Poincaré-Transformations

The inhomogenous Lorentz-transformations of a Minkowski space vector x are given by

$$\mathcal{P} \ni g(\Lambda, a):\; x \mapsto x' = \Lambda x + a \qquad \text{mit} \qquad \Lambda^T \eta \Lambda = \eta\,, \qquad a \in \mathbb{R}^4 \tag{1}$$

This defines Poincaré-group $\mathcal{P}$.

a) Show, that the multipication in $\mathcal{P}$ can be represented by

$$\mathcal{P} \ni g, g':\; g'g \equiv g(\Lambda', b) g(\Lambda, a) = g(\Lambda'\Lambda, \Lambda' a + b)\,. \tag{2}$$

Consider a unitary representation $D : \mathcal{P} \ni g \mapsto U_g \equiv U(\Lambda, a)$

b) Show that U can be decomposed as follows

$$U(\Lambda, a) = T(a) U(\Lambda)\,,\;\; U(\Lambda) \equiv U(\Lambda, 0)\;\; \text{und}\;\; T(a) \equiv U(\mathbb{1}, a) = \exp\{-i a^\mu P_\mu\}\,, \tag{3}$$

where $T(a)$ is a translation, generated by the momentum operator P^μ. Show that the translations are an invariant subgroup of $\mathcal{P}$,

$$U(\Lambda, b)\, T(a)\, U^{-1}(\Lambda, b) = T(\Lambda a)\,. \tag{4}$$

The proper orthochronous Lorentz-group $\mathcal{L}_+^\uparrow$ is represented by the (4×4) matrices Λ, which leave the Minkowski space metric η invariant,

$$\Lambda^T \eta \Lambda = \eta\,, \quad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \text{und} \quad \det\Lambda = 1, \quad \Lambda^0{}_0 \geq 1\,, \tag{5}$$

The proper orthochronous Lorentz-group is generated by the exponential map of the algebra

$$l_+^\uparrow = \{X \in \mathbb{R}^{4,4} \;:\; e^{tX} \in \mathcal{L}_+^\uparrow\,, t \in \mathbb{R}\}\,. \tag{6}$$

c) Show, that

$$X^T \eta = -\eta X \;\Leftrightarrow\; X \in l_+^\uparrow\,, \tag{7}$$

and hence $\dim_{\mathbb{R}} l_+^\uparrow = 6$. A basis of $l_+^\uparrow$ ist given by the 3 generators of rotations and boosts respectively,

$$iJ_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad iJ_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad iJ_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$-iK_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad -iK_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad -iK_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

d) Show that these generators satisfy the following Lie-algebra relations:

$$
\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk} J_k \,, & [N_i, N_j] &= i\epsilon_{ijk} N_k \,, \\
[J_i, K_j] &= i\epsilon_{ijk} K_k \,, & [N_i^\dagger, N_j^\dagger] &= i\epsilon_{ijk} N_k^\dagger \,, \\
[K_i, K_j] &= -i\epsilon_{ijk} J_k \,, & [N_i, N_j^\dagger] &= 0 \,,
\end{aligned}
$$

where we have alos introduced the basis $N_i = (J_i + iK_i)/2$. These relations can be summarised in

$$
\begin{aligned}
M_{0i} = -M_{i0} = K_i \,, \quad M_{ij} = \epsilon_{ijk} J_k \,, \quad \text{mit} \qquad (8) \\
[M_{\mu\nu}, M_{\rho\sigma}] = -i(M_{\mu\rho}\eta_{\nu\sigma} - M_{\mu\sigma}\eta_{\nu\rho} - M_{\nu\rho}\eta_{\mu\sigma} + M_{\nu\sigma}\eta_{\mu\rho}) \,,
\end{aligned}
$$

and we obtain $\Lambda(\omega) = e^{\frac{i}{2}\omega^{\mu\nu} M_{\mu\nu}}$. The tensor $\omega_{\mu\nu} = -\omega_{\nu\mu}$ contains the 6 real parameters of the transformation.

A representation $D: \; g \mapsto U_g$ of a group G as a linear map in a vector space V is called irreducible, if there are no D-invariant sub-spaces of V (except the trivial ones, given by the Null vector and V itself. A sub-vector space $W \subset V$ is D-invariant, if and only if $\forall g \in G, |w\rangle \in W \; : \; U_g|w\rangle \in W$.

e) Show that the operator $P^2 = P_\mu P^\mu$ in an irreducible representation of the Poincaré group on V is provided by a multiple of the unit operator, $P^2 = c_1 \,\mathrm{Id}(V) \,,\; c_1 \in \mathbb{R}$.

P^2 is called a Casimir-operator. The fact, that P^2 is proportional to the unit operator, is a special case of Schur's (first) lemma.

The Pauli-Lubanski-vector W^μ is defined by

$$
W_\mu = \frac{1}{2}\,\epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma} \,. \qquad (9)
$$

f) Show that it has the following properties:

$$
\begin{aligned}
W^\mu P_\mu &= 0 \,, & [M_{\mu\nu}, W_\sigma] &= i(\eta_{\mu\sigma} W_\nu - \eta_{\nu\sigma} W_\mu) \,, \\
[W_\mu, P_\nu] &= 0 \,, & [W_\mu, W_\nu] &= i\epsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma \,. \qquad (10)
\end{aligned}
$$

The operator $W^2 = W^\mu W_\mu$ commutates with all elements of the Poincaré-algebra (why?). It is the second Casimir operator of the Poincaré-group, and $W^2 = c_2 \,\mathrm{Id}(V) \,,\; c_2 \in \mathbb{R}$ in irreducible representations.