Standard Model of Particle Physics

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Problem Sheet 11

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Problem 1: Color algebra

Consider the generators of the $\mathfrak{su}(n)$ Lie algebra in the fundamental representation $t^a, (a = 1, \ldots, n^2 - 1)$ with normalization

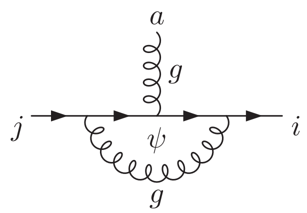
$$\mathrm{Tr}(t^a t^b) = T_r \delta^{ab} \,. \tag{1}$$

The t^a obey the commutation relation $\left[t^a, t^b\right] = i f^{abc} t^c$.

1.) Show that $C_F = \sum_a t^a t^a$ is a casimir operator of the representation, i.e. $[C_F, t^c] = 0$ (from which follows $C_F \sim \mathbb{1}$.)

2.) What is the value of C_F in the fundamental representation (*hint*: use Eq. (1))?

3.) $t^a_{bc} = f^{bac}$ form another representation (the adjoint representation) of $\mathfrak{su}(n)$. What is the value of the Casimir from 1.) in this representation C_A?

4.) Set now $n = 3$, i.e. we look at QCD. Evaluate the color factor of  .

Problem 2: Renormalization of the electromagnetic charge

In the lecture you have been introduced to the running of the strong coupling constant α_s as a result of renormalization and the invariance of "bare" quantities under the renormalization group equations.

We consider the QED Lagrangian in the following. Multiplicative field strength renormalization $\psi_0 = \sqrt{Z_2}\psi_r$, $A^\mu_0 = \sqrt{Z_3}A^\mu_r$ yields for the interaction term

$$\mathcal{L}_{\mathrm{QED}} \supset ie_0 \bar{\psi}_0 A\!\!\!/_0 \bar{\psi}_0 = ie_0 Z_2 \sqrt{Z_3}\, \bar{\psi}_r A\!\!\!/_r \bar{\psi}_r \equiv i Z_1 e_r \bar{\psi}_r A\!\!\!/_r \bar{\psi}_r \,, \tag{2}$$

where the field strength and vertex renormalization constants "multiply" out the QED UV singularities from the bare fields and vertex. "Renormalizability" of a theory (here it is QED) then implies that computing observables in terms of the renormalized quantities ψ_r, A^μ_r, e_r contains no infinities whatsoever anymore. The renormalization constants can be computed from the quantum corrections to the propagators and vertices displayed in Fig. 1 (and this is also the fundamental reasons why we chose constants as roots); shifting the constants Z_i about their tree-level value $Z_i = 1 + \delta_i$ yields new interaction terms which encode the renormalization, the so-called counter terms, see Fig. 2.

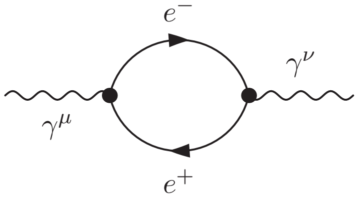


Figure 1: One loop correction to the fermion propagator in QED giving rise to $i\Pi^{\mu\nu}$.

$$= i\delta_3(k^2 g_{\mu\nu} - k^\mu k^\nu)$$

Figure 2: Counter term Feynman rule from the photon field strength renormalization.

From Eq. (2) we can express the renormalized electric charge in terms of the field strength and and the vertex renormalization and the bare charge

$$e_r = \frac{Z_2\sqrt{Z_3}}{Z_1} e_0\,, \tag{3}$$

which is by definition finite! QED is particular nice to us at this point because we have $Z_1 = Z_2$ to all orders of perturbation theory as a consequence of gauge invariance of the quantized and gauge-fixed theory[1], so we can compute the the renormalization of $\alpha = e^2/4\pi$ from the photon field strength renormalization.

Z_3 at one-loop in QED follows from the loop correction to photon propagator shown in Fig. 1. We treat the electrons as massless and the photon off-shell, i.e. $p_\gamma^2 \neq 0$. Loop computations are very technical but there are standard methods to tackle them. The result of the Feynman diagram in Fig. 1 in dimensional regularization $d = 4 - 2\varepsilon$ reads (We urge you to try computing this graph yourself. Ask for more information in the tutorials!)

$$i\Pi^{\mu\nu} = \frac{\alpha}{9\pi} i\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)\left\{3B_0(p^2, 0, 0) - 1\right\} \tag{4}$$

with

$$B_0(p^2, 0, 0) = \frac{(2\pi\mu)^{2\varepsilon}}{i\pi^2}\int \mathrm{d}^{4-2\varepsilon}k\,\frac{1}{k^2(k+p)^2} = \left(\frac{4\pi\mu^2}{-p^2}\right)^{\varepsilon}\Gamma(1+\varepsilon)\left(\frac{1}{\varepsilon} + 2\right) + \mathcal{O}(\varepsilon)\,, \tag{5}$$

where $\Gamma(x)$ is the familiar Gamma function. The divergence of the two-point function shows up as an isolated pole in the self-energy's Laurent series expansion for $d \to 4$.

1.) Renormalize the propagator at an arbitrary renormalization scale μ_R by defining the renormalization constant δ_3.

2.) Derive a differential equation that governs the running of the fine structure constant α by using the invariance of the bare coupling under changes in μ_R in Eq. (3),

$$\mu_R \frac{\mathrm{d}\alpha_0}{\mathrm{d}\mu_R} = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}\alpha_r}{\mathrm{d}\log\mu_R} = \beta(\alpha_r)\,, \tag{6}$$

in the limit $\varepsilon \to 0$, where $\beta(\alpha_R)$ is the notorious "*beta function*" of QED.

3.) Solve this so-called renormalization group equation. What is the behavior of $\alpha_r(\mu_R)$ for $\mu_R \to 0, \infty$? What are the implications if we change the sign of $\beta(\alpha)$ and what is the consequence for *Thomson scattering* qualitatively?

[1] Some of you probably have been exposed to the incarnation of gauge invariance in a quantized abelian gauge theory in terms of *Ward identities* if you have attended the QFT lecture. These identities imply $Z_1 = Z_2$, something which makes our lives easier at this point, but we will not need this during the exercise again. Specifically, this is not valid in non-abelian gauge theories. You can find a taste of that in Tilman's lecture notes and a very concise treatment in the textbook by Böhm, Denner and Joos.