2. Quantisation and Scattering

1) The action of a free real scalar field is given by

$$S[\phi] = \frac{1}{2} \int d^4x \left(\partial_\mu \phi(x) \, \partial^\mu \phi(x) - m^2 \phi^2(x) \right) . \tag{1}$$

The field operator of a real scalar field operator is given by

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \left\{ e^{ikx} a^\dagger(\vec{k}) + e^{-ikx} a(\vec{k}) \right\} , \tag{2}$$

with $\omega = \sqrt{\vec{k}^2 + m^2}$, and annihilation/creation operators $a, a^\dagger$ with

$$[a(\vec{k}), a^\dagger(\vec{k}')] = 2\omega (2\pi)^3 \delta(\vec{k} - \vec{k}') . \tag{3}$$

a) Show that the Klein-Gordon equation $(\partial_\mu \partial^\mu + m^2)\phi(x) = 0$ follows from

$$\frac{\delta S[\phi]}{\delta \phi(x)} = 0 , \qquad \text{with} \qquad \frac{\delta \phi(y)}{\delta \phi(x)} = \delta(x - y) . \tag{4}$$

Show that $\phi(x)$ in (3) satisfies the Klein-Gordon equation, and show, that the canonical momentum operator $\Pi(x)$ is given by $\Pi(x) = \dot{\phi}(x)$.

b) Show with (3) that $\phi(x)$ and $\dot{\phi}(x)$ satisfy the canonical commutation relations

$$[\phi(\vec{x}, t) \, , \, \dot{\phi}(\vec{y}, t)] = i\delta(\vec{x} - \vec{y}) \, , \qquad [\phi(\vec{x}, t) \, , \, \phi(\vec{y}, t)] = [\dot{\phi}(\vec{x}, t) \, , \, \dot{\phi}(\vec{y}, t)] = 0 \, . \tag{5}$$

2) Consider the scattering of one particle in the presence of an interaction Lagrangian $\mathcal{L}'(x) = -1/2 V(\vec{x}) : \phi(x)\phi(x) :$, where $: \, . \, :$ stands for normal ordering, e.g.,

$$: a_1 \cdots a_n a_1^\dagger \cdots a_m^\dagger := a_1^\dagger \cdots a_m^\dagger a_1 \cdots a_n \, . \tag{6}$$

With $H'(t) = -\int d^3x \, \mathcal{L}'(x)$, the time evolution in the interaction picture is given by

$$i\partial_t |t\rangle = H'(t)|t\rangle \, . \tag{7}$$

a) Show by iterating (7) in its infinitesimal form, $|t + \Delta t\rangle = (\mathbb{1} - iH'(t))|t\rangle$, that

$$|t\rangle = U(t, t_0)|t_0\rangle \, , \qquad \text{where} \qquad U(t, t_0) = T e^{-i \int_{t_0}^t dt' H'(t')} , \tag{8}$$

with time ordering T: $T \, H'(t_1)H'(t_2) = H'(t_1)H'(t_2)\theta(t_1 - t_2) + H'(t_2)H'(t_1)\theta(t_2 - t_1)$. Expand U up to the second order.

b) Show that

$$\langle \vec{k}' | : \phi(x)\phi(x) : |\vec{k}\rangle = 2\langle \vec{k}' | : \phi(x)|0\rangle\langle 0|\phi(x) : |\vec{k}\rangle \, , \tag{9}$$

and compute it. This matrix element is relevant for the first order perturbation theory.