3. Fermions and Scattering

1) The Dirac action is given by

$$S[\phi] = \frac{1}{2} \int d^4x \, \bar{\psi} \, (i\partial\!\!\!/ - m) \, \psi(x) \, . \tag{1}$$

The field operator of a fermionic field operator is given by

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \sum_{s=\pm 1/2} \left\{ e^{ip\,x} v_s(p) b^\dagger(\vec{k}) + e^{-ip\,x} u_s(p) a(\vec{p}) \right\} \, , \tag{2}$$

with $p^0 = \sqrt{\vec{p}^2 + m^2}$, and annihilation/creation operators of electrons and fermions $a, a^\dagger$ and $b, b^\dagger$ respectively with

$$\{a_s(\vec{p}) \, , \, a_r^\dagger(\vec{p}')\} = 2p^0(2\pi)^3 \delta_{sr} \delta(\vec{p} - \vec{p}') \, , \qquad \{b_s(\vec{p}) \, , \, b_r^\dagger(\vec{p}')\} = 2p^0(2\pi)^3 \delta_{sr} \delta(\vec{p} - \vec{p}') \tag{3}$$

and

$$\sum_{s=\pm 1/2} u_s(p) \bar{u}_s(p) = p\!\!\!/ + m \, , \qquad \sum_{s=\pm 1/2} v_s(p) \bar{v}_s(p) = p\!\!\!/ - m \, . \tag{4}$$

a) Show with (3) that $\psi(x)$ and $\bar{\psi}(x)$ satisfy the anti-commutation relations

$$\{\psi(\vec{x}, t) \, , \, \bar{\psi}(\vec{y}, t)\} = \gamma^0 \delta(\vec{x} - \vec{y}) \, , \qquad \{\psi(\vec{x}, t) \, , \, \psi(\vec{y}, t)\} = \{\bar{\psi}(\vec{x}, t) \, , \, \bar{\psi}(\vec{y}, t)\} = 0 \, . \tag{5}$$

b) Show that

$$\langle 0 | \, \psi(x) \, | e^-(\vec{p}, s) \rangle = e^{-ipx} u_s(p) \, , \qquad \langle 0 | \, \bar{\psi}(x) \, | e^+(\vec{p}, s) \rangle = e^{-ipx} \bar{v}_s(p) \, , \tag{6}$$

with $|e^-(\vec{p}, s)\rangle = a_s^\dagger(\vec{p})|0\rangle$, and $|e^+(\vec{p}, s)\rangle = b_s^\dagger(\vec{p})|0\rangle$.

2) Consider the scattering of particles in the presence of an interaction Lagrangian $\mathcal{L}'(x) = e \int d^4x \, A_\mu(x) : \bar{\psi} \gamma^\mu \psi(x) :$, with the S-matrix

$$S = T \, e^{ie \int d^4x \, a_\mu : \bar{\psi} \gamma^\mu \psi(x):} . \tag{7}$$

a) Expand the S-matrix up to the second order, and perform the time-ordering.

b) Reduce the matrix element

$$\langle e^+(p_4) e^-(p_3) | : \psi(x) \gamma^\mu \psi(x) :: \psi(x') \gamma^\mu \psi(x') : | e^+(p_2) e^-(p_1) \rangle \tag{8}$$

to products of the matrix elements (6). This matrix element is relevant for $e + e-$-scattering (Bhabba-scattering). Compare the result to that for $e + e- \to \mu + \mu^-$-scattering.

c) Show that

$$\sum_{s,s',r,r'} |\bar{u}_s(p_3) \gamma^\mu v_{s'}(p_4) \, \bar{v}_r(p_2) \gamma_\mu u'_r(p_1)|^2$$
$$= \text{Tr} \, [\gamma^\mu (p\!\!\!/_4 - m) \gamma_\rho (p\!\!\!/_3 + m)] \, \text{Tr} \, [\gamma^\mu (p\!\!\!/_1 + m) \gamma_\rho (p\!\!\!/_2 - m)] \, . \tag{9}$$

The spin sums in (9) are relevant for $e^+e^- \to e^+e^-$ and $e^+e^- \to \mu + \mu^-$-scattering matrix elements.

3) Consider the scattering process a(p_a) + b(p_b) $\to$ c(p_c) + d(p_d), where the 4-momenta p_i are that in the lab-system. The Mandelstam-variables s, t, u are Lorentz-invariant variables,

$$s = (p_a + p_b)^2 \,, \quad t = (p_a - p_c)^2 \,, \quad u = (p_a - p_d)^2 \,, \tag{10}$$

with

$$s = (p_a + p_b)^2 = (p_a^* + p_b^*)^2 \,, \tag{11}$$

where the p_i^* are the momenta of the particles in the centre of mass (CM) system: $\vec{p}_a^{\,*} = -\vec{p}_b^{\,*}$.

a) The particle b is at rest in the lab system. Show that

$$|\vec{p}_a| = \frac{1}{2m_b} w(s, m_a^2, m_b^2) \,, \tag{12}$$

with

$$w(x,y,z) = (x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)^{\frac{1}{2}} \tag{13}$$

Show also that the energy E_a^* (E_a^*) of the particle a (b) in the CMS is given by

$$E_a^* = \frac{s + m_b^2 - m_a^2}{2\sqrt{2}} \quad \text{und} \quad E_b^* = \frac{s + m_a^2 - m_b^2}{2\sqrt{2}} \tag{14}$$

Equivalent relations hold for E_c^* and E_d^*.

b) Show that

$$|\vec{p}_b^{\,*}| = |\vec{p}_a^{\,*}| = \frac{1}{2\sqrt{s}} w(s, m_a^2, m_b^2) \qquad |\vec{p}_c^{\,*}| = |\vec{p}_d^{\,*}| = \frac{1}{2\sqrt{s}} w(s, m_c^2, m_d^2) \,, \tag{15}$$

and write t as a function of s and the scattering angle θ^* in the CMS.

c) The flow F for the above configuration is (b is at rest, volume $V = 1$):

$$F = |\vec{v}| 2E_a 2m_b \,. \tag{16}$$

Express the flow as a function of the momentum $|\vec{p}_a^{\,*}|$ in CMS.

d) Perform the integration of the 2-particle phase space for the above scattering process in the CMS. Show that

$$\int d\Phi_2 = \frac{1}{16\pi^2} \frac{|\vec{p}_c^{\,*}|}{\sqrt{s}} \int d\Omega_C \,, \tag{17}$$

by using the relation

$$\int \delta[f(\omega)] g(\omega) d\omega = \left(g \left| \frac{df}{d\omega} \right|^{-1} \right)_{f=0} \,. \tag{18}$$