac{\not{q}}{q^2} \gamma^\mu \frac{1}{(p+q)^2} (\not{q} + \not{p}) \gamma^\nu \right] = p^2 - p^\mu \tag{3.70}$$

Two Problems:

1. integral is divergent $\sim \int \mathrm{d}^4 q \, \frac{1}{q^2}$
2. poles: $q^2 = 0$, $(q+p)^2 = 0$.

Remedy:

1. Regularisation: $\Pi \to \Pi_\Lambda(p^2, m^2)$ (Casimir)
2. Wick rotation: $t_M \to i t_E$, $p_M^0 \to i p_M^0$, $p_\mu p^\mu \to -p_\mu^E p_\mu^E$.
   There is a one-to-one relation between Euklidean Correlation functions G_E and Minkowski Correlation functions G_M.

How to implement (1):
Photon propagator:

$$A_\mu (\partial_\nu \partial^\nu g^{\mu\rho} - \partial^\mu \partial^\rho) A_\rho \to Z_A A_{ren.\,\mu} (\partial_\nu \partial^\nu g^{\mu\nu} - \partial^\mu \partial^\rho) A_{ren.\,\rho} \tag{3.71}$$

with

$$Z_A = 1 + Z_A^{(1)} \alpha + O(\alpha^2)$$

$\Rightarrow$
Demand

$$\lim_{\Lambda \to \infty} \left(Z_A^{-1} + \Pi_\Lambda(p^2, m^2) \right)$$

finite.

Structure of Π_Λ

$$\begin{aligned} \Pi_\Lambda(p^2, m^2) &= \alpha \left[f_0 \ln \frac{p^2}{\Lambda^2} + f_1 + O\left(\frac{p^2}{\Lambda^2} \right) + \dots \right] \\ \Rightarrow Z_A^{-1} &= 1 - \alpha\, f_0 \, \ln \frac{\Lambda^2}{\mu^2} + \alpha \cdot \text{finite} + O(\alpha^2) \end{aligned}$$

with renormalisation scale μ.

Hence

$$Z_A^{-1} + \Pi_\Lambda(p^2, m^2) = 1 + \alpha \left[\ln \left(\frac{p^2}{\mu^2} \right) + \text{finite} \right] + O(\alpha^2)$$

In summary we demand

$$\boxed{\mu \frac{\mathrm{d}}{\mathrm{d}\mu}\,\text{Observables} = 0}$$

Computation:

$$\Pi^{\mu\nu}(p) = (p^2 g^{\mu\nu} - p^\mu p^\nu)\cdot \Pi(p) \tag{3.72}$$

with renormalised Π: $\Pi_\Lambda(p^2, m^2) - \Pi_\Lambda(0, m^2)$.

$$\begin{aligned}\Pi(p^2) &= -\frac{2\alpha}{\pi}\, i \int_0^1 \mathrm{d}x\, x(1-x)\, \ln \frac{m^2}{m^2 - x(1-x)p^2} \\ &= -\frac{2\alpha}{\pi}\, i \int_0^1 \mathrm{d}x\, x(1-x)\, \ln \frac{m^2/p^2}{m^2/p^2 - x(1-x)}\end{aligned}$$

UV-asymptotics:

$$\Pi(p^2) \simeq \frac{\alpha}{3\pi}\left[\ln(-\frac{p^2}{m^2})\ \underbrace{-\frac{5}{3}}_{\text{finite}=c}\ + O\left(\frac{m^2}{p^2}\right)\right] \tag{3.73}$$

$$\begin{aligned}\Rightarrow \alpha_{eff}(p^2) &= \frac{\alpha}{1 - \frac{\alpha}{3\pi}\ln(\frac{-p^2}{m^2 c})} \\ \lim_{-p^2\to 0} \alpha_{eff} &= 0 \\ \lim_{-p^2 \to m^2 c\, e^{3\pi/\alpha}} \alpha_{eff} &= \infty\end{aligned}$$

with (here)

$$c = e^{\frac{5}{3}}, \quad \boxed{\alpha_{eff}(c\,m^2) = \alpha} \quad \text{1-loop } \beta\text{-function}$$

RG-equation:

$$\frac{\mu \frac{\mathrm{d}}{\mathrm{d}\mu} Z_A}{Z_A} = -\frac{2}{3\pi}\alpha = -\boxed{\frac{1}{6\pi^2}\alpha}$$

Chapter 4

Quantum Chromodynamics (QCD)

The theory of strong interactions provides the nuclear forces that keep nuclear cores together.
Peculiar properties:

$$\alpha_s = \frac{g_s^2}{4\pi}$$

1. asymptotic freedom: $\alpha_s(p^2 \to \infty) \to 0$.
   (Nobel Prize 2004 for Gross, Wilczek and Politzer)
2. confinement: $V_{q\bar{q}}(r) \sim r$ for large distances.
   Millenium Prize riddle (Jaffe, Witten).
3. selfinteraction of gauge fields
   Gauge fields are *colour* charged.

Evidence for $\pm\frac{1}{3}e$, $\pm\frac{2}{3}e$ charged hadronic constituents with spin $\frac{1}{2}$: quarks q (J. Joyce), and gluonic jets (Nobel Prize 1969 for Gell-Mann, Zweig).

4.1 The QCD-Lagrangian

Hadronic current:

$$j_\mu(x) = e \sum_q Q_q \, \bar{q}(x) \, \gamma^\mu \, q(x) \tag{4.1}$$

with

$$Q_{u,c,t} = \frac{2}{3}, \quad Q_{d,s,b} = -\frac{1}{3}.$$

Hadronic states are invariant under $SU(3)$ transformations in colour space:

$$\begin{array}{rcll} q(x) & \to & U\, q(x) & \partial_\mu U = 0 \\ (q_\alpha(x) & \to & U_{\alpha\beta}\, q_\beta(x) & \alpha, \beta = 1,2,3) \end{array}$$

with

$$U \in SU(3)\colon U^\dagger U = U U^\dagger = \mathbb{1}_3, \quad \det U = 1.$$

$SU(3)$ is non-Abelian.

Infinitesimal

$$U = \mathbb{1}_3 + i\,\delta\varphi^a\,\lambda^a, \quad a = 1, \ldots, 8$$

λ^a: generators of $SU(3)$ (Lie-Algebra of $SU(3)$)
with

$$\hbar\lambda = 0, \quad \lambda^\dagger = \lambda$$

Compare to the generators of SU(2) σ^a:

$$[\sigma^a, \sigma^b] = 2i\,\varepsilon^{abc}\,\sigma^c$$

Quarks:

$$q(x) = \begin{pmatrix} q_1(x) \\ q_2(x) \\ q_3(x) \end{pmatrix} \tag{4.2}$$

with Dirac spinors q_i, where i, $i = 1, 2, 3$, indicates the colour.

In table 4.1 the flavours of quarks, their current masses and their charges are listed.

Generation	first	second	third	Charge
Mass [eV]	1.5-4	1150-1350	170	
Quark	u	c	t	$\frac{2}{3}$
Quark	d	s	b	$-\frac{1}{3}$
Mass [eV]	4-8	80-130	4.1-4.4	

Table 4.1: Quarks and some of their properties.

Lagrangian:

$$\mathcal{L}^{(0)}_{quark} = \sum_q \bar{q}(x)\,(i\partial\!\!\!/ - m_q)\,q(x) \tag{4.3}$$

Bound states:

1. Mesons: $q\bar{q}$
2. Baryons: qqq

e.g.:

$$\begin{aligned} \pi^+ &\sim u_1\bar{d}_1 + u_2\bar{d}_2 + u_3\bar{d}_3 \\ p &\sim \varepsilon_{\alpha\beta\gamma}\,u_\alpha\,u_\beta\,d_\gamma \\ (&\to \varepsilon_{\alpha\beta\gamma}\,u_\alpha\,u_\beta\,d_\gamma\,\det U) \end{aligned}$$

π^+ and p are gauge invariant.

$$[\lambda^a, \lambda^b] = 2i\, f^{abc} \lambda^c$$

where f^{abc} are structure constants.

$$\begin{aligned} \mathrm{Tr}\, \lambda &= 0 \\ \mathrm{Tr}\, \lambda^a\, \lambda^b &= 2\, \delta^{ab} \\ \{\lambda^a, \lambda^b\} &= \frac{4}{3}\, \delta^{ab} + 2\, d^{abc}\, \lambda^c \\ d^{abc} &= \mathrm{Tr}\, \lambda^a\, \{\lambda^b \lambda^c\} \end{aligned}$$

The Lagrangian $\mathcal{L}^{(0)}_{quark}(x)$ is invariant under

$$\begin{aligned} q &\to Uq \\ \bar{q} &\to \bar{q}\, U^\dagger. \end{aligned}$$

$$\mathcal{L}^{(0)}_{quark}(x) \to \sum_q \bar{q}\, U^\dagger\, (i\partial\!\!\!/ - m)\, Uq \tag{4.4}$$

With $U^\dagger U = \mathbb{1}_3$:

$$\sum_q \bar{q}\, U^\dagger\, (i\partial\!\!\!/ - m)\, Uq = \mathcal{L}^{(0)}_{quark}(x) \tag{4.5}$$

Gauge principle (as in QED):
We demand

$$\begin{aligned} q(x) &\to U(x)\, q(x) \\ \bar{q}(x) &\to \bar{q}(x)\, U^\dagger(x) \\ \Rightarrow \mathcal{L}_{quark}(x) &\to \mathcal{L}_{quark}(x) \end{aligned}$$

but

$$\begin{aligned} \mathcal{L}^{(0)}_{quark}(x) &\to \mathcal{L}^{(0)}_{quark}(x) + \sum_q \bar{q}\, (U^\dagger\, \partial\!\!\!/ U)\, q \\ \partial\!\!\!/ \delta^{\alpha\beta} &\to \gamma^\mu\, \mathcal{D}^{\alpha\beta}_\mu \end{aligned} \tag{4.6}$$

with

$$\mathcal{D}^{\alpha\beta}_\mu = \partial_\mu \delta^{\alpha\beta} \oplus i\, g_s\, A^{\alpha\beta}_\mu,$$

see page 45.

$SU(3)$:

$$[t^a, t^b] = i\, f^{abc}\, t^c$$

with

$$t^c = \frac{1}{2}\, \lambda^c$$

f^{abc} is anti-symmetric:

$$\begin{aligned} f^{123} &= 1 \\ f^{147} &= -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2} \\ f^{458} &= f^{678} = \frac{\sqrt{3}}{2} \end{aligned}$$

and

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \tfrac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Quark colour charge: $t^a\, t^a = C_F\, \mathbb{1}$ with $C_F = \frac{4}{3}$.
Gluon colour charge: $f^{abc}\, f^{abd} = C_A\, \delta^{cd}$ with $C_A = 3$.

$$\begin{aligned} tr_{adj.}\, t^c\, t^d &= -C_A\, \delta^{cd} \\ tr_{fud.}\, t^c\, t^d &= \frac{1}{2}\, \delta^{cd} \end{aligned}$$

Transformation of $A_\mu^{\alpha\beta} = A_\mu^a \underbrace{\left(\frac{\lambda^a}{2}\right)}_{t^a}^{\alpha\beta}$:

$$\begin{aligned} A_\mu(x) &\rightarrow U(x)\, A_\mu(x)\, U^\dagger(x) - \frac{i}{g_s}\, U(x)\, \partial_\mu U^\dagger(x) \\ &= -\frac{i}{g_s}\, U(x)\, \mathcal{D}_\mu\, U^\dagger(x) \\ \Rightarrow \mathcal{D}_\mu(x) &\rightarrow U(x)\, \mathcal{D}_\mu\, U^\dagger(x) \end{aligned} \tag{4.7}$$

As in QED we define the fieldstrength $F_{\mu\nu}$:

$$i\, g_s\, F_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu] = i\, g_s \left(\partial_\mu\, A_\nu - \partial_\nu\, A_\mu + i g_s\, [A_\mu, A_\nu]\right) \tag{4.8}$$

with

$$F_{\mu\nu} \rightarrow U\, F_{\mu\nu}\, U^\dagger$$

and

$$[A_\mu, A_\nu] = i\, f^{abc}\, A^b\, A^c\, \lambda^a$$

or

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f^{abc} A^b A^c.$$

Pure gauge theory: Yang-Mills

$$\begin{aligned} \mathcal{L}_{YM}(x) &= -\frac{1}{2} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} \\ &= -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \end{aligned} \tag{4.9}$$

Full Lagrangian:

$$\mathcal{L}(x) = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \sum_q \bar{q}\, (i\not{\mathcal{D}} - m_q)\, q \tag{4.10}$$

with gauge symmetry $U \in SU(3)$:

$$\begin{aligned} q &\rightarrow U\, q \\ \bar{q} &\rightarrow \bar{q}\, U^\dagger \\ A_\mu &\rightarrow U\, A_\mu\, U^\dagger - \frac{i}{g_s} U\, \partial_\mu\, U^\dagger \end{aligned}$$

$\Rightarrow \mathcal{L}(x) \rightarrow \mathcal{L}(x)$
with

$$\begin{aligned} -\frac{1}{2} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} &\rightarrow -\frac{1}{2} \mathrm{Tr}\, U\, F_{\mu\nu}\, U\, U^\dagger\, F^{\mu\nu}\, U^\dagger \\ &= -\frac{1}{2} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} \end{aligned}$$

Parameters: g_s or $\alpha_s = \frac{g_s^2}{4\pi}$ where α_s is the strong fine structure constant.
$m_{u,d,s,c,b,t}$ are the quark masses.

Gauge fixing (Lorentz)

$$\int \mathcal{L}(x) \rightarrow \int \mathcal{L}(x) + \frac{1}{2\zeta} \int \mathrm{Tr}\, (\partial_\mu A^\mu)^2 + \underbrace{\int \bar{c}\, \partial \mathcal{D}^\mu\, c}_{\text{ghosts}} \tag{4.11}$$

Feynman rules for QCD
Gauge fixing: $\frac{1}{2\zeta} \int (\partial_\mu A_\mu)^2$
Quark propagator: $i \frac{\not{p}+m}{p^2-m^2+i\varepsilon}\, \delta^{\alpha\beta}$
Gluon propagator: $-i \frac{1}{k^2+i\varepsilon} g_{\mu\nu} \left[-(1-\frac{1}{\zeta})\frac{k_\mu k_\nu}{k^2+i\varepsilon}\right]$
$\left[\text{Ghost propagator:} i\frac{1}{p^2+i\varepsilon}\right]$
Quark-gluon vertex: $-i\, g_s\, (t^a)_{\alpha\beta}\, \gamma^\mu$
3-gluon vertex: $-g_s\, f^{abc}\, [g^{\mu\nu}(p-q)^\rho - g^{\nu\rho}(q-r)^\mu + g^{\rho\mu}(r-p)^\nu]$
4-gluon vertex:

$$\begin{aligned} -ig_s^2 [&f^{ead} f^{ebc} (g^{\mu\nu} g^{\sigma\rho} - g^{\mu\rho} g^{\nu\sigma}) + \\ &+ f^{eac} f^{ebd} (g^{\mu\nu} g^{\sigma\rho} - g^{\mu\sigma} g^{\nu\rho}) + \\ &+ f^{eab} f^{ecd} (g^{\mu\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\nu\sigma})] \end{aligned}$$

4.2 Running coupling

As in QED we compute the running coupling.
Reminder QED page 50 (Euklidean)

$$p^2 \gg m^2_{Cpt.} \quad \alpha(p^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi} \ln\left(\frac{p^2}{\mu^2}\right)} \tag{4.12}$$

computed from
β-function:

$$\begin{aligned} \beta_{QED}(\alpha) &\simeq p^2\, \partial_{p^2}\, \alpha(p^2) = \frac{\alpha^2}{3\pi} + \mathrm{O}(\alpha^3) > 0 \\ \beta_{QED} &= -\beta_0\, \alpha^2 - \beta_1\, \alpha^3 + \dots \end{aligned}$$

By comparison of coefficients one gets:

$$\boxed{\beta_0 = -\frac{1}{3\pi}} \tag{4.13}$$

QCD:

$$\beta_{QCD} \simeq p^2 \partial_{p^2} \alpha_s(p^2) = -\frac{1}{12\pi}(33 - 2N_f)\, \alpha_s^2 + \mathrm{O}(\alpha_s^3) \tag{4.14}$$

$$\Rightarrow \boxed{\beta_0 = \frac{1}{12\pi}(33 - 2N_f)} \tag{4.15}$$

$$\begin{aligned} \Rightarrow \alpha_s(p^2) &= \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)\beta_0 \ln\left(\frac{p^2}{\mu^2}\right)} \\ &= \frac{1}{\beta_0 \ln\left(\frac{p^2}{\Lambda^2_{QCD}}\right)} \end{aligned} \tag{4.16}$$

with

$$\mu^2 = \Lambda^2_{QCD} \cdot \exp\left\{-\frac{\beta_0}{\alpha_s(\mu)}\right\}$$

asymptotic freedom: $\alpha_s(p^2) \sim \frac{1}{\beta_0 \ln\left(\frac{p^2}{\mu^2}\right)} \to 0$

$$\Lambda_{QCD} \simeq 217^{+25}_{-23}\,\mathrm{MeV}$$

4.3 Confinement

no coloured asymptotic states.
Example: $q\bar{q}$-pair

1. Definition of $q\bar{q}$-state:
   $\bar{q}(y)\,q(x)$ not gauge invariant
   but
$$\bar{q}(y)\underbrace{\mathcal{P}\exp\left\{+i\,g_s\int\limits_y^x \mathrm{d}z^\mu\,t^a\,A^a_\mu(z)\right\}}_{\prod\limits_{z=y}^{x}\left(\mathbb{1}-i\,g_s\,\mathrm{d}z^\mu\,t^a\,A^a_\mu(z)\right)}q(x)$$
   gauge trafo:
$$A_\mu(x)\to U(x)\,A_\mu(x)\,U^\dagger(x)-\frac{i}{g_s}\,U(x)\,\partial_\mu\,U^\dagger(x)\quad\text{see page 54.}$$
$$\Rightarrow\left(\mathbb{1}+i\,g_s\,\mathrm{d}z^\mu\,t^a\,A^a_\mu(z)\right)\quad=\quad U(z)\,\left(\mathbb{1}+i\,g_s\,\mathrm{d}z^\mu\,t^a\,A^a_\mu(z)\right)\cdot\underbrace{\left(U^\dagger(z)+\partial_\mu U^\dagger(z)\,\mathrm{d}z^\mu\right)}_{U^\dagger(z+\mathrm{d}z)}$$
$$\Rightarrow\mathcal{P}\exp\left\{+i\,g_s\int\limits_y^x\mathrm{d}z^\mu\,A_\mu(z)\right\}\quad\to\quad U(y)\,\mathcal{P}\exp\left\{+i\int\limits_y^x g_s\,\mathrm{d}z^\mu\,A_\mu(z)\right\}U^\dagger(x)$$

2.
$$\lim_{|x-y|\to\infty}\left\langle\bar{q}(y)\mathcal{P}\exp\left\{+i\,g_s\int\limits_y^x\mathrm{d}z^\mu\,A_\mu(z)\right\}q(x)\right\rangle\to 0$$
   in quenched QCD (no dynamical quarks)
   Three quarks:
   Confinement
$$V(r)=V_0+\kappa\cdot r-\frac{e}{r}+\frac{f}{r^2}$$
   Remarks:
   - strong coupling is not enough!!
   - mass gap in Yang-Mills pure glue [Millenium Prize (Jaffe, Witten)]
   - perturbation theory fails ⇒ non-perturbation methods
     - lattice: space-time grid ($\sim 126^4$ lattices)
     - operator product expansions/sum rules ...
     - renormalisation group methods solve theory via relations between correlation functions.

3. Area law of Wilson loop
$$\mathcal{W}(\mathcal{C}_{x,y})=\mathrm{Tr}\,\mathcal{P}\exp\left\{+i\,g_s\int\limits_{\mathcal{C}_{x,y}}\mathrm{d}z_\mu\,A^\mu(z)\right\}$$

QED:

$$\exp\left\{-i\,e\int \mathrm{d}^4x\, j_\mu(x)\, A^\mu(x)\right\}$$

with

$$j_\mu = \int\limits_{\mathcal{C}_{x,y}} \mathrm{d}z_\mu\, \delta(x-z)$$

wordline of an electron

$$\begin{aligned}\Rightarrow \langle \mathcal{W}(\mathcal{C}_{x,y})\rangle &\sim \exp\left\{-F_{q_x\,\bar{q}_y}\right\} \to 0 \\ &\sim e^{-\sigma A}\end{aligned}$$

4. dynamical quarks

4.4 Phase diagram of QCD

Order parameter:

- chiral condensate: $\langle \bar{q}q\rangle = \sigma$
- Polyakov loop $L \sim e^{-F_q}$

Remark on phase transitions:

Chapter 5

Electroweak Theory (Quantum Flavourdynamics, QFD)

In 1930 Pauli suggested the existence of the neutrino ν. It was discovered from 1953 to 1959 by Reines. In the years 1933 and 1934 Fermi worked out a theory of the β-decay.
β-decay:

$$n \to p + e^- + \bar{\nu}_e$$

Fermi interaction:

$$H = G \int \mathrm{d}^3x \, [p(x) \, \gamma^\mu \, n(x)][e(x) \, \gamma_\mu \, \nu(x)] + \text{h.c.} \tag{5.1}$$

with G is the Fermi constant: $G = 1.1 \cdot 10^{-5} \, \mathrm{GeV}^{-2}$.
Important: parity violation in β-decay !

$$H = \frac{G_\beta}{\sqrt{2}} \left[p(x) \, \gamma^\mu \left(1 - \frac{g_A}{g_V} \gamma_5 \right) n(x) \right] \left[e(x) \, \gamma_\mu \, (1 - \gamma_5) \, \nu(x) \right] + \text{h.c.}$$

with

$$\begin{aligned} G_\beta &= 1.147 \cdot 10^{-5} \, \mathrm{GeV}^{-2} \\ \frac{g_A}{g_V} &= 1.255 \end{aligned}$$

Weak interaction distinguishes between left- and right-handed particles.

Universality of weak interaction.
γ_5 and handedness.
Fermions revisited: compare to page 22.

$$U(p) = \sqrt{p^0 + m} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\chi_s \end{pmatrix}$$

with

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and standard representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}$$

For $m = 0$ we equate $|\vec{p}| = p_0 = p$. With $\hat{p} = \frac{\vec{p}}{p}$ we get for $U(p)$:

$$U(p) = \sqrt{p} \begin{pmatrix} \chi_s \\ \vec{\sigma}\cdot\hat{p}\,\chi_s \end{pmatrix}$$

Spin-orientation/Helicity: $\vec{\sigma}\cdot\hat{p}\,\chi_\pm = \pm\chi_\pm$.
Define:

$$U_\pm(p) = \sqrt{p} \begin{pmatrix} \chi_\pm \\ \pm\chi_\pm \end{pmatrix}$$

$m = 0$: with $\gamma_5\, U_\pm(p) = \pm U_\pm(p)$.
We define left- and right-handed spinors:

$$\begin{aligned} \psi_{L/R} &= \frac{1 \mp \gamma_5}{2}\,\psi \\ &= \psi_\pm \quad \text{for}\, m = 0 \end{aligned}$$

with

$$\gamma_5\,\psi_{L/R} = \mp\,\psi_{L/R} \quad \text{chirality}$$

5.1 Lagrange density of electroweak theory

Fermi interaction via gauge theory:
Consider

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$$

Gauge principle (for the time being we consider massless fermions):

Leptons: $\Psi_e = \begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix}$, $\Psi_\mu = \begin{pmatrix} \psi_{\nu_\mu} \\ \psi_\mu \end{pmatrix}$, $\Psi_\tau = \begin{pmatrix} \psi_{\nu_\tau} \\ \psi_\tau \end{pmatrix}$

Quarks: $\Psi_q = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$, $\Psi_q = \begin{pmatrix} \psi_c \\ \psi_s \end{pmatrix}$, $\Psi_q = \begin{pmatrix} \psi_t \\ \psi_b \end{pmatrix}$

Free Lagrangian for the electron:

$$\mathcal{L}_0(x) = \bar{\Psi}_{e_L}\, i\,\gamma^\mu\partial_\mu\,\Psi_{e_L} + \bar{\psi}_{e_R}\, i\,\gamma^\mu\partial_\mu\,\psi_{e_R}(x)$$

The second summand is necessary because of QED.
$\mathcal{L}_0$ is invariant under global SU(2) rotations of Ψ_L:

$$\Psi_L \to U\,\Psi_L$$

with

$$\begin{aligned} U &= e^{i\omega} \in \mathrm{SU}(2) \\ \omega &= \omega^a \, \frac{\sigma^a}{2} \end{aligned}$$

Singlet ψ_R: $\psi_R \to \psi_R$.
$\frac{\sigma^a}{2}$ are the generators of SU(2) with Lie-algebra

$$\left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}\right] = i\,\varepsilon^{abc}\,\frac{\sigma^c}{2}, \quad \varepsilon^{123} = 1$$

and Pauli-matrices σ^i, $i = 1, 2, 3$ (see 10).
Local symmetry (gauging) via minimal coupling

$$\mathcal{L}_0 \to \mathcal{L}(x) = \bar{\Psi}_{e_L}\, i\,\gamma^\mu \mathcal{D}_\mu\, \Psi_{e_L} + \psi_{e_R}\, i\,\gamma^\mu \partial_\mu\, \psi_{e_R}$$

with

$$\begin{aligned} \mathcal{D}_\mu &= \partial_\mu + i\,g\,\mathcal{W}_\mu \\ \mathcal{W}_\mu &= \mathcal{W}_\mu^a \, \frac{\sigma^a}{2} \end{aligned}$$

and gauge transformations

$$\begin{aligned} \mathcal{W}_\mu(x) &\to U(x)\,\mathcal{W}_\mu(x)\,U^\dagger(x) - \frac{i}{g}\,U(x)\,\partial_\mu\,U^\dagger(x) \\ &= -\frac{i}{g}\,U(x)\,\mathcal{D}_\mu\,U^\dagger(x) \\ \Psi(x) &\to \exp\left\{i\,\omega(x)\,\frac{1-\gamma_5}{2}\right\}\,\Psi(x) \\ &= \begin{pmatrix} U(x)\,\Psi_L \\ \psi_R \end{pmatrix} \end{aligned}$$

with $\Psi(x) = \begin{pmatrix} \Psi_L \\ \psi_R \end{pmatrix}$.
Within this notation $\mathcal{L}$ reads

$$\mathcal{L} = \bar{\Psi}\,i\,\gamma^\mu\,\mathcal{D}_\mu\,\Psi$$

with

$$\begin{aligned} \mathcal{D}_\mu &= \partial_\mu + i\,g\,\mathcal{W}_\mu \\ \mathcal{W}_\mu &= \mathcal{W}_\mu^a \cdot T^a \\ T^a &= \begin{pmatrix} \frac{\sigma^a}{2} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Coupling via

$$\mathcal{W}_\mu^a \, \sigma^a = \begin{pmatrix} \mathcal{W}_\mu^3 & \mathcal{W}_\mu^1 - i\,\mathcal{W}_\mu^2 \\ \mathcal{W}_\mu^1 + i\,\mathcal{W}_\mu^2 & -\mathcal{W}_\mu^3 \end{pmatrix}$$

where $\mathcal{W}_\mu^3$ is neutral and $\mathcal{W}_\mu^1$, $\mathcal{W}_\mu^2$ are charged.
With

$$\mathcal{W}^\pm = \frac{1}{\sqrt{2}} \left(\mathcal{W}^1 \pm i\,\mathcal{W}^2 \right)$$

we have the interaction, e.g.:

$$\begin{aligned} \frac{g}{\sqrt{2}} \, \bar{\Psi}_{e,L} \, \gamma^\mu \, \mathcal{W}_\mu^- \, \psi_{\nu_e,L} &= \frac{g}{\sqrt{2}} \, \bar{\psi}_e \, \gamma^\mu \, \mathcal{W}_\mu^- \, \underbrace{\frac{1-\gamma_5}{2} \, \psi_{\nu_e}}_{\psi_{\nu_e,L}} \\ &= \frac{g}{\sqrt{2}} \, \bar{\psi}_e \, \frac{1+\gamma_5}{2} \, \gamma^\mu \, \mathcal{W}_\mu^- \, \psi_{\nu_e} \\ &= \frac{g}{\sqrt{2}} \, \psi_e^\dagger \, \frac{1-\gamma_5}{2} \, \gamma^0 \, \gamma^\mu \, \mathcal{W}_\mu^- \, \psi_\nu \end{aligned} \tag{5.2}$$

Diagrammatically
neutral gauge boson $\mathcal{W}_\mu^3$:

- is not the photon: no L-R-symmetry
  $\Rightarrow$ additional U(1) gauge boson $\sim Z^0$ (triumph of theory)
- is not Z^0: no coupling to right-handed fermions

Existence of neutral currents,
e.g. $\bar{\nu}_\mu + e^- \to \bar{\nu}_\mu + e^-$
or $\nu_\mu + n \to \nu_\mu + n$.
Consider

$$\mathcal{W}_\mu^3 = \cos\theta_W \, Z_\mu^0 + \sin\theta_W \, A_\mu$$

where A_μ represents the photon and θ_W is the Weinberg angle (weak mixing angle).

$$\sin^2\theta_W = 0.23117(6), \quad \text{result at SLAC: } \sin^2\theta_W = 0.23098 \pm 0.00028$$

Orthogonal combination:

$$B_\mu = -\sin\theta_W \, Z_\mu^0 + \cos\theta_W \, A_\mu$$

with U(1) gauge transformation

$$\left. \begin{array}{lcl} \Psi_L & \to & e^{i\,Y_L\,\omega} \\ \psi_R & \to & e^{i\,Y_R\,\omega} \end{array} \right\} \begin{pmatrix} \Psi_L \\ \psi_R \end{pmatrix} \to e^{iY\omega} \begin{pmatrix} \Psi_L \\ \psi_R \end{pmatrix}$$

with Hypercharge

$$Y = \begin{pmatrix} Y_L & 0 & 0 \\ 0 & Y_L & 0 \\ 0 & 0 & Y_R \end{pmatrix}. \tag{5.3}$$

B_μ commutes with $\mathcal{W}_\mu$!

$$\boxed{\mathrm{SU}(2) \times \mathrm{U}(1)}$$

The hypercharge Y equals the difference between the electric charge Q in $|e|$ and the third component of the isospin I_3:

$$Y = Q - I_3.$$

For right-handed fermions: $Y = Q$.

For left-handed fermions, e.g.:

Particle	Y	Q	I_3
ν_L	$-\frac{1}{2}$	0	$-\frac{1}{2}$
e_L	$-\frac{1}{2}$	-1	$-\frac{1}{2}$
u_L	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{2}$
d_L	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$

Interaction term: $\Psi = \begin{pmatrix} \Psi_L \\ \psi_R \end{pmatrix}$

$$\bar{\Psi}\, i\, \gamma^\mu\, \mathcal{D}_\mu\, \Psi$$

with

$$\mathcal{D}_\mu = \partial_\mu + i\, g\, \mathcal{W}_\mu + i\, g'\, B_\mu\, Y$$

where $\mathcal{W}_\mu$ is explained on page 61 and for the Hypercharge Y see page 63.
Full Lagrangian:

$$\begin{aligned} \mathcal{W}^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g\, \varepsilon^{abc}\, W^b_\mu\, W^c_\nu \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \end{aligned}$$

$$\mathcal{L}_{EW} = -\frac{1}{4}\, \mathcal{W}^a_{\mu\nu}\, \mathcal{W}^{a\mu\nu} - \frac{1}{4}\, B_{\mu\nu}\, B^{\mu\nu} + \bar{\psi}\, i\, \gamma^\mu\, \mathcal{D}_\mu\, \psi \tag{5.4}$$

General gauge transformation:

$$\Psi \to e^{i\, g\, \omega^a\, I^a + i\, g\, \omega\, Y}\, \Psi \tag{5.5}$$

Neutral gauge bosons:

$$-\bar{\Psi}\, \gamma^\mu \left[g\, (\cos\theta_W\, Z^0_\mu + \sin\theta_W\, A_\mu)\, T^3 + g'\, (-\sin\theta_W\, Z^0_\mu + \cos\theta_W\, A_\mu)\, Y \right] \Psi$$

Coupling to Photon:

$$\begin{aligned} g \sin\theta_W T^3 + g' \cos\theta_W Y &= e\,Q \\ &= e\,T^3 + e\,Y \\ \Rightarrow g \sin\theta_W &= e \\ g' \cos\theta_W &= e \end{aligned}$$

Current: $-e\,A^0_\mu\,j^\mu_{em}$.

$$j^\mu_{em} = \bar{\Psi}\,\gamma^\mu\,\Psi = (\bar{\psi}_R\,\gamma^\mu\,\psi_R + \bar{\Psi}_L\,\gamma^\mu\,\Psi_L)$$

Coupling to Z^0:

$$\begin{aligned} g \cos\theta_W T^3 - g' \sin\theta_W Y &= \frac{e}{\sin\theta_W\,\cos\theta_W}\left(\cos^2\theta_W\,T^3 - \sin^2\theta_W\,Y\right) \\ &= \boxed{\frac{2e}{\sin 2\theta_W}}\left(\Gamma^3 - \sin^2\theta_W\,Q\right) \end{aligned} \tag{5.6}$$

Current: $-Z^0_\mu\,j^\mu_{nc}\cdot\frac{2e}{\sin 2\theta_W}$

$$\begin{aligned} j^\mu_{nc} &= \bar{\Psi}_L\,\gamma^\mu\,\underbrace{\left(T^3 - \sin^2\theta_W\,Q\right)}_{\sim g_L=\frac{1}{2}(g_V-g_R)}\,\Psi_L + \bar{\psi}_R\,\gamma^\mu\,\underbrace{\left(-\sin^2\theta_W\,Q\right)}_{g_R=\frac{1}{2}(g_V+g_R)}\,\psi_R \\ &= \frac{1}{2}\,\bar{\psi}\,\gamma^\mu\left(\Gamma^3_L\left(1-\gamma_5\right) - 2\,Q\,\sin^2\theta_W\right)\psi \end{aligned}$$

Electron: $= \frac{1}{2}\,\bar{\psi}_{\nu_{e_L}}\,\gamma^\mu\,\psi_{\nu_{e_L}} - \frac{1}{2}\bar{\psi}_{e_L}\,\gamma^\mu\,\psi_{e_L} - \sin^2\theta_W\,j^\mu_{em}$
Problems:

1. Masses for $W^\pm$, Z^0: explicit mass-terms break gauge invariance!
2. Masses for matter fields: couple left- to right-handed fields $\rightarrow$ break weak gauge invariance!
3. Neutrino mixing

Resolution to 1. and 2.: Higgs-mechanism: masses via spontaneous symmetry breaking.

5.2 The Higgs sector

1. $W^\pm$, Z^0 are massive, e.g. $m_Z = 91.1882(22)$GeV, $m_W^2 = m_Z^2\,\cos^2\theta_W$.
2. Matter-fields are massive:
   (a) $\sim m_W^2\,\mathrm{Tr}\,W^2$
   (b) $\sim -m_\psi\,\bar{\psi}\,\psi = m_\psi\,\bar{\psi}_R\,\psi_L + m_\psi\,\bar{\psi}_L\,\psi_R$

(a) and (b) are not gauge invariant:

$$\psi \to \exp\Big\{ i\,\frac{1-\gamma_5}{2}\,\omega \Big\}\,\psi \tag{5.7}$$

$$\begin{aligned} m_W^2\,\mathrm{Tr}\,\mathcal{W}^2 &\to -\frac{i}{g}\,m_W^2\,\mathrm{Tr}\,\mathcal{D}\,\mathcal{W} \\ m_\psi\,\bar\psi\,\psi &\to m_\psi\,\bar\psi\,\exp\Big\{-i\,\frac{1+\gamma_5}{2}\,\omega\Big\}\,\exp\Big\{i\,\frac{1-\gamma_5}{2}\,\omega\Big\}\,\psi \end{aligned}$$

Explanation to the last transformation:

$$\begin{aligned} \bar\psi \to \psi^\dagger\,\exp\Big\{-i\,\frac{1-\gamma_5}{2}\Big\}\,\gamma^0 &= \psi^\dagger\,\gamma^0\,\exp\Big\{-i\,\frac{1+\gamma_5}{2}\Big\} - \bar\psi\,\exp\Big\{-i\,\frac{1+\gamma_5}{2}\Big\} \\ &= m_\psi\,\bar\psi\,e^{-i\,\gamma_5\,\omega}\,\psi \\ &= m_\psi\,\big(\bar\psi_R\,e^{i\omega}\,\psi_L + \bar\psi_L\,e^{-i\omega}\,\psi_R\big) \end{aligned} \tag{5.8}$$

Infinitesimal:

$$\bar\psi\,\psi \to -i\,\bar\psi\,\gamma_5\psi$$

Assume we have scalar field ϕ with $\langle\phi\rangle = \frac{v}{\sqrt{2}} \neq 0$.
$\Rightarrow$ "Massterms":

$$\mathcal{L}_Y(x) = -h_\psi\,(\psi_R\,\phi^\dagger\,\Psi_L + \bar\Psi_L\,\phi\,\psi_R) \qquad \text{Yukawa term} \tag{5.9}$$

with doublet ϕ:

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}, \quad \phi(x) \to U(x)\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}, \quad U \in \mathrm{SU}(2)$$

$$\Rightarrow \phi^\dagger(x)\,\Psi_L(x) \to \phi^\dagger(x)\,U^\dagger(x)\,U(x)\,\Psi_L(x) = \phi^\dagger(x)\,\psi_L(x) \tag{5.10}$$

$\Rightarrow$ Yukawa term $\mathcal{L}_Y(x) \xrightarrow{U} \mathcal{L}_Y(x)$ gauge invariant under SU(2)!
Hypercharge:

$$\phi(x) \to e^{i\,Y_H\,\omega}\,\phi(x)$$

with

$$\boxed{Y_H = Y_L - Y_R = \frac{1}{2}.}$$

$$\Rightarrow \mathcal{L}_Y(x) \overset{e^{i\,Y\,\omega}}{\to} \mathcal{L}_Y(x)$$

$$\begin{aligned} \bar\psi_R\,\phi^\dagger\,\Psi_L &= \bar\psi_R\,e^{-i\,Y_R\,\omega}\,\phi^\dagger\,e^{-i\,(Y_L-Y_R)\omega}\,e^{i\,Y_L\,\omega}\,\Psi_L \\ &= \bar\psi_R\,\phi^\dagger\,\Psi_L \end{aligned} \tag{5.11}$$

Electric charge:

$$\begin{aligned}\phi_1(x) : Y_H + I_3 &= 1 \\ \phi_2(x) : Y_H + I_3 &= 0\end{aligned}$$

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

In summary $\mathcal{L}(x)$ is gauge invariant, ϕ couples to $\mathcal{W}_\mu$ and to B_μ!

Mass term for fermion: ( ) $\quad \phi_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$

$$-h_e\left(\bar{\psi}_{e_R}\,\phi_0^\dagger\,\Psi_L + h.c.\right) = -h_e\,\frac{v}{\sqrt{2}}\left(\bar{\psi}_{e_R}\,\psi_L + h.c.\right) \tag{5.12}$$

$$\boxed{m_e = h_e\,\frac{v}{\sqrt{2}}}$$

Kinetic term:

$$\partial_\mu\phi^\dagger\,\partial^\mu\phi \to \mathcal{D}_\mu\,\phi^\dagger\,\mathcal{D}^\mu\,\phi$$

with

$$\begin{aligned}\mathcal{D}_\mu &= \partial_\mu + i\,g\,\mathcal{W}_\mu + i\,g'\,B_\mu\,Y_H \\ W_\mu &= W_\mu^a\,\frac{\sigma^a}{2}\end{aligned}$$

Higgs Lagrangian: (for electron)

$$\mathcal{L}_H(x) = \mathcal{D}_\mu\,\phi^\dagger\,\mathcal{D}^\mu\,\phi - h_e\left(\psi_{e_R}\,\phi^\dagger\,\Psi_{e_L} + h.c.\right) - V\left(\phi^\dagger\,\phi\right) \tag{5.13}$$

V is gauge invariant, as

$$\begin{aligned}\phi^\dagger\,\phi &\overset{U}{\to} \phi^\dagger\,U^\dagger\,U\,\phi = \phi^\dagger\,\phi \\ \phi^\dagger\,\phi &\overset{e^{i\,Y_H\,\omega}}{\to} \phi^\dagger\,e^{-i\,Y_H\,\omega}\,e^{i\,Y_H\,\omega}\,\phi\end{aligned} \tag{5.14}$$

Mass of Z^0, $W^\pm$: Take $\phi_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$

$$\mathcal{D}^\mu\,\phi_0 = \frac{iv}{2\sqrt{2}}\begin{pmatrix} \sqrt{2}\,g\,W^{+\,\mu} \\ g'\,B^\mu - g\,W_3^\mu \end{pmatrix} \tag{5.15}$$

with

$$\begin{aligned}g'\,B^\mu - g\,W_3^\mu &= \frac{g}{\cos\theta_W}\left(\sin\theta_W\,B^\mu - \cos\theta_W\,W_3^\mu\right) \\ &= -\frac{g}{\cos\theta_W}\,Z^{0\,\mu}\end{aligned}$$

$$\Rightarrow \mathcal{D}_\mu \phi_0^\dagger \mathcal{D}^\mu \phi_0 = \frac{v^2 g^2}{8} \left(2W_\mu^- W^{+\mu} + \frac{1}{\cos^2\theta_W} Z_\mu Z^{0\mu} \right) \tag{5.16}$$

This provides mass terms for Z^0, $W^\pm$:

$$\boxed{m_Z = \frac{1}{2} \frac{v\,g}{\cos\theta_W}}$$

$$\boxed{m_W = \frac{1}{2} v\,g}$$

and

$$\sin^2\theta_W = 1 - \frac{m_W^2}{m_Z^2}$$

Full Lagrange density:

$$\begin{aligned} \mathcal{L}(x) &= \mathcal{L}_{EW}(x) + \mathcal{L}_H(x) \\ &\quad \text{see page 63 and page 65} \\ &= -\frac{1}{4} \mathcal{W}^a_{\mu\nu} \mathcal{W}^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \bar{\Psi}\, i\,\gamma^\mu \mathcal{D}_\mu \Psi - \\ &\quad - h_\psi \left(\bar{\psi}_R \phi^\dagger \Psi_L + \bar{\Psi}_L \phi \psi_R \right) + \mathcal{D}_\mu \phi^\dagger \mathcal{D}^\mu \phi - V(\phi^\dagger \phi) \end{aligned}$$

with

$$\begin{aligned} Z^0_\mu &= \cos\theta_W W^3_\mu - \sin\theta_W B_\mu \\ A_\mu &= \sin\theta_W W^3_\mu + \cos\theta_W B_\mu \end{aligned}$$

and currents $-\frac{2e}{\sin 2\theta_W} Z^0_\mu j^\mu_{nc}$, $-e\,A_\mu\, j^\mu_{em}$
with

$$\begin{aligned} j^\mu_{em} &= \bar{\psi}_R \gamma^\mu \psi_R + \bar{\Psi}_L \gamma^\mu \Psi_L \\ j^\mu_{nc} &= \frac{1}{2} \bar{\psi} \gamma^\mu \left(\Gamma^3_L (1-\gamma_5) - 2\,Q \sin^2\theta_W \right) \psi \end{aligned}$$

Parameters:

$$g, \sin\theta_W, \nu, h_\psi$$

Measurements:

1. Fine structure constant

$$\alpha = \frac{e^2}{4\pi} = \frac{g^2 \sin^2\theta_W}{4\pi} = 137.0359\ldots \quad \text{see QED section}$$

2. Fermi coupling constant

$$G_F = \frac{g^2 \sqrt{2}}{8 m_W^2} = \frac{1}{\sqrt{2}\, v^2} = 1.16639(1) \cdot 10^{-5}\,\text{GeV}^{-2}$$

3. The Z^0-boson mass

$$m_Z = \frac{g\,v}{2\cos\theta_W} = 91.1882(22)\,\mathrm{GeV}$$

4. Fermion masses

$$m_f = h_f\,\frac{v}{\sqrt{2}}$$

Mass hierarchy is not understood.

5.3 Spontaneous Symmetry Breaking

1. Simple example: O(2)-model, ϕ complex field

$$\mathcal{L} = \partial_\mu\phi^*\,\partial^\mu\phi - \mu^2\,\phi^*\,\phi - \lambda(\phi^*\,\phi)^2 \tag{5.17}$$

with invariance (global)

$$\begin{array}{rcl} \phi & \rightarrow & e^{i\omega}\,\phi, \qquad \partial_\mu\omega = 0 \\ \rightarrow \phi^* & \rightarrow & \phi^*\,e^{-i\omega} \end{array} \tag{5.18}$$

Hamiltonian density

$$\mathcal{H} = \partial_\mu\phi^*\,\partial_\mu\phi + V\,(\phi^\dagger\,\phi) \tag{5.19}$$

with

$$\begin{array}{rcl} \partial_\mu\phi^*\,\partial_\mu\phi & = & \partial_t\phi^*\,\partial_t\phi + \vec{\nabla}\phi^*\,\vec{\nabla}\phi \\ V(\phi\,\phi^*) & = & \mu^2\,\phi^*\,\phi + \lambda\,(\phi^*\,\phi)^2 \end{array}$$

Minimum: $\phi_0 = 0$, Minimum: $\phi_0 = \sqrt{\frac{-\mu^2}{2\lambda}\,e^{i\theta}}$
Mass: $m^2 = \frac{\partial^2 V}{\partial\phi\,\partial\phi^*}|_{\phi_0} = \mu^2$
Masses: $\theta = 0, \phi = \phi_1 + i\phi_2$, $\frac{1}{2}\,\frac{\partial^2 V}{\partial\phi_1^2}|_{\phi_0} = -2\mu^2$, $\frac{1}{2}\,\frac{\partial^2 V}{\partial\phi_2^2}|_{\phi_0} = 0$.
$\Rightarrow$ 1 massive boson (radial mode)
1 massless boson (Goldstone boson)
Spontaneous Symmetry breaking: theory rests in given minimum.

Remark:
In QM the ground state is *symmetric*! In QFT for $d > 2$ spontaneous symmetry breaking (SSB) is possible, for $d <= 2$ no SSB for a cont. symmetry can occur (Mermin-Wagner(-Coleman)), but discrete SSB. For $d < 2$ no SSB can occur: QM: $d = 0$.

Lagrangian:

$$\phi(x) = \frac{1}{\sqrt{2}}\left(v + \underbrace{\sigma(x)}_{radialmode} + \underbrace{i\,\pi(x)}_{Goldstone}\right) \tag{5.20}$$

with

$$v = \sqrt{\frac{-\mu^2}{\lambda}}.$$

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left[\partial_\mu\sigma\,\partial^\mu\sigma + \partial_\mu\pi\,\partial^\mu\pi\right] - \frac{1}{2}\mu^2\left[(v+\sigma)^2+\pi^2\right] - \frac{1}{4}\lambda\left[(v+\sigma)^2+\pi^2\right]^2 \\ &= \frac{1}{2}\left[\partial_\mu\sigma\,\partial^\mu\sigma - m_\sigma^2\,\sigma^2\right] + \frac{1}{2}\,\partial_\mu\pi\,\partial^\mu\pi + \text{interaction-terms } (+\,\text{const}) \qquad (5.21)\end{aligned}$$

with

$$\boxed{m_\sigma^2 = -2\mu^2}$$

2. U(1) gauge theory: (Abelian Higgs model)

$$\begin{aligned}\mathcal{L}(x) &= -\frac{1}{4}\,B_{\mu\nu}\,B^{\mu\nu} + \mathcal{D}_\mu\,\phi^*\,\mathcal{D}^\mu\,\phi - \mu^2\,\phi^*\,\phi - \lambda(\phi^*\,\phi)^2, \qquad \boxed{\mu^2 < 0} \\ B^{\mu\nu} &= \partial^\mu B^\nu - \partial^\nu B^\mu \\ \mathcal{D}^\mu &= \partial^\mu + i\,g'\,Y\,B^\mu\end{aligned}$$

Again we have $\phi_0 = \frac{v}{\sqrt{2}} \in \mathbb{R}$ as minimum:
Mass-term for B_μ:

$$\mathcal{D}_\mu\,\phi_0\,\mathcal{D}^\mu\,\phi_0 = -(g'\,Y v)^2\,B_\mu B^\mu \qquad (5.22)$$

with mass

$$m_B^2 = (g'\,Y v)^2.$$

dofs:	ϕ_1,	ϕ_2,	$B_\mu^\pm$		$\rightarrow$	σ,	π,	$B_\mu^\pm$,	B_μ^L
	1	+1	+2	$[+\omega]$		1	+1	+2	+1

$\pm$ in $B_\mu^\pm$ in order to distinguish between the two helicity states and L in B_μ^L stands for longitudinal.
Perform gauge trafo on ϕ with $e^{i\omega}$, $\omega = -\arctan\frac{\pi}{v+\sigma}$.

$$\begin{aligned}\phi \rightarrow e^{i\omega}\,\phi &= (\cos\omega + i\sin\omega)\,\frac{1}{\sqrt{2}}\,(v+\sigma+i\pi) \\ &= \frac{1}{\sqrt{2}}\,\sqrt{(v+\sigma)^2+\pi^2} \\ &= \frac{1}{\sqrt{2}}\,(v+\sigma') \qquad (5.23)\end{aligned}$$

with

$$\sigma' = \sqrt{(v+\sigma)^2+\pi^2} - v = \sigma + \frac{\pi^2}{2v} + \ldots$$

Unitary gauge.
The gauge field has 'eaten up' the Goldstone Boson
$\Big($Higgs(-Kibble) dinner$\Big)$.

3. Electroweak theory:

$$SU(2)\times U(1) \stackrel{\text{SSB}}{\rightarrow} U_{em}(1)$$
$$\text{4 gen.} \rightarrow \text{1 gen.}$$

3 Goldstone bosons are eaten up.

$$\phi_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$
$$e^{i\omega(T^3+Y_H)}\,\phi_0 = \phi_0 \tag{5.24}$$

where ϕ_2 is neutral.
$\Rightarrow U_{em}(1)$ is unbroken Symmetry.
For the mass terms of $W^{\pm}$ and Z^0 see page 67.
We have used that

$$\phi = (e^{i\,\omega^a\,T^a + Y_H\,\omega})\phi_0$$

to gauge away the Goldstones.
We chose SU(2) gauge transformation such that

$$U\,\phi(x) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}\,(\rho+v) \end{pmatrix} \tag{5.25}$$

This is left invariant under the U(1)-transformations

$$e^{i\,\omega\,(T^3+Y_H)}$$

It follows that

(a)

$$\begin{aligned}\mathcal{D}_\mu\,\phi^\dagger\,\mathcal{D}^\mu\,\phi &= \frac{1}{2}\,\partial_\mu\rho\,\partial^\mu\rho + m_W^2\left(1+\frac{\rho}{v}\right)^2 W_\mu^+\,W^{-\mu} + \\ &+\frac{1}{2}\,m_Z^2\left(1+\frac{\rho}{v}\right)^2 Z_\mu^0\,Z^{0\mu}\end{aligned} \tag{5.26}$$

For m_W and m_Z see page 67.

(b)

$$\begin{aligned}V(\phi^\dagger\,\phi) &= V\Big(\frac{1}{2}\,(v+\rho)^2\Big) \\ &= \frac{1}{2}\,\mu^2(v+\rho)^2 + \frac{\lambda}{4}\left(v+\rho\right)^4 \\ &= \frac{1}{2}\,m_\rho^2\,\rho^2\left(1+\frac{\rho}{v}+\frac{1}{4}\left(\frac{\rho}{v}\right)^2\right)\end{aligned} \tag{5.27}$$

with the Higgs mass:

$$\boxed{m_\rho^2 = 2\,\lambda\,v^2} \tag{5.28}$$

where λ is a parameter.

(c) Leptons: electron

$$\begin{aligned}
\bar{\Psi}_e\, i\,\gamma^\mu\, \mathcal{D}_\mu\, \Psi_e &= \bar{\Psi}_e\, i\,\gamma^\mu\, \partial_\mu\, \Psi_e - e\, A^0_\mu\, j^\mu_{em} - \\
&\quad - \frac{2e}{\sin 2\theta_W}\, Z^0_\mu\, j^\mu_{nc} + \qquad \text{see page 64} \\
&\quad + \frac{1}{\sqrt{2}}\, \frac{e}{\sin\theta_W}\, \left(W^+_\mu\, j^\mu_{cc} + W^-_\mu\, j^{\mu +}_{cc}\right) \qquad (5.29)
\end{aligned}$$

with

$$\begin{aligned}
j^\mu_{cc} &= \bar{\Psi}_e\, \gamma^\mu\, (T^1 + i\, T^2) \Psi_e \\
&= \Psi_e\, \gamma^\mu\, (T^1 + i\, T^2)\, \frac{1-\gamma_5}{2}\, \Psi_e \qquad (5.30)
\end{aligned}$$

$$-h_\psi\, (\psi_R\, \phi^\dagger\, \Psi_L - \bar{\Psi}_L\, \phi\, \psi_R) = m_e\, \bar{\psi}_e\, \psi_e \left(1 + \frac{\rho}{v_0}\right) \qquad (5.31)$$

In general

$$\psi = \begin{pmatrix} \Psi_e \\ \Psi_\mu \\ \Psi_\tau \\ \Psi_q \end{pmatrix}$$

allows for mass matrix.
$\Rightarrow$ Mixing!
Kobayashi-Maskawa matrix (v_1, v_2, v_3, δ).

Feynman rules:
Propagators:

: $\frac{-i\, g_{\mu\nu}}{q^2 + i\,\varepsilon}$ Feynman gauge

: $\frac{-i\left(g_{\mu\nu} - \frac{q_\mu\, q_\nu}{m_W^2}\right)}{q^2 - m_W^2 + i\,\varepsilon}$

: $\frac{-i\left(g_{\mu\nu} - \frac{q_\mu\, q_\nu}{m_Z^2}\right)}{q^2 - m_Z^2 + i\,\varepsilon}$

: $\frac{i}{q^2 - m_\rho^2 + i\,\varepsilon}$

Selected vertices:

: $(ie)\left\{(k_1 - k_2)_{\mu_3}\, g_{\mu_1\mu_2} + (k_2 - k_3)_{\mu_1}\, g_{\mu_2\mu_3} + (k_3 - k_1)_{\mu_2}\, g_{\mu_3\mu_1}\right\}$

: $(ie)\,\frac{\cos\theta_W}{\sin\theta_W}\left\{(k_1 - k_2)_{\mu_3}\, g_{\mu_1\mu_2} + (k_2 - k_3)_{\mu_1}\, g_{\mu_2\mu_3} + (k_3 - k_1)_{\mu_2}\, g_{\mu_3\mu_1}\right\}$

: $(ie^2)\left\{g_{\mu_1\mu_3}\, g_{\mu_2\mu_4} + g_{\mu_1\mu_4}\, g_{\mu_2\mu_3} - 2 g_{\mu_1\mu_2}\, g_{\mu_3\mu_4}\right\}$

: $(ie^2)\,\frac{\cos\theta_W}{\sin\theta_W}\left\{g_{\mu_1\mu_3}\, g_{\mu_2\mu_4} + g_{\mu_1\mu_4}\, g_{\mu_2\mu_3} - 2 g_{\mu_1\mu_2}\, g_{\mu_3\mu_4}\right\}$

: $ie^2\,\frac{\cos^2\theta_W}{\sin^2\theta_W}\left\{g_{\mu_1\mu_3}\, g_{\mu_2\mu_4} + g_{\mu_1\mu_4}\, g_{\mu_2\mu_3} - 2 g_{\mu_1\mu_2}\, g_{\mu_3\mu_4}\right\}$

: $-ie^2\,\frac{1}{\sin^2\theta_W}\left\{g_{\mu_1\mu_3}\, g_{\mu_2\mu_4} + g_{\mu_1\mu_4}\, g_{\mu_2\mu_3} - 2 g_{\mu_1\mu_2}\, g_{\mu_3\mu_4}\right\}$

Gauge Higgs:

$: \; i\, g_{\mu\nu}\, \frac{2m_W^2}{v}$

$: \; i\, g_{\mu\nu}\, \frac{2m_W^2}{v}$

$: \; i\, g_{\mu\nu}\, \frac{2m_Z^2}{v}$

$: \; i\, g_{\mu\nu}\, \frac{2m_Z^2}{v^2}$

$: \; -3i\, \frac{m_\rho^2}{v}$

$: \; -3i\, \frac{m_\rho^2}{v^2}$

W^- im Anfangszustand	einlaufende W-Linie
$\epsilon(k)$	Skizze
W^- im Endzustand	auslaufende W-Linie
$\epsilon^*(k)$	Skizze
W^+ im Anfangszustand	auslaufende W-Linie
$\epsilon(k)$	Skizze
W^+ im Endzustand	einlaufende W-Linie
$\epsilon^*(k)$	Skizze
Z im Anfangs-(End-)zustand	outer Z-line
$\epsilon(k)\,\epsilon^*(k)$	Skizze
Higgs-particle in beginning-(end-)state	outer v-line
1	Skizze
virtual W-boson	inner W-line
$\frac{i\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{m_W^2}\right)}{k^2-m_W^2+i\,\epsilon}$	Skizze
virtual Z-boson	inner Z-line
$\frac{i\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{m_Z^2}\right)}{k^2-m_Z^2+i\,\epsilon}$	Skizze
virtual Higgs-particle	inner v-line
$\frac{i}{k^2-m_v^2+i\,\epsilon}$	Skizze

Fermion-Boson-Vertices:

$: \; -i\, e\, Q_f\, \gamma^\mu$

$: \; -i\, \frac{e}{\sin\theta_W\, \cos\theta_W} \left\{ T_3^f\, \gamma^\mu\, \frac{1-\gamma_5}{2} - \sin^2\theta_W\, Q_f\, \gamma^\mu \right\}$

$: \; -i\, \frac{e}{\sqrt{2}\, \sin\theta_W}\, \gamma^\mu\, \frac{1-\gamma_5}{2}$

$: \; -i\, \frac{e}{\sqrt{2}\, \sin\theta_W}\, \gamma^\mu\, \frac{1-\gamma_5}{2}$

$: \; -i\, \frac{e}{\sqrt{2}\, \sin\theta_W}\, V_{ij}\, \gamma^\mu\, \frac{1-\gamma_5}{2}$

$: \; -i\, \frac{e}{\sqrt{2}\, \sin\theta_W}\, V_{ij}^*\, \gamma^\mu\, \frac{1-\gamma_5}{2}$

$: \; -i\, \frac{m_f}{v}$

5.4 The mass matrix and the Cabibbo angles

So far we have treated diagonal Yukawa-terms. In general Isospin doublets need not to be mass eigenstates!
Quantum numbers (Flavour): $\psi' = V\,\psi$, where ψ' is the (weak) isospin eigenstate, ψ is the mass eigenstate and V is unitary.

Families			T	T_3	Y	Q
1	2	3				
$\psi_{\nu_{e_L}}$	$\psi_{\nu_{\mu_L}}$	$\psi_{\nu_{\tau_L}}$	1/2	1/2	−1/2	0
ψ_{e_L}	ψ_{μ_L}	ψ_{τ_L}	1/2	−1/2	−1/2	−1
ψ_{e_R}	ψ_{μ_R}	ψ_{τ_R}	0	0	−1	−1
u_L	c_L	t_L	1/2	1/2	1/6	2/3
d'_L	s'_L	b'_L	1/2	−1/2	1/6	−1/3
u_R	c_R	t_R	0	0	2/3	2/3
d'_R	s'_R	b'_R	0	0	−1/3	−1/3

Iso scalars: $\Psi_L = \begin{pmatrix} \psi_{1_L} \\ \psi_{2_L} \end{pmatrix}$, e.g. for leptons: $\psi_{1_L} = \psi_{\nu_{e_L}}$, $\psi_{2_L} = \psi_{e_L}$ or for quarks: $\psi_{1_L} = u_L$, $\psi_{2_L} = d'_L$.

1. $\phi^\dagger \, \Psi_L = \phi_1^\dagger \, \psi_{1_L} + \phi_2^\dagger \, \psi_{2_L}$
2. $\phi^T \, \varepsilon \, \psi_L = \phi_1 \, \psi_{2_L} - \phi_2 \, \psi_{1_L}$
   with $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

- Isospin transformations
  1., 2. are invariant under SU(2) Isospin rotations.
  U:

$$\begin{aligned} \phi &\rightarrow U \, \phi \\ \Psi_L &\rightarrow U \, \Psi_L \\ \psi_R &\rightarrow \psi_R \\ \varepsilon \, U^T \varepsilon &= U^\dagger \end{aligned}$$

as

$$\boxed{\varepsilon^T \, \vec{\sigma}^T \, \varepsilon = -\vec{\sigma}}.$$

For $\vec{\sigma}$ see page 10.
1.:

$$\phi^\dagger \, \Psi_L \rightarrow \phi^\dagger \, U^\dagger \, U \, \Psi_L = \phi^\dagger \, \Psi_L$$

2.:

$$\begin{aligned} \phi^T \, \varepsilon \, \Psi_L &\rightarrow \phi^T \, U^T \, \varepsilon \, U \, \Psi_L \\ &= \phi^T \, \underbrace{\varepsilon \, \varepsilon^T}_{\mathbb{1}_2} \, U \, \varepsilon \, \Psi_L \\ &= \phi^T \, \varepsilon \, \Psi_L \end{aligned}$$

with $\varepsilon^T \, U \, \varepsilon = U^\dagger$.

- Hypercharge $\boxed{Y_H = \frac{1}{2}}$:

$$\begin{array}{rcl} \Psi_L & \rightarrow & e^{i\,\omega\,Y_{\Psi_L}}\;\Psi_L \\ \psi_R & \rightarrow & e^{i\,\omega\,Y_{\psi_R}}\;\psi_R \\ \phi & \rightarrow & e^{i\,\omega\,Y_H} \end{array}$$

Leptons:

$$\begin{array}{rcl} \phi^\dagger\,\Psi_{L_{Leptons}} & \rightarrow & e^{-i\,\omega}\,\phi^\dagger\,\Psi_{L_{Leptons}} \\ \phi^T\,\varepsilon\,\Psi_{L_{Leptons}} & \rightarrow & \phi^T\,\varepsilon\,\Psi_{L_{Leptons}} \end{array}$$

$\Rightarrow$ Only $\boxed{\bar{\psi}_R\,\phi^\dagger\,\Psi_L}$ invariant under Hypercharge transformations.
Quarks:

$$\begin{array}{rcl} \phi^\dagger\,\Psi_{L_{quarks}} & \rightarrow & e^{-i\,\frac{1}{3}\,\omega}\,\phi^\dagger\,\Psi_{L_{quarks}} \\ \phi^T\,\varepsilon\,\Psi_{L_{quarks}} & \rightarrow & e^{i\,\frac{2}{3}\,\omega}\,\phi^T\,\varepsilon\,\Psi_{L_{quarks}} \end{array}$$

$\Rightarrow$
$\bar{u}_R\,\phi^T \begin{pmatrix} u_L \\ d'_L \end{pmatrix}$, $\bar{c}_R\,\phi^T\,\varepsilon \begin{pmatrix} u_L \\ d'_L \end{pmatrix}$, $\bar{t}_R\,\phi^T\,\varepsilon \begin{pmatrix} u_L \\ d'_L \end{pmatrix}$, $\ldots$, $\bar{d}_R\,\phi^\dagger \begin{pmatrix} u_L \\ d'_L \end{pmatrix}$, $\bar{s}_R\,\phi^\dagger \ldots$

In summary:

$$\begin{array}{rcl} \mathcal{L}_Y(x) & = & -\begin{pmatrix} \bar{\psi}_{e_R} \\ \bar{\psi}_{\mu_R} \\ \bar{\psi}_{\tau_R} \end{pmatrix} H_l \begin{pmatrix} \phi^\dagger\,\psi_{e_R} \\ \phi^\dagger\,\psi_{\mu_R} \\ \phi^\dagger\,\psi_{\tau_R} \end{pmatrix} + \\ & & + \begin{pmatrix} \bar{u}_R \\ \bar{c}_R \\ \bar{t}_R \end{pmatrix} H'_q \begin{pmatrix} \phi^T\,\varepsilon \begin{pmatrix} u_L \\ d'_L \end{pmatrix} \\ \phi^T\,\varepsilon \begin{pmatrix} c_L \\ s'_L \end{pmatrix} \\ \phi^T\,\varepsilon \begin{pmatrix} t_L \\ b'_L \end{pmatrix} \end{pmatrix} - \\ & & - \begin{pmatrix} \bar{d}'_R \\ \bar{s}'_R \\ \bar{b}'_R \end{pmatrix} H_q \begin{pmatrix} \phi^\dagger \begin{pmatrix} u_L \\ d'_L \end{pmatrix} \\ \phi^\dagger \begin{pmatrix} c_L \\ s'_L \end{pmatrix} \\ \phi^\dagger \begin{pmatrix} t_L \\ b'_L \end{pmatrix} \end{pmatrix} + \\ & & + \, h.c. \end{array}$$

Change of basis in fieldspace: $\boxed{u, u', v \in U(3)}$.

$$\begin{aligned}
\psi_{R_{l/q}} &\rightarrow U_{l/q}\, \psi_{R_{l/q}} \\
\psi_{L_{l/q}} &\rightarrow V_{l/q}\, \psi_{L_{l/q}} \\
\psi'_{R_q} &\rightarrow U'_q\, \psi'_{R_q}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow H_l &\rightarrow U_l^\dagger\, H_l\, V_l = \begin{pmatrix} h_e & 0 & 0 \\ 0 & h_\mu & 0 \\ 0 & 0 & h_\tau \end{pmatrix} \\
H'_q &\rightarrow U'^{\dagger}_q\, H'_q\, V_q = \begin{pmatrix} h_u & 0 & 0 \\ 0 & h_c & 0 \\ 0 & 0 & h_t \end{pmatrix} \\
H_q &\rightarrow U_q^\dagger\, H_q\, V_q = V \begin{pmatrix} h_d & 0 & 0 \\ 0 & h_s & 0 \\ 0 & 0 & h_b \end{pmatrix} V^\dagger
\end{aligned}$$

with

$$H_q = \tilde{U} \begin{pmatrix} h_d & 0 & 0 \\ 0 & h_s & 0 \\ 0 & 0 & h_b \end{pmatrix} \tilde{V},\; V = V_q^\dagger\, \tilde{V}^\dagger,\; U_q^\dagger = V\, \tilde{U}^\dagger$$

Furthermore the first transformation (H_l) is bi-unitary.

V: Cabibbo-Kobayashi-Maskawa-Matrix (CKM-Matrix)

- $V \in U(3)$ carries phase redundancy

  $$V^\dagger \rightarrow U_\varphi^\dagger\, V^\dagger\, U_\theta$$

  with

  $$U_\varphi = \begin{pmatrix} e^{i\varphi_1} & 0 & 0 \\ 0 & e^{i\varphi_2} & 0 \\ 0 & 0 & e^{i\varphi_3} \end{pmatrix}$$

  and

  $$U_\varphi \begin{pmatrix} h_d & 0 & 0 \\ 0 & h_s & 0 \\ 0 & 0 & h_b \end{pmatrix} U_\varphi^\dagger = \begin{pmatrix} h_d & 0 & 0 \\ 0 & h_s & 0 \\ 0 & 0 & h_b \end{pmatrix}, \quad \psi \rightarrow U_\theta\, \psi$$

  5 phases (global phase drops out)
  parameters: $9 - 5 = 4$.

  $$V = \begin{pmatrix} c_1 & s_1\, c_3 & s_1\, s_3 \\ -s_1\, c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ -s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{pmatrix}$$

Cabibbo angles θ_i: $c_i = \cos\theta_i, \quad i = 1,2,3$
$s_i = \sin\theta_i, \quad \theta_i \in \left[0, \frac{\pi}{2}\right]$
$\delta \in [0, 2\pi]$

$\boxed{\delta}$ CP-violation

- two families

$$V \in U(\alpha)$$

phase redundancy: 3 phases (global phase drops out)

$$\begin{aligned} U_\varphi^\dagger\, V\, U_\theta &= \begin{pmatrix} e^{-i(\varphi_1-\theta_1)}\, V_{11} & e^{-i(\varphi_1-\theta_2)}\, V_{12} \\ e^{-i(\varphi_2-\theta_1)}\, V_{21} & e^{-i(\varphi_2-\theta_2)}\, V_{22} \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \end{aligned}$$

e.g. Nachtmann page 314.
$\Rightarrow$ no CP-violation.

Total Yukawa Lagrangian: $\phi = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}\,(\rho + v) \end{pmatrix}$

$$\begin{aligned} \mathcal{L}_y(x) &= -\Bigg\{ \begin{pmatrix} \bar\psi_{e_R} \\ \bar\psi_{\mu_R} \\ \bar\psi_{\tau_R} \end{pmatrix} \begin{pmatrix} h_e & 0 & 0 \\ 0 & h_\mu & 0 \\ 0 & 0 & h_\tau \end{pmatrix} \begin{pmatrix} \psi_{e_L} \\ \psi_{\mu_L} \\ \psi_{\tau_L} \end{pmatrix} + \\ &\quad + \begin{pmatrix} \bar u_R \\ \bar c_R \\ \bar t_R \end{pmatrix} \begin{pmatrix} h_u & 0 & 0 \\ 0 & h_c & 0 \\ 0 & 0 & h_t \end{pmatrix} \begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix} + \\ &\quad + \begin{pmatrix} \bar d'_R \\ \bar s'_R \\ \bar b'_R \end{pmatrix} V \begin{pmatrix} h_d & 0 & 0 \\ 0 & h_s & 0 \\ 0 & 0 & h_b \end{pmatrix} V^\dagger \begin{pmatrix} \bar d'_L \\ \bar s'_L \\ \bar b'_L \end{pmatrix} + h.c. \Bigg\} \cdot \frac{v}{\sqrt{2}} \left(1 + \frac{\rho}{v}\right) \\ &= -\Bigg[m_e\, \bar\psi_e\, \psi_e + m_\mu\, \bar\psi_\mu\, \psi_\mu + m_\tau\, \bar\psi_\tau\, \psi_\tau + m_u\, \bar u\, u + m_c\, \bar c\, c + m_t\, \bar t\, t + \\ &\quad + \begin{pmatrix} \bar d' \\ \bar s' \\ \bar b' \end{pmatrix} V \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} V^\dagger \begin{pmatrix} \bar d' \\ \bar s' \\ \bar b' \end{pmatrix} \Bigg] \cdot \left(1 + \frac{\rho}{v}\right) \end{aligned} \tag{5.32}$$

with

$$\boxed{m = \frac{h\, v}{\sqrt{2}}}.$$

Charged quark current:

$$
\begin{aligned}
j^\mu_{cc} &= \bar\Psi_q\,\gamma^\mu\left(T^1 + i\,T^2\right)\Psi_q \\
&= \begin{pmatrix} \bar u_L \\ \bar c_L \\ \bar t_L \end{pmatrix} \gamma^\mu \begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix} \\
&= \begin{pmatrix} \bar u_L \\ \bar c_L \\ \bar t_L \end{pmatrix} V \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}
\end{aligned}
\tag{5.33}
$$

$$
\Rightarrow \quad
\begin{array}{ll}
d \to u + W^- & V_{11} \\
s \to u + W^- & V_{12} \\
b \to u + W^- & V_{13}
\end{array}
$$

e.g. $n \to p + W^-$

$$
V = \begin{pmatrix}
0.97383^{+0.00024}_{-0.00023} & 0.2272 \pm 0.001 & (3.69 \pm 0.09)\cdot 10 \\
0.2271^{+0.0010}_{-0.0010} & 0.97296 \pm 0.00024 & \left(42.21^{+0.10}_{-0.80}\right)\cdot 10 \\
\left(8.14^{+0.32}_{-0.64}\right)\cdot 10^{-3} & \left(41.61^{+0.12}_{-0.78}\right)\cdot 10^{-3} & 0.999100^{+0.0003}_{-0.000}
\end{pmatrix}
\tag{5.34}
$$

Unitary triangle: $\sum_i V_{ij}\,V^*_{ik} = \delta_{jk}$
e.g.

$$
\boxed{V_{ud}\,V^*_{ub} + V_{cd}\,V^*_{cb} + V_{td}\,V^*_{tb} = 0}
$$

Jarlskog invariant:

$$
Jm\left[V_{ij}\,V_{kl}\,V^*_{il}\,V^*_{kj}\right] = J\sum_{m,n}(\varepsilon_{ikm}\,\varepsilon_{jln})
$$

$\left|\frac{V_{ud}\,V^*_{ub}}{V_{cd}\,V^*_{cb}}\right|, \ \left|\frac{V_{td}\,V^*_{tb}}{V_{cd}\,V^*_{cb}}\right|$
Neutral current

$$
\begin{aligned}
j^\mu_{nc_{quark}} &= \bar\Psi\,\gamma^\mu\left(T^3 - Q\,\sin^2\theta_W\right)\Psi \\
&= \begin{pmatrix} \bar u \\ \bar c \\ \bar t \end{pmatrix} \gamma^\mu \left(\frac{1}{2}\,\frac{1-\gamma_5}{2} - \frac{2}{3}\,\sin^2\theta_W\right) \begin{pmatrix} u \\ c \\ t \end{pmatrix} + \\
&\quad + \begin{pmatrix} \bar d \\ \bar s \\ \bar b \end{pmatrix} \gamma^\mu \left(-\frac{1}{2}\,\frac{1-\gamma_5}{2} + \frac{1}{3}\,\sin^2\theta_W\right) \begin{pmatrix} d \\ s \\ b \end{pmatrix}
\end{aligned}
\tag{5.35}
$$

The factors $\pm\frac{1}{2}$ in front of $\frac{1-\gamma_5}{2}$ equal T^3 and the factors in front of $\sin^2\theta_W$ are the negative charges of the quarks.

$$
\boxed{\text{No flavour changing neutral currents.}}
$$

In addition to the parameters on page 67.

5. Cabbibo angles + phase
   3 1

5.5 CP-Violation in the Standard model

CP:

$$\psi' \to e^{i\varphi_\psi}\, \gamma_0\, S(C)\, \bar{\psi}^T(-\vec{x}, t) \quad \text{see page 26 to page 27}$$

$j_{cc\,Leptons}$

$$\bar{\Psi}\, \gamma^\mu\, T^+\, \Psi \to -\bar{\Psi}_{Leptons}\, \gamma^\mu\, T^-\, \Psi_{Leptons}\, e^{i\chi}$$

with

$$\chi = \varphi_{Leptons} - \varphi_{\nu_{Leptons}}$$

Please notice the charge conjugation of T.
$j^\mu_{cc\,quarks}$:

$$\bar{\Psi}_q\, \gamma^\mu\, T^+\, \Psi_q \to -\bar{\Psi}_q\, \gamma^\mu\, T^-\, \Psi_q\, e^{i\chi}$$

If

$$\boxed{\begin{pmatrix} e^{i\varphi_d} & 0 & 0 \\ 0 & e^{i\varphi_s} & 0 \\ 0 & 0 & e^{i\varphi_b} \end{pmatrix} V^T \begin{pmatrix} e^{-i\varphi_u} & 0 & 0 \\ 0 & e^{-i\varphi_c} & 0 \\ 0 & 0 & e^{-i\varphi_t} \end{pmatrix} = e^{i\chi} V^\dagger}$$

$$\Rightarrow \boxed{V = V^*} \quad \text{or} \quad \boxed{\delta = 0, \pi}$$

Remarks:

1. strong CP-problem: θ-angle in QCD, U(1)-problem

$$\mathcal{L}_\theta = \frac{\theta\, g^2}{32\,\pi^2}\, \mathrm{Tr}\, \underbrace{\varepsilon^{\mu\nu\rho\sigma}\, F_{\rho\sigma}}_{\tilde{F}^{\mu\nu}}\, F_{\mu\nu}$$

(Euclidean $\frac{g^2}{32\,\pi^2} \int \mathrm{Tr}\, F_{\mu\nu}\, \tilde{F}_{\mu\nu} = n \in \mathbb{Z}$)
$|\theta| < 10^{-9}$

$$\mathcal{L}_{\text{'t Hooft}} \sim \left[\det_{s,t} \bar{\psi}_s\, \frac{1 - \gamma_5}{2}\, \psi_t\right] \boxed{e^{i\theta n}} \quad \text{U(1)-phase}$$

2. Neutrino masses:

Neutrino oszillations

$\Rightarrow$ Missing neutrinos from the sun.

Chapter 6

Beyond the Standard Model (SM)

Despite its great successes, the SM has its problems: (? mostly aesthetic ?)

1. Unification(s) (GUT)
   running couplings, fine tuning

weak scale:	10^2 GeV	$\sim m_{W/Z}$
GUT scale:	10^{15} GeV	
Planck scale:	10^{19} GeV	$E_{pl} : c^2\sqrt{\frac{\hbar c}{G}} \sim 2.4 \cdot 10^{16}$ GeV

2. Hierarchy problem, radiative corrections (time tuning)

   - Mass of the Higgs-particle $\gtrsim 115$ GeV is 'bad' for the SM.
     According to the SM it should be smaller than about 1 TeV.
     SM: $\delta_{m_H^2} = O\left(\frac{\alpha}{\pi}\right) \cdot \Lambda^2$
     from
     Integrals: $\sim \alpha \int_{k^2 \leq \Lambda^2} \mathrm{d}^4k \, \frac{1}{k^2} \sim \alpha\, \Lambda^2$
     (for fermions $\delta_{m_f} \sim O\left(\frac{\alpha}{\pi}\right) m + \ln\left(\frac{\Lambda^2}{m_f}\right)$)

aesthetics 2:	in RG theory no problem bare mass/ par. encode cancellations
If Λ is natural cut-off:	
	$\Lambda = 10^3\,\text{TeV},\ 10^{15}\,\text{GeV},\ 10^{19}\,\text{GeV}$ For $\Lambda = 10^3$ TeV the SM 'naively' breaks down ('heavy' tops). Higgs part gets negative (Unitarity). ϕ^4-corrections

   - Where do the scales come from? (aesthetics 3)

3. Quantisation of Gravity

   - quantum gravity perturbatively non-renormalisable

- unification of gravity and quantum physics
- (quantum) cosmology,
  early universe
  - Inflation
  - Baryon asymmetry
  - cosmological constant
  - dark energy

Possible resolutions:

1. (amongst) candidates: Susy theories
   strength: 'naturality', effective low en.
   theories of String theory
   other possibilities: fine-tuning

2. (amongst) candidates: Susy theories
   strength: 'naturality', connection to string theory, extra-dimensions
   other possibilities: fine-tuning
   RG-theory: UV-fixed point
   (extra-dimensions)

3. (amongst) candidates: Sugra/String theory
   (UV-cut-off string scale)
   other possibilities: RG-theory: UV-fixed point
   non-perturbatively ren. (also lattice)
   loop quantum gravity

6.1 A hint of Supersymmetry

Coleman-Mandula theorem:
'Internal symmetries B (Lie group/algebra) commute with Poincaré'
$[P^2, B] = 0$ O'Raifeartaigh: Int. Sym. cannot relate diff. mass-shells.
$[\mathcal{W}^2, B] = 0$ $\mathcal{W} \rightarrow$ Pauli-Ljubanski

Way out: (Haag-Lopuszanski-Sohnius)
Lie algebra $\rightarrow$ super Lie algebra (Z_2 graduated)
$[B_i, B_j] \rightarrow \{Q_i, Q_j\}$: Q fermionic

		chiral dofs	Multiple V
$\Rightarrow$	Q boson = fermion	2	ϕ
			$\downarrow Q$
	Q fermion = boson	2	ψ (Majorana)
			$\downarrow Q$
	$Q^2 = 0$	2	F

Remark:
O'Raifeartaigh still intact, but Q does not commute with spin!

Properties/Notation:

- Susy theories have as many fermions as bosons
  - fermionic partners: sfermions (sleptons, squarks), e.g. stop.
  - bosonic partners: bosinos: wino, zino, photino, gluino, ...
- radiative corrections: (Higgs)

Appendix A

Auxiliary calculation to Fermi's trick

Compare to page 6.

$$\begin{aligned}
\int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{d}t\ e^{iEt} &= \int_{0}^{\frac{T}{2}} \mathrm{d}t\ e^{i(E+i\varepsilon)t} + \int_{-\frac{T}{2}}^{0} \mathrm{d}t\ e^{i(E-i\varepsilon)t} \\
\lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{d}t\ e^{iEt} &= -\frac{1}{i(E+i\varepsilon)} + \frac{1}{i(E-i\varepsilon)} \\
&= i\ \frac{1}{E^2+\varepsilon^2} \cdot (-2i\varepsilon) \\
&= \frac{2\varepsilon}{E^2+\varepsilon^2}
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{R}} \mathrm{dE}\ f(E) \frac{2\varepsilon}{E^2+\varepsilon^2} &= \int_{\mathbb{R}} \mathrm{dE}' f(E' \cdot \varepsilon) \frac{2}{E'^2+1} \text{ with } E' = \frac{E}{\varepsilon}\ . \\
\to 2f(0) \cdot \int_{\mathbb{R}} \mathrm{dE}' \frac{1}{1+E'^2} &= f(0) \frac{1}{i} \int \mathrm{dE}' \left(\frac{1}{1+iE'} + \frac{1}{1-iE'} \right) \\
&= 2\pi f(0)\ \forall\ f
\end{aligned}$$

$$\Rightarrow \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{d}t\ e^{iEt} = 2\pi\delta(E)$$

Appendix B

Supplement

B.1 Landé-factor

The Landé-factor is also called gyromagnetic ratio.
Dirac-equation:

$$(\not{D} + im)\,\psi = 0 \tag{B.1}$$

with

$$\begin{aligned} \not{D} &= \not{\partial} + ie\not{A} \\ &= -\gamma_\mu \mathcal{D}_\mu \end{aligned}$$

with

$$\mathcal{D}_\mu = \partial_\mu + ieA.$$

From equation (B.1) follows:

$$(\not{D} - im)(\not{D} + im)\,\psi = (\not{D}^2 + m^2)\,\psi = 0 \tag{B.2}$$

with

$$\begin{aligned} \not{\mathcal{D}}^2 &= \gamma_\mu\,\gamma_\nu\,(\partial_\mu + ieA_\mu)\,(\partial_\nu + ieA_\nu) \\ &= \left[\frac{1}{2}\{\gamma_\mu, \gamma_\nu\} + 2i\,\underbrace{\frac{1}{4i}[\gamma_\mu, \gamma_\nu]}_{\sigma_{\mu\nu}}\right](\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu) \\ &= -\mathcal{D}^2 - e\,\sigma_{\mu\nu}\,F_{\mu\nu} \end{aligned} \tag{B.3}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Reminder:

$$\begin{aligned}
\sigma &= (i\,\mathbb{1},\vec{\sigma}), \quad \bar{\sigma} = (i\,\mathbb{1},-\vec{\sigma}) \\
\gamma_\mu &= \begin{pmatrix} 0 & \bar{\sigma} \\ \sigma & 0 \end{pmatrix} \\
\Rightarrow \sigma_{\mu\nu} &= \frac{1}{4i}\begin{pmatrix} \bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu & 0 \\ 0 & \sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu \end{pmatrix}
\end{aligned} \tag{B.4}$$

$$[\sigma_i,\sigma_j] = 2i\,\varepsilon_{ijk}\,\sigma_k$$

$$\begin{aligned}
\Rightarrow e\,\sigma_{\mu\nu}\,F_{\mu\nu} &= 2e\,\sigma_{oi}\,F_{oi} + \sigma_{ij}\,F_{ij} \\
&= 2\,\frac{e}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}\vec{E} - 2\,\frac{e}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\vec{B}
\end{aligned}$$

with

$$E_i = F_{oi}, \quad B_i = \frac{1}{2}\,\varepsilon_{ijk}\,F_{jk}.$$

Magnetic moment

$$\vec{\mu} = g\,\frac{e}{2mc}\,\vec{L} \tag{B.5}$$

and

$$\mathrm{H}_{magn.} = -\vec{\mu}\cdot\vec{B}. \tag{B.6}$$

$$\begin{aligned}
\Rightarrow \vec{\mu} &= 2\,\frac{e}{2}\,\frac{1}{mc}\vec{S} \quad \text{with } \vec{S} = \hbar\,\frac{\vec{\sigma}}{2} \\
\Rightarrow g &= 2 \\
|\vec{\mu}| &= \frac{e\hbar}{2mc} = 5.79\cdot 10^{-9}\mathrm{eV/G}
\end{aligned} \tag{B.7}$$

where G stands for Gauß and $|\vec{S}| = \frac{1}{2}$.

Pauli equation:

$$i\,\frac{\partial\phi}{\partial t} = \left[\frac{(\vec{\sigma}\cdot\vec{\pi})^2}{2m} + eA^0\right]\phi \tag{B.8}$$

with

$$\vec{\pi} = \vec{p} - e\vec{A}.$$

Quantum corrections?

$$\begin{aligned}
(\not{D} + im)\,\psi &\to \left(\not{D} + im - i\,\frac{\Delta g}{2}\,\frac{e}{2m}\,\sigma_{\mu\nu}\,F_{\mu\nu}\right)\psi \\
\mathcal{L} = \bar{\psi}\,(-i\not{D} + m)\,\psi &\to \bar{\psi}\left(-i\,\not{D} + m - \frac{\Delta g}{2}\,\frac{e}{4m}\,\sigma_{\mu\nu}\,F_{\mu\nu}\right)\psi
\end{aligned}$$

$$\Rightarrow \bar{\psi}\,e\,A^\rho_{ce}\,\gamma_\rho\,\psi$$

$$\rightarrow \bar{\psi}\, e\, A^{\rho}_{ce}\, (\gamma_\rho + \Gamma_\rho + \bar{\omega}_{\rho\nu}\, G^{\nu\rho}\, \gamma_\sigma)\, \psi$$

$$\bar{\psi}\, e\, A^{\rho}_{ce}\, (\gamma_\rho + \Gamma_\rho + \Sigma_{\rho\nu}\, G_0^{\nu\sigma}\, \gamma_\sigma)\, \psi, \quad G = G_0 + G_0\, \Sigma\, G_0 + \mathrm{O}(g^3)$$

with ψ, $\bar{\psi}$ on-shell
absorbed in the definition of mass and normalisation.

$\langle \bar{\psi}\, A\, \psi \rangle$:

1. Calculate vacuum pole
2. Calculate vertex correction
3. Project on $\sigma_{\mu\nu}\, k_\nu$ terms (on-shell)

No contributions of ... to $\sigma_{\mu\nu}\, k_\nu$
but $\Gamma_\rho = \sigma_{\mu\nu}\, k_\nu$.

$\Gamma_\rho(p, p')$:

$$= i\, (e\, \mu^{2-\omega})^3 \int \frac{\mathrm{d}^{2\omega} q}{(2\pi)^{2\omega}}\, \gamma_\mu\, \frac{1}{\not{p} - \not{q} + m}\, \gamma_\rho\, \frac{1}{\not{p}' - \not{q} + m} \cdot \gamma_\mu\, \frac{1}{q^2}$$

with

$$e \rightarrow e\, \mu^{2-\omega}.$$

Dimensional regularisation:
with

$$i\, e\, \mu^{2-\omega}\, \gamma_\mu\, (2\pi)^4\, \delta^{(4)}(k + q - p)$$

$\frac{1}{\not{p}+m}$

$$\frac{1}{p^2}\, \delta_{\mu\nu} - \frac{p_\mu\, p_\nu}{(p^2)^2}\, (1 - \zeta) \overset{\zeta=1}{\rightarrow} \frac{1}{p^2}\, \delta_{\mu\nu}$$

$\omega = 2 - \varepsilon$ with $\varepsilon \rightarrow 0$. Blatt 9
With $\Gamma[1 + 1 + 1] = 2$ and Blatt 9(6)

$$= 2i\, (e\, \mu^{2-\omega})^3 \int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta \int \frac{\mathrm{d}^{2\omega} q}{(2\pi)^{2\omega}} \cdot \frac{\gamma_\mu\, (\not{p} + \not{q} - m)\, \gamma_\rho\, (\not{p}' + \not{q} - m)\, \gamma_\mu}{\left(q^2 + m^2(\alpha + \beta) + 2q(p\alpha + p'\beta) + p^2\alpha + p'^2\beta\right)^3}$$

Quadratic addition: $\bar{q} = q + p\alpha + p'\beta$

$$= 2i\, (e\, \mu^{2-\omega})^3 \int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta \int \frac{\mathrm{d}^{2\omega} \bar{q}}{(2\pi)^{2\omega}} \cdot \frac{\gamma_\mu\, \left(\bar{\not{q}} - \not{p}'\beta + \not{p}(1 - \alpha) - m\right) \gamma_\rho\, \left(\bar{\not{q}} - \not{p}\alpha + \not{p}'(1 - \beta)\right) \gamma_\mu}{\left(\bar{q}^2 + m^2(\alpha + \beta) + p^2\alpha(1 - \alpha) + p'^2\beta(1 - \beta) - 2pp'\alpha\beta\right)^3}$$

$\Rightarrow$ only even terms in $\bar{q}$ of the counter can contribute $(\bar{q}^2, \bar{q}^0)$
$\bar{\not{q}}^2$:

$$\begin{aligned} \gamma_\mu\, \bar{\not{q}}\, \gamma_\rho\, \bar{\not{q}}\, \gamma_\mu &= \gamma_\mu\, \gamma_\alpha\, \gamma_\rho\, \gamma_\beta\, \gamma_\mu\, \bar{q}_\alpha\, \bar{q}_\beta \\ &\simeq \gamma_\mu\, \gamma_\nu\, q_\rho\, \gamma_\nu\, \gamma_\mu \\ &= (2\omega - 2)^2\, \gamma_\rho \sim \gamma_\rho \end{aligned} \tag{B.9}$$

with

$$\{\gamma_\mu, \gamma_\alpha\} = -2\,\delta_{\mu\alpha}$$

$$\gamma_\mu\,\gamma_\alpha\,\gamma_\rho\,\gamma_\beta\,\gamma_\mu = 2\,\gamma_\beta\,\gamma_\rho\,\gamma_\alpha - 2(2-\omega)\,\gamma_\alpha\,\gamma_\rho\,\gamma_\beta \tag{B.10}$$

with dimension of γ's: $\mathrm{Tr}\,\mathbb{1} = 2^\omega$

$$\begin{aligned} \mathrm{Tr}\,\gamma_\mu\,\gamma_\nu &= -2^\omega\,\delta_{\mu\nu} \\ \mathrm{Tr}\,\gamma_\mu\,\gamma_\rho\,\gamma_\nu\,\gamma_\sigma &= 2^\omega\,(\delta_{\mu\rho}\,\delta_{\nu\sigma} + \delta_{\mu\sigma}\,\delta_{\rho\nu} - \delta_{\mu\nu}\,\delta_{\rho\sigma}) \end{aligned}$$

From

$$\{\gamma_\mu, \gamma_\nu\} = -2\,\delta_{\mu\nu}$$

follows

$$\gamma_\mu\,\gamma_\mu = -2\omega\,\mathbb{1}$$

and

$$\begin{aligned} \gamma_\mu\,\gamma_\rho\,\gamma_\mu &= [2 - 2(2-\omega)]\,\gamma_\rho \\ \gamma_\mu\,\gamma_\alpha\,\gamma_\rho\,\gamma_\beta\,\gamma_\mu &= 2\,\gamma_\beta\,\gamma_\rho\,\gamma_\alpha - 2(2-\omega)\,\gamma_\alpha\,\gamma_\rho\,\gamma_\beta \end{aligned}$$

Remain

$$\gamma_\mu\left(\not{p}(1-\alpha) - \not{p}'\beta - m\right)\gamma_\rho\left(\not{p}'(1-\beta) - \not{p}\alpha - m\right)$$

For the calculation we use

1. Electrons on mass shell

$$\begin{aligned} (\not{p}' + m)\,|\psi(p')\rangle &\simeq 0, \quad (\not{p} - m)(\not{p} + m) = -p^2 - m^2 \\ \langle\psi(p)|\,(\not{p} + m) &\simeq 0 \end{aligned}$$

2. Gordon identities

$$\begin{aligned} \gamma_\rho\,\not{p}' &= -\gamma_\rho\,\gamma_\mu\,p_\mu \\ &= p'_\rho - 2i\,\sigma_{\rho\mu}\,p'_\mu \\ &= -m\gamma_\rho + \gamma_\rho\,(\not{p}' + m) \\ \not{p}\,\gamma_\rho &= p_\rho + 2i\,\sigma_{\rho\mu}\,p_\mu \end{aligned}$$

with

$$\sigma_{\rho\mu} = \frac{1}{40}[\gamma_\rho, \gamma_\mu]$$

and

$$\gamma_\rho\,\gamma_\mu = \frac{1}{2}\{\gamma_\rho, \gamma_\mu\} + \frac{1}{2}[\gamma_\rho, \gamma_\mu] = -\delta_{\rho\mu} + 2i\,\sigma_{\rho\mu}.$$

$$\boxed{\not{p}' = -\gamma_\mu p'_\mu}$$

Furthermore

$$\begin{aligned} \not{p}'\,\gamma_\rho &= -m\,\gamma_\rho + (\not{p}' + m)\,\gamma_\rho \\ &= -m\,\gamma_\rho + \gamma_\rho\,(\not{p}' + m) + [\not{p}', \gamma_\rho] \\ &= -m\,\gamma_\rho + \gamma_\rho\,(\not{p} + m) + 4i\,\sigma_{\rho\mu}\,p'_\mu \end{aligned}$$

In the same way:

$$\begin{aligned} \gamma_\rho\,\not{p} &= -m\,\gamma_\rho + (\not{p} + m)\,\gamma_\rho - 4i\,\sigma_{\rho\mu}\,p_\mu \\ \Rightarrow \not{p}'\,\gamma_\rho\,\not{p} &= m^2\,\gamma_\rho - k^2\,\gamma_\rho - 4im\,\sigma_{\rho\mu}\,k_\mu \end{aligned}$$

with

$$k = p - p'.$$

$$\begin{aligned} &\Rightarrow \gamma_\mu\,\big(\not{p}(1-\alpha) - \not{p}'\beta - m\big)\,\gamma_\rho\,\big(\not{p}'(1-\beta) - \not{p}\alpha - m\big)\gamma_\mu \simeq \\ \simeq\;& \gamma_\rho\Big\{2m^2\big[(\alpha+\beta)^2 - 2(1-\alpha-\beta)\big] - 2k^2(1-\alpha)(1-\beta)\Big\} + \\ &+ 8i\,\sigma_{\rho\mu}\Big\{p'_\mu\,\big(\alpha - \beta(\alpha+\beta)\big) - p_\mu\big(\beta - \alpha(\alpha+\beta)\big)\Big\}m \end{aligned}$$

$$\int \mathrm{d}\alpha \int \mathrm{d}\beta[\,] \sim (p'_\mu - p_\mu)$$

σ-part (finite):

$$\begin{aligned} &-16e^3 \int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta \int \frac{\mathrm{d}^4\bar{q}}{(2\pi)^4}\,\frac{1}{\big(\bar{q}^2 + m^2(\alpha+\beta) + p^2(\alpha(1-\alpha) + p'^2(p(1-\beta) - 2\cdot p'\alpha\beta)))\big)}\,\cdot \\ &\cdot\sigma_{\rho\mu}\Big[p'_\mu\big(\alpha - \beta(\alpha+\beta)\big) - p_\mu\,\big(\beta - \alpha(\alpha+\beta)\big)\Big] = \\ =\;& -\frac{16e^3}{2(4\pi)^2}\cdot\frac{1}{m^2}\,\sigma_{\rho\mu}\,(p'_\mu - p_\mu)\int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta\,\frac{\alpha - \beta(\alpha+\beta)}{(\alpha+\beta)^2} \\ =\;& \sigma_{\rho\mu}\,k_\mu\,\frac{e^3}{2m\pi^2}\int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta\,\frac{\alpha - \beta(\alpha+\beta)}{(\alpha+\beta)^2} \\ =\;& \frac{e^3}{8m\pi^2}\,\sigma_{\rho\mu}\,k_\mu \end{aligned} \tag{B.11}$$

$$\begin{aligned} \Rightarrow\;& \frac{e}{m}\cdot\frac{e^2}{8\pi^2}\,\bar{\psi}(p)\,\sigma_{\rho\mu}\,k_\mu\,A_\rho(k)\,\psi(q) \\ \rightarrow\;& -2\,\frac{ie}{4m}\cdot\underbrace{\frac{e^2}{4\pi}}_{\alpha}\cdot\frac{1}{2\pi}\,\bar{\psi}(x)\,\sigma_{\rho\mu}\,F_{\rho\mu}\,\psi(x) \\ \Rightarrow\;& g = 2 \rightarrow 2\left(1 + \frac{\alpha}{2\pi}\right) \end{aligned}$$

In general: $g = 2(1+a)$ where a is the anomalous magnetic moment

$$a = \frac{\alpha}{2\pi} - 0.328\ldots\left(\frac{\alpha}{\pi}\right)^2 + 1.183\ldots\left(\frac{\alpha}{\pi}\right)^3$$

$O(\alpha^4)$: 891diagrams

Full result for $\langle\bar{\psi}\, A\, \psi\rangle$:

$$\gamma_\rho + \Gamma_\rho + \Sigma_{\rho\nu}\, G_0^{\nu\sigma}\, \gamma_\sigma \simeq \gamma_\rho \left[1 + \frac{\alpha k^2}{3\pi m^2}\left(\ln\frac{m}{\mu} - \frac{3}{8} - \frac{1}{5}\right)\right] + \frac{1}{2m}\cdot\frac{\alpha}{2\pi}\,\sigma_{\rho\mu}\, k_\mu \quad \text{for}\, k^2 \ll m$$