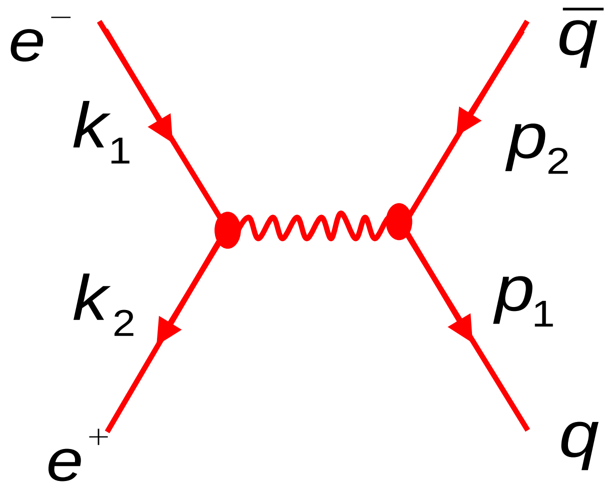
Experimental tests of QED:

1. From the matrix element to the measurement
2. e^+e^- scattering experiments

QED result for transition amplitude:

$$e^+e^- \rightarrow q\,\bar{q}$$



$$|\mathcal{M}|^2 = \sum_{\text{spin, color}} e^4 Q_e^2 Q_q^2 \frac{1}{(k_1+k_2)^4} (\bar{v}_4\gamma_\nu u_3)(\bar{u}_3\gamma_\mu v_4)\,(\bar{u}_1\gamma^\nu v_2)\,(\bar{v}_2\gamma^\mu u_1)$$

$$= 32 e^4 Q_e^2 Q_q^2 N_c \frac{1}{(k_1+k_2)^4} \left[(k_1p_1)(k_2p_2) + (k_1p_2)(k_2p_1)\right]$$

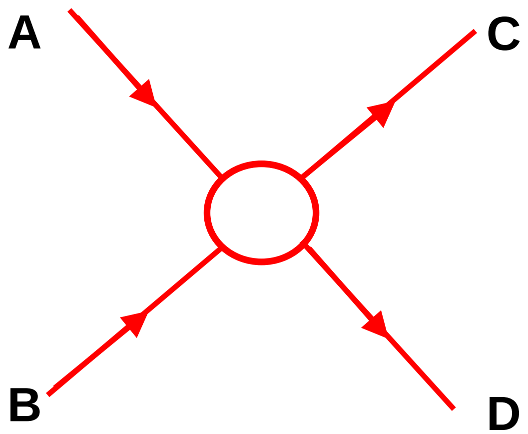
What is the (differential) cross section for the reaction ?

1. From the matrix element to the measurement

1.1 Kinematics

$$A + B \rightarrow C + D$$

Unpolarized particles



What are the Lorentz scalars the cross section can depend on ?

$$p_i p_k \quad \text{with} \quad p_{i,k \geq i} = p_A, p_B, p_C, p_D$$

(unpolarized particles)

10 combinations

$p_i^2 = m_i^2$ 4 constraints

4-mom. conservation: 4 constraints

2 independent products

Instead of $p_i p_k$ use 2 out of the 3 **Mandelstam variables**

$$s = (p_A + p_B)^2$$

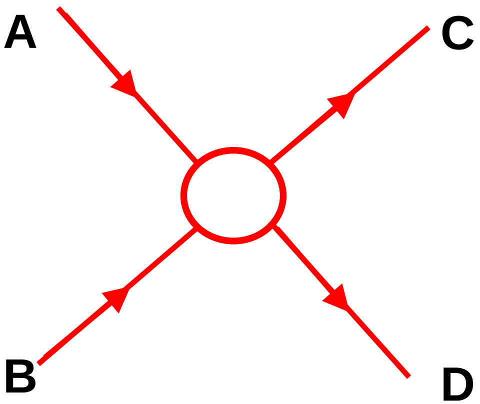
$$t = (p_A - p_c)^2$$

$$u = (p_A - p_D)^2$$

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$$

Crossing

$$A + B \to C + D$$

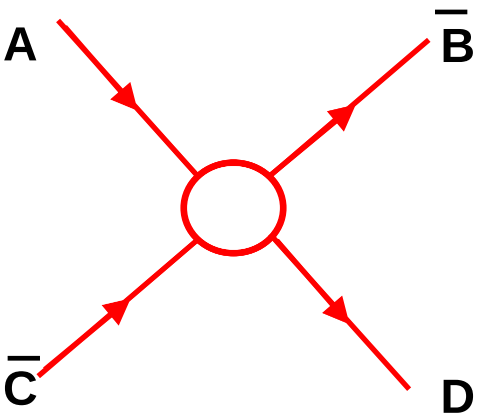


$$s = (p_A + p_B)^2$$

$$t = (p_A - p_c)^2$$

$$u = (p_A - p_D)^2$$

$$A + \overline{C} \to \overline{B} + D$$



$$s' = (p_A + p_{\overline{C}})^2$$

$$t' = (p_A - p_{\overline{B}})^2$$

$$u' = (p_A - p_D)^2$$

<u>Crossing:</u>

$$p_{\overline{C}} \to -p_C$$

$$p_{\overline{B}} \to -p_B$$

$$s' \to t$$

$$t' \to s$$

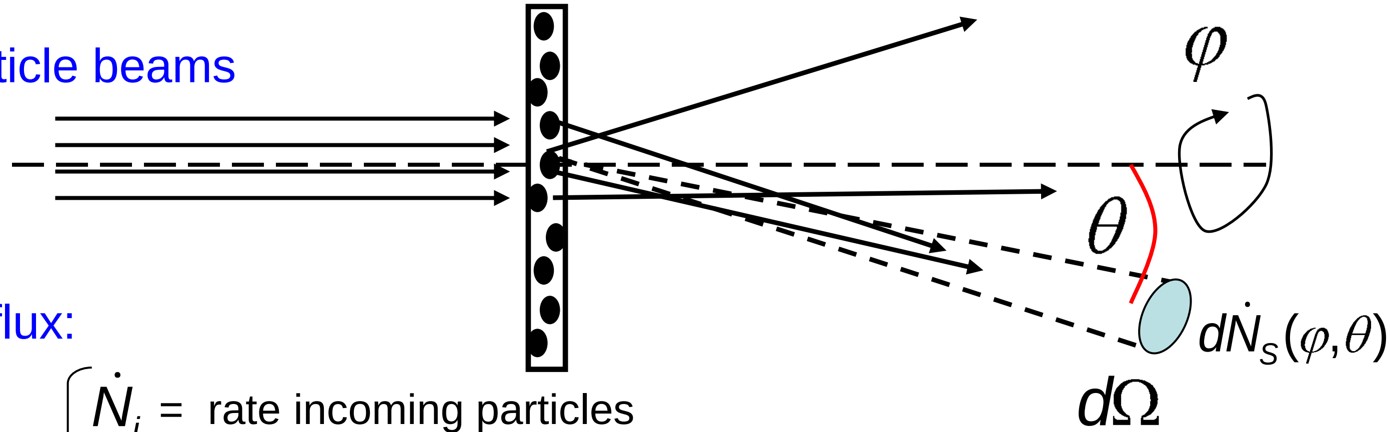
$$u' \to u$$

1.2 Cross section – experimental definition

→ Most important observable to describe scattering processes.

Target:
N_t scattering centers

particle beams



Incident particle flux:

$$F = \frac{\dot{N}_i}{A} = n_i v_i$$

- $\dot{N}_i$ = rate incoming particles
- n_i = particle density in beam
- v_i = velocity of particles

Particle rate scattered

into $d\Omega(\varphi,\theta)$: $d\dot{N}_s(\varphi,\theta)$

$$d\sigma = \frac{d\dot{N}_s(\varphi,\theta)}{F \cdot N_t} = \frac{d\dot{N}_s(\varphi,\theta)}{\Phi}$$

Differential cross section:

$$\frac{d\sigma(\varphi,\theta)}{d\Omega} = \frac{d\dot{N}_s(\varphi,\theta)}{F N_t \cdot d\Omega} = \frac{d\dot{N}_s(\varphi,\theta)}{\Phi \cdot d\Omega}$$

$N_t=1$

Total cross section:

The total cross section is obtained from the total rate of scattered particles:.

$$\sigma_{tot} = \frac{\dot{N}_s}{\Phi}$$

respectively:

$$\sigma_{tot} = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega = \frac{1}{\Phi} \int \left(\frac{d\dot{N}_s}{d\Omega}\right) d\Omega$$

$d\cos\theta\, d\varphi$ of scattered particle

$$\text{Dimension } \sigma = \frac{\text{time}^{-1}}{(\text{time}^{-1} \times \text{area}^{-1})} = \text{area}$$

Units: $[\sigma] = 1\,\text{b} = 10^{-28}\,\text{m}^2 = 10^{-24}\,\text{cm}^2$

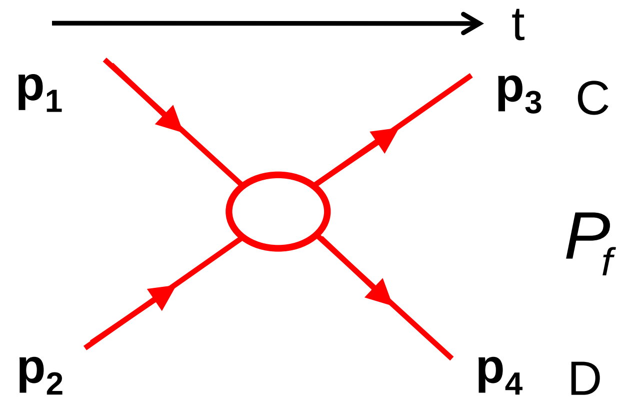
1 b = 1 barn

1.3 Scattering matrix and transition amplitude

Scattering process:

$$1+2 \rightarrow 3+4$$

$$P_i = p_1 + p_2$$



$$P_f = p_3 + p_4$$

Initial and final states:

$$|i\rangle \quad \rightarrow \quad |t\rangle$$

Scattering operator (S matrix):

$$\lim_{t\rightarrow+\infty} |t\rangle = \mathbf{S}|i\rangle$$

Measurement selects specific state f.
Probability S_{fi} to find f:

$$\langle f\,|\,t\rangle = \langle f\,|\mathbf{S}|\,i\rangle = S_{fi}$$

$$\boxed{S_{fi} = \delta_{fi} + \mathrm{i}(2\pi)^4\,\delta^4(P_f - P_i)\,\mathcal{M}_{fi}} \qquad \mathcal{M}_{fi} = \langle f\,|\,\mathcal{M}\,|\,i\rangle$$

Probability (density) that collection of states **i** will make the transition to a final state **f**:

$$\mathcal{P}_{fi} = \left| S_{fi} \right|^2$$

$$\mathcal{P}_{fi} = (2\pi)^8 \left[\delta^4 (P_f - P_i)\right]^2 \left| \mathcal{M}_{fi} \right|^2$$

… or to all possible final states **f**:

$$\mathcal{P}_{fi} = \sum_f \left| S_{fi} \right|^2$$

Final-state phase-space:

The calculation of the transition probability has to consider the number of possible states for each of the out-going particles: → phase-space factor

dN_f = number of states within $\vec{p}$ and $\vec{p} + d\vec{p}$:

$$dN_f = \frac{d^3 p_C}{2E_c (2\pi)^3} \cdot \frac{d^3 p_D}{2E_D (2\pi)^3} \cdot \ldots$$

1.4 Amplitude, cross section and phase space

$$d\sigma = \frac{\text{transition probability}}{\text{Volume} \times \text{Time}} \times \frac{1}{\text{incident flux}} \qquad (*)$$

$$d\sigma = \frac{|S_{fi}|^2 \, dN_f}{VT} \times \frac{1}{\Phi'}$$

incident flux

w_{fi} = Transition rate / V

$$w_{fi} = \frac{|S_{fi}|^2 \, dN_f}{VT} = \frac{(2\pi)^8 \left[\delta^4(P_f - P_i)\right]^2 |\mathcal{M}_{fi}|^2}{VT} dN_f$$

Fermi's trick ⟶ $= (2\pi)^4 \delta^4(P_f - P_i) |\mathcal{M}_{fi}|^2 dN_f$

$*$) We use the probability density, but we divide by a redefined flux: $\Phi' = \Phi / V$
Ignore the "prime" in the following.

Fermi‘s Trick:

$$\left[\delta^4(p_f - p_i)\right]^2 = \frac{VT}{(2\pi)^4}\,\delta^4(p_f - p_i)$$

$$\left[2\pi\,\delta^4(x - x')\right]^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt\; e^{i(x-x')t}\cdot\delta(x - x')$$

$$= \frac{1}{2\pi}\left(\int_{-\infty}^{+\infty} dt\right)\cdot\delta(x - x')$$

Incident flux Φ:

Lab: 1 $\longrightarrow$ 2

CMS: 1 $\longrightarrow \longleftarrow$ 2 $\quad \vec{p}_1 = -\vec{p}_2$

Lab: $$\Phi\Big|_{\text{unit V}} = \rho_1 v_1 N_2 / \text{V} = 2\text{E}_1 2E_2 \left(\frac{|\vec{p}_1|}{E_1} \right)$$

CMS: $$\Phi\Big|_{\text{unit V}} = 2E_1 2E_2 |\vec{v}_1 - \vec{v}_2| = 2E_1 2E_2 \left| \frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2} \right| = 4|\vec{p}_1|(E_1 + E_2)$$

$$= 4\left[(p_1 \cdot p_2)^2 - m_1^2 m_2^2\right]^{1/2}$$

Reminder: Normalization of states

$$\langle (p,s) | (p',s') \rangle = \delta_{ss'} (2\pi)^3 2E_p \, \delta(\vec{p} - \vec{p}')$$

States are delta-function in momentum space.

$$= \delta_{ss'} 2E_p \int d^3x \, e^{i(\vec{p}-\vec{p}')\vec{x}} = 2E_p V \Big|_{s=s'}$$

Differential cross section:

Lorentz invariant phase-space factor

$$d\text{LIPS}_2(P_i, p_C, p_D)$$

$$d\sigma = \frac{|\mathcal{M}_{fi}|^2}{4\left((p_1 \cdot p_2)^2 - m_1^2 m_2^2\right)^{1/2}} \overbrace{(2\pi)^4 \delta^4(P_f - P_i) \frac{d^3 p_C}{2E_c (2\pi)^3} \cdot \frac{d^3 p_D}{2E_D (2\pi)^3}}$$

Lorentz invariant phase-space factor for n particles:

$$d\text{LIPS}_n(P, \underbrace{p_1, p_2, \ldots, p_n}_{\text{Final state}}) = (2\pi)^4 \delta^4(P - (p_1 + p_2 + \ldots + p_n)) \prod_{final} \frac{d^3 p_f}{(2\pi)^3 2E_f}$$

See also PDG http://pdg.lbl.gov/2010/reviews/rpp2010-rev-kinematics.pdf

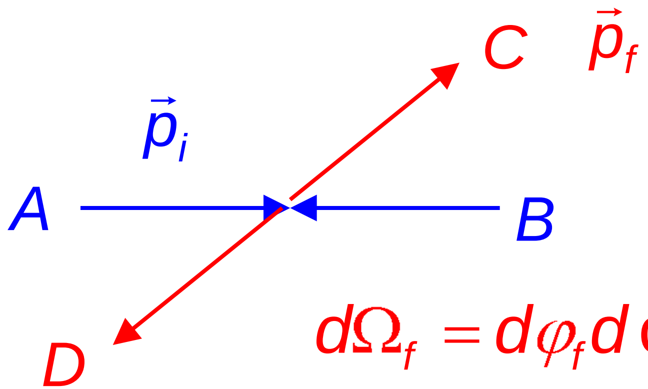
(uses unfortunately different normalization)

Phase space integration for two-particles final-state (CMS)

Center of Mass System:

$\vec{p}_i = \vec{p}_A = -\vec{p}_B \quad \vec{p}_f = \vec{p}_C = -\vec{p}_D$

$s = (E_A + E_B)^2$



$d\Omega_f = d\varphi_f d\cos\theta_f$

$$dLIPS_2 \xrightarrow{\int}$$

$$\int dLIPS_2 = \frac{1}{4\pi^2}\int \delta^3(\vec{p}_C + \vec{p}_D)\delta(E_A + E_B - E_C - E_D)\frac{d^3p_C}{2E_C}\frac{d^3p_D}{2E_D}$$

$$\int d\text{LIPS}_2 = \frac{1}{4\pi^2}\int \delta^3(\vec{p}_C+\vec{p}_D)\,\delta(E_A+E_B-E_C-E_D)\,\frac{d^3p_C}{2E_C}\frac{d^3p_D}{2E_D}$$

In the CMS and 2-particle final-state need to integrate only over p_c: $d^3p = d\Omega\, p^2\, dp$

$$\vec{p}_f = \vec{p}_C = -\vec{p}_D$$

$$\int d\text{LIPS}_2 = d\Omega_C \frac{1}{16\pi^2}\int \delta(E_A+E_B-E_C-E_D)\,\frac{|\vec{p}_C|^2\, d|\vec{p}_C|}{E_C E_D}$$

With $\int \delta[h(\omega)]\, g(\omega)\, d\omega = \left(g \left| \frac{dh}{d\omega} \right|^{-1} \right)_{h=0}$

and $h = \sqrt{s} - \sqrt{\vec{p}_C^{\,2} + m_C^2} - \sqrt{\vec{p}_C^{\,2} + m_C^2}$

$$\left|\frac{dh}{d|p_C|}\right|_{h=0} = 2\frac{1}{2}\frac{2|\vec{p}_C|}{\sqrt{\vec{p}_C^{\,2}+m_C^2}}$$

$$h = 0 = \sqrt{s} - 2\sqrt{\vec{p}_C^{\,2}+m_C^2} \Rightarrow \sqrt{s} = 2E_C$$

$$\left(\left|\frac{dh}{d|p_C|}\right|_{h=0} \right)^{-1} = \frac{\sqrt{s}}{4|\vec{p}_C|}$$

$$\int d\mathrm{LIPS}_2 \quad = \quad d\Omega_C \frac{1}{16\pi^2} \underbrace{\int \delta(E_A + E_B - E_C - E_D) \frac{|\vec{p}_C|^2 \, d|\vec{p}_C|}{E_C E_D}}$$

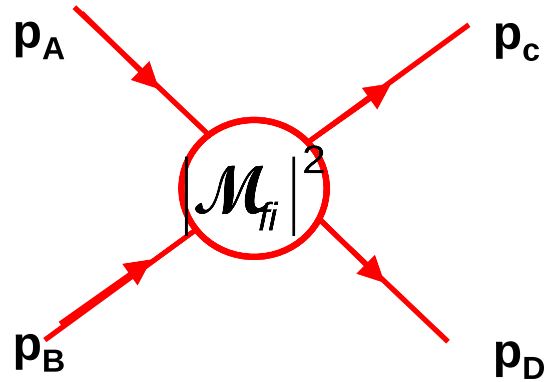
$$\int \delta[h(\omega)] g(\omega) d\omega = \left(g \left| \frac{dh}{d\omega} \right|^{-1} \right)_{h=0} \Rightarrow \left(g \left| \frac{dh}{d|p_C|} \right|^{-1}_{h=0} \right) = \frac{\sqrt{s}}{4|\vec{p}_C|} \underbrace{\frac{|\vec{p}_C|^2 d|\vec{p}_C|}{E_C E_d}} = \frac{|\vec{p}_C|}{\sqrt{s}}$$

$$4E_C E_D = s$$

$$\Rightarrow \quad \boxed{\int d\mathrm{LIPS}_2 \quad = \quad \frac{1}{16\pi^2} \int \frac{|\vec{p}_f|}{\sqrt{s}} d\Omega_f}$$

1.5 Differential cross section …putting everything together

$$A + B \rightarrow C + D$$



CMS

$$d\sigma = \frac{|\mathcal{M}_{fi}|^2}{\Phi} d\mathrm{LIPS}_2 = \frac{1}{64\pi^2} \cdot \frac{1}{s} \cdot \frac{|\vec{p}_f|}{|\vec{p}_i|} \cdot |\mathcal{M}_{fi}|^2 d\Omega_f$$

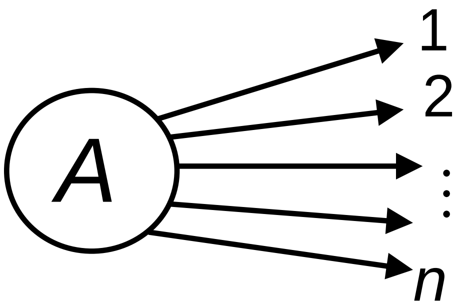
$$\frac{d\sigma}{d\Omega_f} = \frac{1}{64\pi^2} \cdot \frac{1}{s} \cdot \frac{|\vec{p}_f|}{|\vec{p}_i|} \cdot |\mathcal{M}_{fi}|^2$$

$$\Phi = 4|\vec{p}_i|(E_1 + E_2)$$
$$= 4|\vec{p}_i|\sqrt{s}$$

- The dynamics of the scattering process is contained in the matrix element M_{fi} which can be calculated using Feynman rules
- **1/s dependence** of the cross section because of **initial/final state kinematics**

1.6 Decay width, lifetime and Dalitz plots

Decay width



$$\tau = \frac{1}{\Gamma} \qquad \Gamma = \sum \Gamma_i$$

Differential decay width (rate):

$$d\Gamma_i(A \to 1+2+\ldots+n) = \frac{W_{fi}}{n_A} d\mathrm{LIPS}_n$$

$$d\Gamma_i = \frac{|\mathcal{M}_{fi}|^2}{2E_A} \cdot (2\pi)^4 \delta^4(p_A - p_1 - p_2 - \ldots - p_n) \cdot \frac{d^3p_1}{2E_1(2\pi)^3} \cdot \frac{d^3p_2}{2E_2(2\pi)^3} \cdot \ldots \cdot \frac{d^3p_n}{2E_n(2\pi)^3}$$

Two-body decay:

$$A \to 1+2$$

$$d\Gamma_i(A \to 1+2) = \frac{|M_{fi}|^2}{2E_A} \cdot d\mathrm{LIPS}_2 = \frac{|M_{fi}|^2}{2E_A} \frac{1}{16\pi^2} \frac{|\vec{p}_f|}{\sqrt{s}} d\Omega_f$$

CMS: $= \frac{1}{16\pi^2} \frac{|\vec{p}_f|}{\sqrt{s}} d\Omega_f$ $\qquad d\Omega_f = d\varphi_f \, d\cos\theta_f$

$$\sqrt{s} = E_A = m_A$$

$$d\Gamma_i(A \to 1+2) = \frac{|\vec{p}_f|}{32\pi^2 m_A^2} |\mathcal{M}_{fi}|^2 d\Omega_f$$

flat in E_1 and E_2

Three-body decay:

$$A \rightarrow 1 + 2 + 3$$

In rest frame of A:
1,2,3 in one plane

$$\int dLIPS_3 = \frac{1}{8(2\pi)^5} \overbrace{\int dE_1 dE_2 \, d\alpha \, d(\cos\beta) \, d\gamma}$$

3 angles define the orientation

for scalar A or
averaged over spins

$$d\Gamma_i(E_1, E_2) = \frac{1}{64\pi^3} \frac{1}{m_A} |M_{fi}|^2 dE_1 dE_2$$

<u>Remark:</u> Instead of variables E_1 and E_2 one can use variables m_{12}^2 and m_{23}^2 = invariant mass of pairs (i,j)

$$m_{ij}^2 = (p_i + p_j)^2$$

$$dE_1 dE_1 = \underbrace{C \cdot dm_{12}^2 dm_{23}^2}$$

$$d\Gamma_i(m_{12}^2, m_{23}^2) = \frac{1}{256\pi^3} \frac{1}{m_A^3} |M_{fi}|^2 dm_{12}^2 dm_{23}^2$$

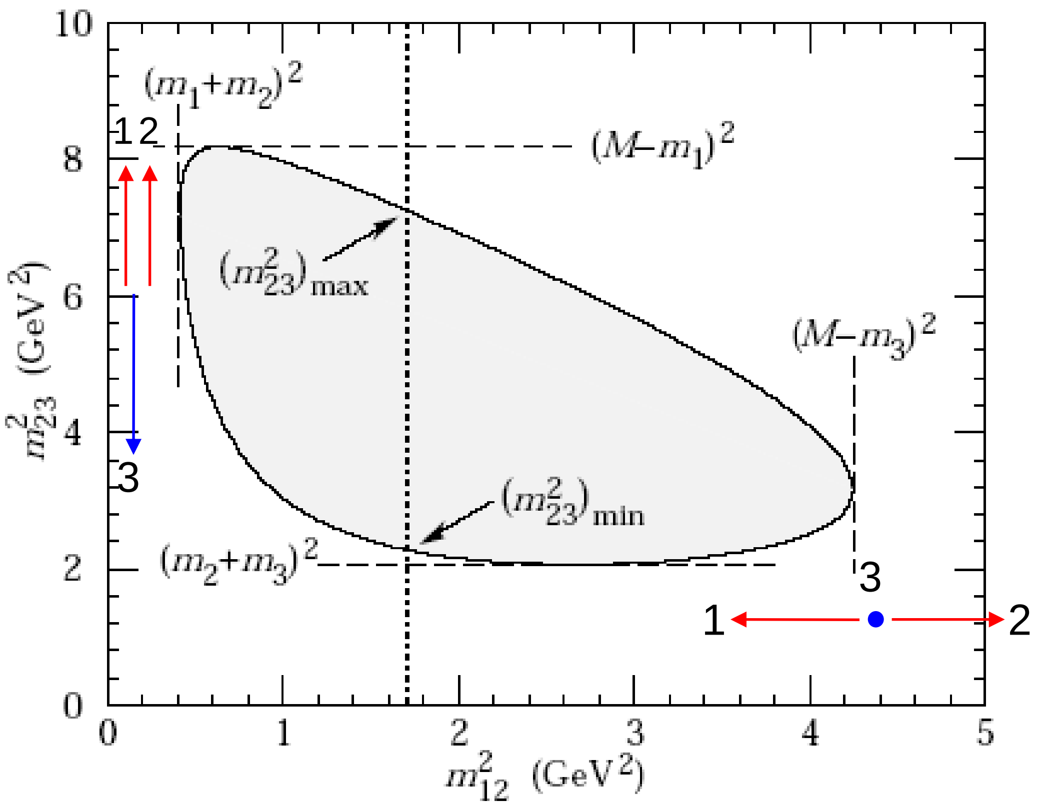
If phase space is flat in E_i then it is also flat in m_{ij}

(for A being a scalar, or average over all spin states)

Experimental method to explore behavior of $\boldsymbol{M_{fi}}$: **Dalitz Analysis**

Dalitz Plots

$$A \rightarrow 1 + 2 + 3$$

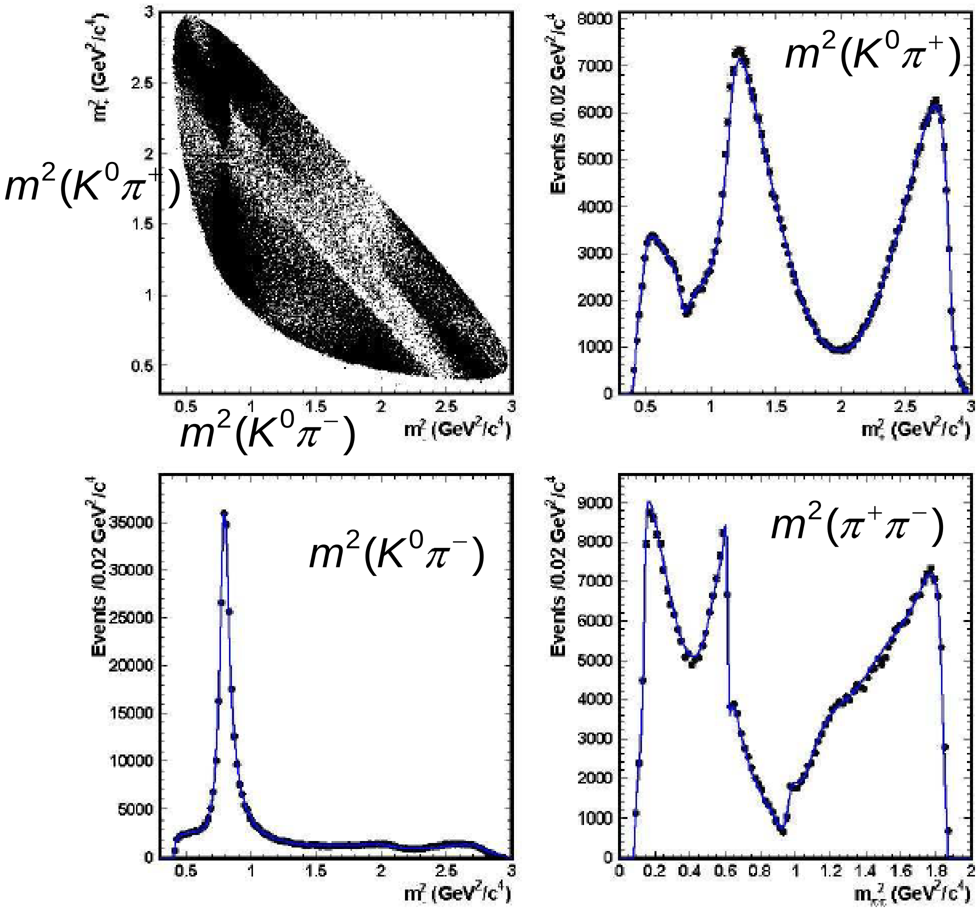


Method:

Put every measured decay into a 2-dim,. (E_1,E_2) or (m_1^2, m_2^2) distribution. A flat distribution over the allowed region corresponds to a "flat matrix element". Structures in the distribution point to a varying matrix element

Dalitz-Plot at Work:

$$D^0 \rightarrow K_s^0 \pi^+ \pi^-$$



M. Staric, HEP 2007 (Manchester)

Resonance	Amplitude	Phase (deg)	Fit fraction
$K^*(892)^-$	1.629 ± 0.005	134.3 ± 0.3	0.6227
$K_0^*(1430)^-$	2.12 ± 0.02	-0.9 ± 0.5	0.0724
$K_2^*(1430)^-$	0.87 ± 0.01	-47.3 ± 0.7	0.0133
$K^*(1410)^-$	0.65 ± 0.02	111 ± 2	0.0048
$K^*(1680)^-$	0.60 ± 0.05	147 ± 5	0.0002
$K^*(892)^+$	0.152 ± 0.003	-37.5 ± 1.1	0.0054
$K_0^*(1430)^+$	0.541 ± 0.013	91.8 ± 1.5	0.0047
$K_2^*(1430)^+$	0.276 ± 0.010	-106 ± 3	0.0013
$K^*(1410)^+$	0.333 ± 0.016	-102 ± 2	0.0013
$K^*(1680)^+$	0.73 ± 0.10	103 ± 6	0.0004
$\rho(770)$	1 (fixed)	0 (fixed)	0.2111
$\omega(782)$	0.0380 ± 0.0006	115.1 ± 0.9	0.0063
$f_0(980)$	0.380 ± 0.002	-147.1 ± 0.9	0.0452
$f_0(1370)$	1.46 ± 0.04	98.6 ± 1.4	0.0162
$f_2(1270)$	1.43 ± 0.02	-13.6 ± 1.1	0.0180
$\rho(1450)$	0.72 ± 0.02	40.9 ± 1.9	0.0024
σ_1	1.387 ± 0.018	-147 ± 1	0.0914
σ_2	0.267 ± 0.009	-157 ± 3	0.0088
NR	2.36 ± 0.05	155 ± 2	0.0615