Standard Model of Particle Physics

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Problem Sheet 1

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Problem 1

Let $\psi(x)$ denote a spin 1/2 field of mass m satisfying the Dirac Equation,

$$(i\,\partial\!\!\!/ - m)\,\psi(x) = 0. \tag{1}$$

where $\partial\!\!\!/ \equiv \gamma^\mu\,\partial_\mu$.

1. Prove that $\psi(x)$ fulfills the Klein-Gordon Equation, provided that the γ-matrices obey the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2\,g^{\mu\nu}. \tag{2}$$

(*hint*: apply the operator $(i\partial\!\!\!/ + m)$ to Eq. (1)).

2. One possible representation of the Clifford algebra is the *Weyl* (or *Chiral*) representation,

$$\gamma^0 = \tau^1 \otimes \mathcal{I} = \begin{pmatrix} 0 & \mathcal{I} \\ \mathcal{I} & 0 \end{pmatrix}, \qquad \gamma^k = -i\,\tau^2 \otimes \sigma^k = \begin{pmatrix} 0 & -\sigma^k \\ \sigma^k & 0 \end{pmatrix}, \tag{3}$$

where $\mathcal{I}$ stands for the 2×2 identity matrix and σ^i, τ^i denote the usual Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{4}$$

Show explicitly that the Weyl representation fulfills the Clifford algebra (2).

3. Using the Weyl representation (3), derive an explicit expression for the chirality projectors $P_{L(R)} \equiv (1 \mp \gamma_5)/2$ (recall that $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$).

4. Show explicitly that the projectors $P_{L(R)}$ you just constructed satisfy the following properties:

   (a) $P_{L(R)}^2 = P_{L(R)}$ (idempotence).
   (b) $P_L\,P_R = 0$ (orthogonality).
   (c) $P_L + P_R = 1$ (completeness).

5. Again with the help of the Weyl representation (3), show that the Dirac equation (1) can be rewritten in the following way:

$$(i\,\partial\!\!\!/ - m)\,\psi(x) = \begin{pmatrix} -m & i(\partial^0 + \vec{\sigma}\cdot\vec{\nabla}) \\ i(\partial^0 - \vec{\sigma}\cdot\vec{\nabla}) & -m \end{pmatrix} \begin{pmatrix} \psi_R(x) \\ \psi_L(x) \end{pmatrix} = 0 \tag{5}$$

What is the interesting feature of the massless case?

Problem 2

Let us consider the Lagrangian density of a free spin 1/2 particle of mass m,

$$\mathcal{L} = \bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi. \tag{6}$$

1. Check that the kinetic term in the above Lagrangian density preserves the chirality of the fermion field, whereas the mass term does not.

   (*hint*: use the chirality projectors $P_{L,R} \equiv (1 \mp \gamma_5)/2$ to rewrite Eq. (6) in terms of the left-handed and right-handed components of the spinor field, $\psi_{L(R)} \equiv P_{L(R)}\,\psi$.)

2. The following bilinear terms describe generic types of interactions in the Standard Model involving spinor fields:

   (a) vector: $\bar{\psi}\,\gamma^\mu\,\psi$.

   (b) axial-vector: $\bar{\psi}\,\gamma^\mu\,\gamma_5\,\psi$

   (c) Yukawa: $\bar{\psi}\,\phi\,\psi$, with ϕ denoting a generic scalar field

   which of the above bilinear structures preserve the chirality of the fermions, and which of them do not? Check it explicitly.

3. A Parity Transformation acting on a spin 1/2 field is implemented through

$$\psi(t,\vec{x}) \xrightarrow{\mathcal{P}} \eta\,\gamma^0\,\psi(t,-\vec{x}), \tag{7}$$

   where $\eta \in \mathcal{C}$. Which of the above bilinear structures are invariant under a Parity transformation, and which of them are not ? Check it explicitly.

   (*hints:* first of all, check that $\bar{\psi}(t,\vec{x}) \xrightarrow{\mathcal{P}} \eta^*\,\bar{\psi}(t,-\vec{x})\,\gamma^0$. Then, you should make use of the properties of the γ matrices as representations of the Clifford algebra (2), in particular $(\gamma^0)^2 = 1$; $\{\gamma^0,\gamma^k\} = 0$.)

Contact information

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Time and place of the tutorial:

Group1: Tuesdays 9:15 - 11:00, Albert-Ueberle Str. 3-5, SR 1.
Group2: Thursdays 14:15 - 16:00, Philosophenweg 12, kHS.
