Standard Model of Particle Physics

Lectures: Tilman Plehn, Ulrich Uwer
Exercises: James Barry, Christoph Englert, Dorival Gonçalves Netto, David López Val

Problem Sheet 5

hand in until: 23.05.2011, 11:15

Problem 1: Spontaneous symmetry breaking of global symmetries

Consider a theory with three real fields (ϕ_1, ϕ_2, ϕ_3), invariant under a global SO(3) symmetry, i.e., rotations in three-dimensional space. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial^\mu \Phi^T \partial_\mu \Phi - V(\Phi^T \Phi)\,, \tag{1}$$

where $\Phi = (\phi_1, \phi_2, \phi_3)^T$ is a column matrix and the potential is

$$V(\Phi^T \Phi) = \frac{\lambda}{4!}(\Phi^T \Phi - v^2)^2\,. \tag{2}$$

The three fields are massless in the SO(3) symmetric Lagrangian in Eq. (1), which can be seen by expanding the potential and noting that the bare mass terms have the wrong sign.

1. Break the symmetry by giving the VEV $\langle \phi_3 \rangle = v \neq 0$ to ϕ_3, while leaving the other two fields unchanged. Rewrite the potential in terms of the new fields $\phi_1' = \phi_1$, $\phi_2' = \phi_2$ and $\phi_3' = \phi_3 - v$.
2. Show that the new potential $V(\Phi'^T \Phi')$ is still invariant under rotations about the 3-axis, but not under rotations about the first and second axes. Use the 3×3 orthogonal rotation matrices that generate the SO(3) symmetry.

This simple example illustrates Goldstone's theorem: the two broken generators of SO(3) correspond to the massless particles ϕ_1' and ϕ_2', which are called Goldstone bosons.

Problem 2: Broken U(1) symmetry and massive gauge bosons

The Lagrangian of electrodynamics coupled to a complex scalar field can be written as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D^\mu \phi)^*(D_\mu \phi) - V(\phi^* \phi)\,, \tag{3}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$. Eq. (3) is invariant under the local transformation

$$\phi(x) \to e^{i\omega(x)}\phi(x), \quad A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \omega(x)\,, \tag{4}$$

where $\omega(x)$ is an arbitrary real function. The potential $V(\phi^*\phi)$ can be chosen to have the form

$$V(\phi^*\phi) = -\mu^2 \phi^* \phi + \frac{\lambda}{2}(\phi^*\phi)^2\,, \tag{5}$$

with $\mu^2 > 0$, which will lead to spontaneous symmetry breaking. This is known as the Boulware-Gilbert model or Abelian Higgs model, and demonstrates that the breaking of a *local gauge symmetry* leads to massive gauge bosons.

1. Expand the potential in Eq. (5) (to second order in the fields ϕ_i) about the vacuum state given by the minimum

$$\langle \phi \rangle = v = \left(\frac{\mu^2}{\lambda} \right)^{1/2} , \tag{6}$$

i.e., rewrite $V(\phi^*\phi)$ in terms of $\phi(x) = v + \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$. Hence, show that the field ϕ_1 acquires mass, whereas ϕ_2 remains massless.

2. Do the same for the kinetic term of ϕ, but omit terms that are cubic and quartic in the fields A_μ, ϕ_1 and ϕ_2. What is the mass of the gauge boson?

Although your answer to question 2 should contain interaction terms between the gauge boson A_μ and the Goldstone boson ϕ_2, the latter (non-physical) field can be gauged away by choosing the correct U(1) gauge transformation, i.e., by working in the *unitary gauge*.

Problem 3: Gauge boson masses in the Standard Model

In the Standard Model Lagrangian, the kinetic term of the Higgs field contains Higgs-gauge field interactions, which lead to masses for the electroweak gauge bosons. At the tree level, the (non-diagonal) effective mass terms are

$$\Delta\mathcal{L}_{D2} = \frac{1}{2}\frac{v^2}{4} \left[g^2 (W_\mu^1)^2 + g^2 (W_\mu^2)^2 + (-gW_\mu^3 + g' B_\mu)^2 \right] , \tag{7}$$

where W_μ^a and B_μ are the SU(2) and U(1) gauge bosons, respectively.

1. Use the relations

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp i W_\mu^2), \quad \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix} , \tag{8}$$

to diagonalise Eq. (7) and obtain the physical masses of the W and Z bosons, in terms of g, v and $\cos\theta_w$, where $\cos\theta_w = g/\sqrt{g^2 + g'^2}$. You should find that A_μ is massless.

2. By identifying A_μ with the photon field, the electric charge $e = \sqrt{4\pi\alpha}$ can be expressed as

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}} . \tag{9}$$

In addition, consistency of the Standard Model (at $q^2 \ll M_W^2$) with the Fermi model requires the identification

$$\frac{G_\mu}{\sqrt{2}} = \frac{e^2}{8\sin^2\theta_w M_W^2} , \tag{10}$$

where $G_\mu = 1.166364(5) \times 10^{-5}$ GeV^{-2} is the Fermi constant, measured via the muon lifetime. Use Eq. (10) to show that the relationship

$$M_W^2 \left(1 - \frac{M_W^2}{M_Z^2} \right) = \frac{\pi\alpha}{\sqrt{2} G_\mu} \tag{11}$$

between the gauge boson masses holds, and hence provide an estimate on M_W, given M_Z, α and G_μ.

Additional information:
Christoph (`c.englert@thphys.uni-heidelberg.de`)
David (`d.lopez@thphys.uni-heidelberg.de`),
Dorival (`d.goncalves@thphys.uni-heidelberg.de`)
James (`james.barry@mpi-hd.mpg.de`)
