# Renormalisation

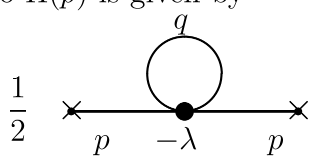
**1)** In the lecture we have studied the vacuum polarisation $\Pi^{\mu\nu}$ in QED at one loop. A much simpler example is the one loop correction to the scalar propagator in a $\phi^4$-theory with action

$$S[\phi] = \int d^4x \left\{ \frac{1}{2}\phi(x)(-\partial_\mu\partial_\mu + m^2)\phi + \frac{\lambda}{4!}\phi^4(x) \right\} \tag{1}$$

in Euclidean space-time with metric $g^{\mu\nu} = -\delta^{\mu\nu}$. The full scalar propagator is given by

$$\int d^4x \langle\phi(x)\phi(0)\rangle e^{ipx} = \frac{1}{p^2 + m^2 + \Pi(p)}\,, \tag{2}$$

where $p^2 = p_\mu p_\mu$, and $\Pi(p)$ stands for the quantum contribution, the self-energy. The one loop diagram contributing to $\Pi(p)$ is given by




The regularised one loop contribution reads

$$\Pi_\Lambda = -\frac{\lambda}{2}\int_{q^2\leq\Lambda^2} \frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2}\,, \tag{3}$$

with ultraviolet (UV) cut-off $\Lambda$. $\Pi_\Lambda$ does not depend on the external momentum $p$.

**a)** Compute $\Pi_\Lambda$. Choose $\Delta m^2$ with $m^2 = m^2_{\rm ren} + \lambda\Delta m^2(\Lambda)$ such that $m^2_{\rm ren}$ and $m^2 + \Pi_\Lambda$ are $\Lambda$-independent. Rewrite this relation as $m^2 = Z_m m^2_{\rm ren}$ with $Z_m = 1 + O(\lambda)$.

**b)** What is the scalar wave function renormalisation $Z_\phi$ at one loop, where $\phi = Z_\phi^{1/2}\phi_{\rm ren}$.

**2)** In Euclidean space-time with metric $g^{\mu\nu} = -\delta^{\mu\nu}$ the regularised vacuum polarisation reads

$$\begin{aligned}\Pi^{\mu\nu}(p) &= -e^2\int_{q^2<\Lambda}\frac{d^4q}{(2\pi)^4}\,{\rm Tr}\left[\frac{1}{\not q - m}\gamma^\mu\frac{1}{\not p + \not q - m}\gamma^\nu\right]\\ &= -e^2\int_{q^2<\Lambda}\frac{d^4q}{(2\pi)^4}\,{\rm Tr}\left[\frac{\not q + m}{q^2+m^2}\gamma^\mu\frac{(\not p + \not q)+m}{(p+q)^2+m^2}\gamma^\nu\right],\end{aligned} \tag{4}$$

where $q^2 = q_\mu q_\mu$. With the help of the trace relations

$${\rm Tr}\,\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu = 4\left(g^{\rho\mu}g^{\sigma\nu} + g^{\mu\sigma}g^{\nu\rho} - g^{\nu\mu}g^{\sigma\rho}\right)\,, \qquad {\rm Tr}\,\gamma^\mu\gamma^\nu = 4g^{\mu\nu}\,, \tag{5}$$

the regularised vacuum polarisation can be written as

$$\Pi^{\mu\nu}(p) = -4e^2\int_{q^2<\Lambda}\frac{d^4q}{(2\pi)^4}\frac{q^\mu(p+q)^\nu + q^\nu(p+q)_\mu - g^{\nu\mu}q(p+q) + m^2g^{\mu\nu}}{(q^2+m^2)((p+q)^2+m^2)}\,. \tag{6}$$

**a)** Compute the orders $(p^2)^0$ and $p^2$.

**b)** Renormalise $\Pi^{\mu\nu}$.