Electro-weak unification and the Standard Model (SM)

- Requisites
- Local gauge invariance and weak isospin
- Electro-weak unification and hypercharge
- Couplings to gauge fields
- Feynman rules
- Higgs mechanism
- Parameters of the SM
- Grand Unification

1. Requisites

a) Fundamental fermions

Leptons

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L$$

Left-handed doublets

$$e_R^- \quad \mu_R^- \quad \tau_R^-$$

right-handed singlets

Quarks

$$\begin{pmatrix} u \\ d' \end{pmatrix}_L \quad \begin{pmatrix} c \\ s' \end{pmatrix}_L \quad \begin{pmatrix} t \\ b' \end{pmatrix}_L$$

$$u_R, d_R \quad c_R, s_R \quad t_R, b_R$$

b) Fundamental interaction

- Charged current interaction: transitions inside LH doublets
- Neutral current interaction: couples to LH and RH fermions

weak

- Electromagnetic interaction couples equally to LH and RH fermions

2. Local Gauge Invariance
Quantum Electrodynamics

Reminder

QED Lagrangian for free spin ½ particle:

$$L(x,t) = i\overline{\psi}(x,t)\gamma^{\mu}\partial_{\mu}\psi(x,t) - m\overline{\psi}(x,t)\psi(x,t)$$

Applying the Euler-Lagrange formalism leads to the Dirac equation.

Invariance under Local Gauge Transformation

Demanding invariance under local phase transformation of the free Langrangian (local gauge invariance):

$$\psi(x) \rightarrow \psi(x) = e^{i\alpha(x)}\psi(x)$$

requires the substitution:

$$i\partial_{\mu} \rightarrow i\partial_{\mu} + eA_{\mu}(x)$$

If *A* transforms under local gauge transformation as

Reminder

$$A_{\mu}(x) \rightarrow A_{\mu}(x) + \partial_{\mu}\alpha(x)$$

Then *L* remains invariant:

$$L(x) \xrightarrow{\psi \rightarrow \psi\, e^{i\alpha(x)}} L(x)$$

To interpret the introduced field A_μ as photon field requires to complete the Langrangian by the corresponding field energy:

$$\boxed{L = \overline{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi + e\overline{\psi}\gamma^{\mu}\psi A_{\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$$

The requirement of local gauge invariance has automatically led to the interaction of the free electron with a field.

Quantum Chromodynamics – SU(3) Theory

Reminder

Lagrangian is constructed with quark wave functions $\psi = \begin{pmatrix} \psi_R \\ \psi_G \\ \psi_B \end{pmatrix}$

Invariance of the QCD Lagrangian under

<u>local SU(3) gauge transformation</u>

$$\psi(x) \to \psi'(x) = U(x)\psi(x) = e^{i\frac{\alpha_k(x)}{2}\lambda_k}\psi(x)$$

with any unitary (3 x 3) matrix $U(x)$.

$U(x)$ can be given by a linear combination of
8 Gell-Mann matrices $\lambda_1 \ldots \lambda_8$ [SU(3) group generators]

requires interaction fields – 8 gluons corresponding to these matrices

Electroweak Theory – SU(2) $\times$ U(1) Theory

Describe the particles of the LH doublet as two states of one particle.
In analogy to the strong isospin one can introduce a new quantum number: **weak isospin** $T = ½$. The two states are then given by $T_3 = \pm ½$.

$$\chi_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L \quad T = \frac{1}{2}, T_3 = \pm\frac{1}{2}$$

As in case of the (iso)spin, the transformations are defined by the SU(2) group - the group of all unitary (2 x 2) matrices $U(x)$. The generators of the group are the three Pauli matrices $\tau^k = \sigma^k$.

$$\chi_L(x) \to \chi'_L(x) = U(x)\chi_L(x) = e^{i\frac{\alpha_k(x)}{2}\tau^k}\chi_L(x)$$

Three gauge bosons W^1, W^2, W^3 should correspond to the three group generators.

This means

as in the QED lagrangian

$$L = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + e\underbrace{\bar{\psi}\gamma^\mu\psi}_{\text{fermion current}}\underbrace{A_\mu}_{\text{e.m. field}} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

we expect three fermion currents in the weak lagrangian:

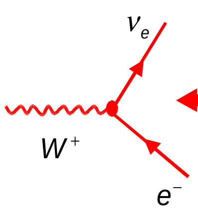
Weak isospin Triplett
of LH fermion currents

$$J_\mu^i = \bar{\chi}_L \gamma_\mu \frac{1}{2}\tau^i \chi_L$$

i. e. we expect the terms $gJ^i_\mu W_i^\mu$ with some coupling *g*.

What do we see in reality?

Charged current weak interaction

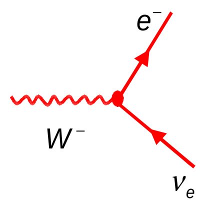


Charge raising current:

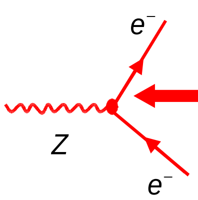
$$J_\mu^+ = \bar{u}_\nu \gamma_\mu \frac{1-\gamma^5}{2} u_e = \bar{\nu}\gamma_\mu \frac{1-\gamma^5}{2} e = \bar{\nu}_L \gamma_\mu e_L$$

Charge lowering current:

$$J_\mu^- = \bar{e}\gamma_\mu \frac{1-\gamma^5}{2}\nu = \bar{e}_L \gamma_\mu \nu_L$$



Neutral current weak interaction

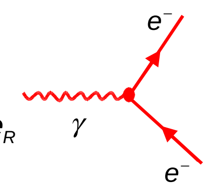


$$J_\mu^{NC} = \bar{e}\gamma_\mu \frac{1}{2}(g_V - g_A\gamma^5)e = \bar{e}\gamma_\mu \left[\overbrace{\frac{1}{2}(g_V + g_A)}^{g_L}\frac{1-\gamma^5}{2} + \overbrace{\frac{1}{2}(g_V - g_A)}^{g_R}\frac{1+\gamma^5}{2} \right]$$

$$= \bar{e}_L g_L \gamma_\mu e_L + \bar{e}_R g_R \gamma_\mu e_R$$

Electromagnetic interaction

$$J_\mu^{em} = \underset{\substack{\uparrow \\ \text{units of e}}}{q}\bar{e}\gamma_\mu e = q\bar{e}_L\gamma_\mu e_L + q\bar{e}_R\gamma_\mu e_R$$



Weak isospin and weak currents

J^+ and J^- describe state transitions along the T_3 axis. $\chi_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L \quad T = \frac{1}{2}, T_3 = \pm\frac{1}{2}$

As in the case of the (iso)spin, one can use the raising and lowering operators defined by the Pauli matrices to express the state transitions.

$$\tau^\pm = \frac{1}{2}(\tau^1 \pm i \cdot \tau^2)$$

$$\tau^i = \sigma^i = \text{Pauli-matrices}$$

The charge current can then be written in the compact form:

$$J^\pm_\mu = \overline{\chi}_L \gamma_\mu \tau^\pm \chi_L$$

and the corresponding fields are: $W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \mp i W^2_\mu)$

From the SU(2) structure of the isospin formalism one expects that in addition to the currents $J^\pm$

there exists a 3rd neutral current J^3 of the form:

$$J^3_\mu = \overline{\chi}_L \gamma_\mu \cdot T_3 \tau^3 \chi_L = \frac{1}{2}\overline{\nu}_L \gamma_\mu \nu_L - \frac{1}{2}\overline{e}_L \gamma_\mu e_L$$

➡

Weak isospin Triplett of LH fermion currents

$$J^i_\mu = \overline{\chi}_L \gamma_\mu \frac{1}{2}\tau^i \chi_L$$

3. Electro-weak unification

The new current J^3 is not equal to the current J^{NC}:
J^{NC} contains LH and RH fermion contributions

⬇

Treat both neutral currents, J^{em} and J^{NC}, simultaneously:

As both currents contain RH contributions it should be possible to construct a linear combination which couples only to LH fermions:

two linear combinations of J^{em} and J^{NC}

$$J^3_\mu = \sin^2\theta_W J^{em}_\mu + J^{NC}_\mu$$

$$\frac{1}{2}J^Y_\mu = \cos^2\theta_W J^{em}_\mu - J^{NC}_\mu$$

⬅ Choose θ_W such that RH fermions components in J^3 vanish.

- J^3 completes the isospin triplet J^i
- J^Y is called hypercharge current
  It couples via hypercharge Y $\quad J^Y_\mu = 2J^{em}_\mu - 2J^3_\mu$

Electro-weak hypercharge

From the above definitions follows:

$$J_\mu^Y = 2J_\mu^{em} - 2J_\mu^3 = \overline{\psi}\gamma_\mu\left[2Q - 2T_3\right]\psi = \overline{\psi}\gamma_\mu Y\psi$$

Hypercharge operator: $Y = 2[Q-T_3]$

(Gell-Mann – Nishijima formula)

Electro-weak quantum numbers

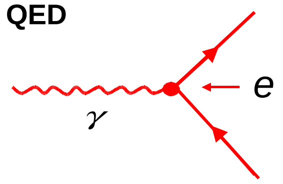
Leptons	T	T_3	Q	Y
ν_e	$1/2$	$+1/2$	0	-1
e_L	$1/2$	$-1/2$	-1	-1
e_R	0	0	-1	-2

Quarks	T	T_3	Q	Y
u_L	$1/2$	$+1/2$	$2/3$	$1/3$
d'_L	$1/2$	$-1/2$	$-1/3$	$1/3$
u_R	0	0	$2/3$	$4/3$
d_R	0	0	$-1/3$	$-2/3$

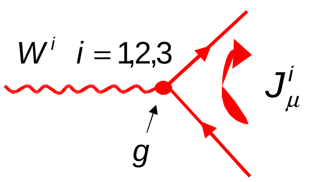
4. Current coupling to the gauge fields/bosons

In the **electro-weak theory** the coupling between bosons and fermions is defined in analogy to the coupling of the photon to the fermions currents in QED. There are in total 4 boson fields:

QED



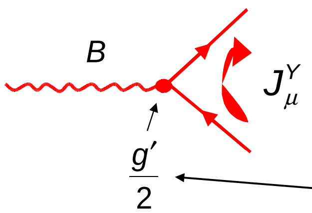
$$\sim -ieJ_\mu^{em}\cdot A^\mu$$



$$\sim -igJ_\mu^i \cdot W^{i,\mu}$$

Corresponding to $J^\pm$ and J^3 there are fields

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2) \quad \text{and} \quad W_\mu^3$$



$$\sim -i\frac{g'}{2}J_\mu^Y\cdot B^\mu$$

convention

g, g′ are coupling constants.

Note: B corresponds to an additional local gauge transformation $U(1)_Y$ of L

Gauge bosons

While the charged boson fields $W^{\pm}$ correspond to the observed W bosons, the neutral fields B and W^3 only correspond to linear combinations of the observed photon and Z boson:

$$\boxed{\begin{aligned} A_\mu &= B_\mu \cos\theta_W + W_\mu^3 \sin\theta_W \\ Z_\mu &= -B_\mu \sin\theta_W + W_\mu^3 \cos\theta_W \\ B_\mu &= A_\mu \cos\theta_W - Z_\mu \sin\theta_W \\ W_\mu^3 &= A_\mu \sin\theta_W + Z_\mu \cos\theta_W \end{aligned}}$$

massless photon

massive Z boson

The weak mixing angle θ_W (Weinberg angle) is defined by the coupling constants to A^μ and Z^μ.

The fermion couplings to the neutral fields are given by

$$-igJ_\mu^3 \cdot W^{3,\mu} - i\frac{g'}{2}J_\mu^Y \cdot B^\mu$$
$$= i\left[g\sin\theta_W J_\mu^3 + \frac{g'}{2}\cos\theta_W J_\mu^Y\right]A^\mu - i\left[g\cos\theta_W J_\mu^3 - \frac{g'}{2}\sin\theta_W J_\mu^Y\right]Z^\mu$$

Fermion coupling to the photon

$$[\quad]A^\mu \equiv eJ_\mu^{em}A^\mu = e\left[J_\mu^3 + \frac{1}{2}J_\mu^Y\right]A^\mu$$

$$J_\mu^Y = 2\left[J_\mu^{em} - J_\mu^3\right]$$

Comparison of the coefficients gives:

$$\boxed{e = g\cdot\sin\theta_W \qquad e = g'\cdot\cos\theta_W \qquad g = \frac{e}{\sin\theta_W} \qquad g' = \frac{e}{\cos\theta_W}}$$

The couplings to the different boson types have similar strength.

Fermion coupling to the Z boson

From $-i[\qquad]Z^\mu$ follows with
$$\begin{cases} J^Y_\mu = 2\left[J^{em}_\mu - J^3_\mu\right] \\ g\cdot\sin\theta_W = g'\cdot\cos\theta_W \end{cases}$$

$$-i[\qquad]Z^\mu = -i\frac{g}{\cos\theta_W}\left[J^3_\mu - \sin^2\theta_W J^{em}_\mu\right]Z^\mu$$
$$= -i\frac{g}{\cos\theta_W}J^{NC}_\mu Z^\mu$$

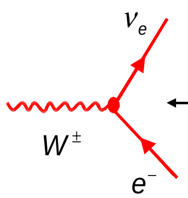
Using
$$\begin{cases} J^3_\mu = \bar{\chi}_L\gamma_\mu\cdot T_3\tau^3\chi_L \\ J^{em}_\mu = \bar{e}\gamma_\mu Qe \end{cases}$$
one finds

$$-i[\qquad]Z^\mu = -i\frac{g}{\cos\theta_W}\left[T_3\cdot\bar{e}\gamma_\mu\frac{1-\gamma^5}{2}e - q\cdot\sin^2\theta_W\,\bar{e}\gamma_\mu e\right]Z^\mu$$
$$-i[\qquad]Z^\mu = -i\frac{g}{\cos\theta_W}\left[\bar{e}\gamma_\mu\frac{1}{2}[g_V - g_A\gamma^5]e\right]Z^\mu$$

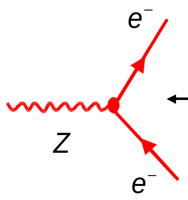
with $g_V = T_3 - 2q\sin^2\theta_W$ and $g_A = T_3$

5. Feyman rules

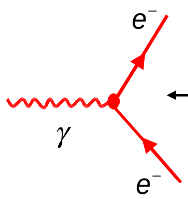
Vertex factors | Propagator



$$-i\frac{g}{\sqrt{2}}\gamma_\mu\frac{1}{2}(1-\gamma^5) \qquad \frac{g_{\mu\nu} - q_\mu q_\nu/M_W^2}{q^2 - M_W^2}$$

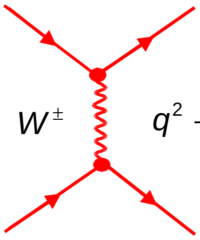


$$-i\frac{g}{\cos\theta_W}\gamma_\mu\frac{1}{2}(g_V - g_A\gamma^5) \qquad \frac{g_{\mu\nu} - q_\mu q_\nu/M_Z^2}{q^2 - M_Z^2}$$



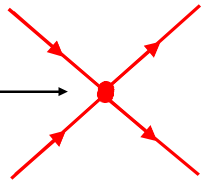
$$-ie\gamma_\mu \qquad \frac{1}{q^2}$$

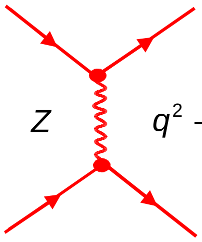
Comparison of the $q^2 \to 0$ limit with the 4-femion ansatz



$W^{\pm}$ $\quad q^2 \to 0:$

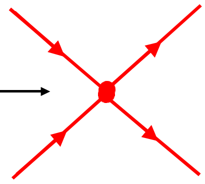
$$\boxed{\frac{g^2}{8M_W^2} = \frac{G_F}{\sqrt{2}}}$$





Z $\quad q^2 \to 0:$

$$\boxed{\frac{g^2}{8\cos^2\theta_W M_Z^2} = \frac{G_{NC}}{\sqrt{2}}}$$



For $\frac{G_F}{\sqrt{2}} \equiv \frac{G_{NC}}{\sqrt{2}}$ follows $\boxed{\cos^2\theta_W = \frac{M_W^2}{M_Z^2}}$ ⟸ follows also from Higgs mechanism

6. Higgs boson and the parameters of the SM

The original EW lagrangian

$$L = i\bar{\psi}\gamma^\mu\partial_\mu\psi + e\bar{\psi}\gamma^\mu\psi\, A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

with $\quad e A_\mu = \frac{g}{2}\tau_i W_\mu^i + \frac{g'}{2} Y B_\mu$

$$F_{\mu\nu}F^{\mu\nu} = W_{\mu\nu}W^{\mu\nu} + B_{\mu\nu}B^{\mu\nu}$$

is invariant under local gauge transformations only **with massless fields**.
But particles have masses. Ad-hoc mass terms for

fermions $\quad m\bar{\psi}\psi$

bosons $\quad m^2 A_\mu A^\mu$

destroy the gauge invariance under $SU(2)_L \times U(1)_Y$.

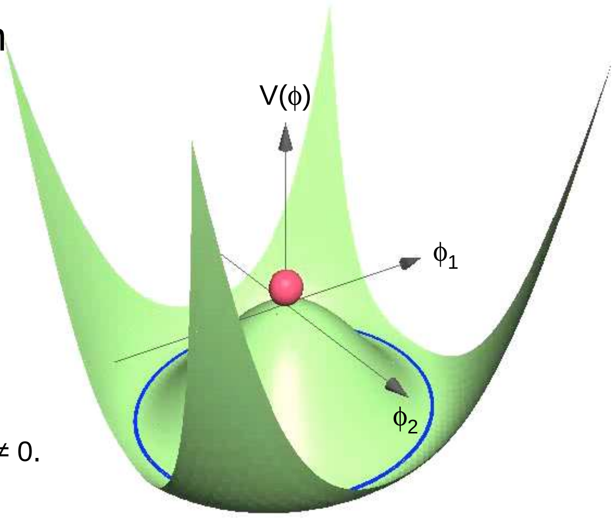
To generate boson and fermion masses in an gauge invariant way the Standard Model uses the Higgs mechanism.

Higgs–Kibble Mechanism

Introduce a new doublet of complex scalar fields (4 degrees of freedom) with the ‘mexican hat’ potential:

$$V(\phi) = -\mu^2 |\phi^+ \phi| + \lambda |\phi^+ \phi|^2$$

with μ, $\lambda > 0$



Spontanous symmetry breaking:
System falls in to minimum of V at $\phi \neq 0$.

This results in:
Three massless excitations along the valley $\rightarrow$ masses for W± and Z
One massive exciation out of the valley $\rightarrow$ „physical“ Higgs boson

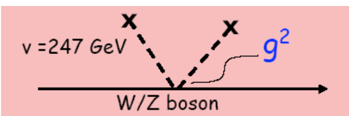
Higgs field has two components: $\phi = v + H$.

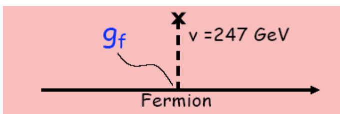
1. omnipresent, constant background condensate with non-vanishing vacuum expectation value $v = \mu/\sqrt{\lambda} = 247$ GeV (from G_F)
2. Higgs boson H with unknown mass $M_H = \mu \cdot \sqrt{2} = v\sqrt{2\lambda}$

Interactions of the condensate field result in Lagrangian terms:

$$\frac{g^2 v^2}{4} W_\mu W^\mu + \frac{(g^2 + g'^2) v^2}{8} Z_\mu Z^\mu - g_f \frac{v}{\sqrt{2}} \bar{\Psi} \Psi$$

boson mass terms fermion mass terms





For the boson masses one finds:

$$M_W = \frac{1}{2} \upsilon g \qquad \frac{M_W}{M_Z} = \frac{g}{\sqrt{g^2 + g'^2}} = \cos\theta_w$$

$$M_Z = \frac{1}{2} \upsilon \sqrt{g^2 + g'^2} \qquad g \sin\theta_w = g' \cos\theta_w$$

Fermion masses are added via uknown Yukawa couplings $m_f \sim g_f v$ specific for each fermion

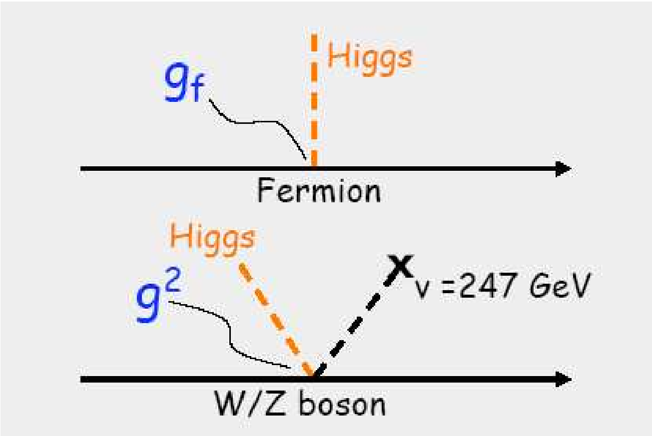
Omnipresent Higgs field

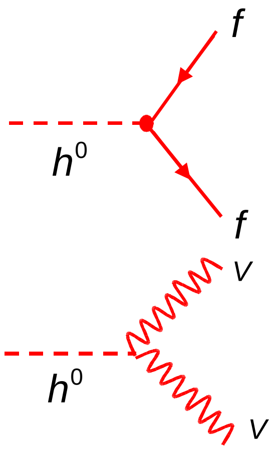
Higgs field – generation of masses

Excitation of the Higgs field

Higgs boson gets mass

Higgs boson couplings



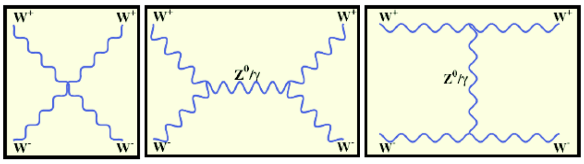


Fermions: $g_f \sim m_f$ / v - coupling proportional to the fermion mass

W/Z bosons: $g_V \sim g^2$ v = M_V^2 / v

Higgs and WW scattering

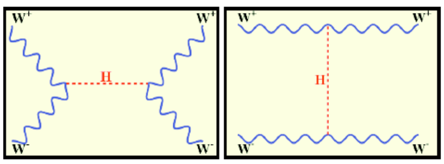
$F_{\mu\nu}F_{\mu\nu}$ term contains self couplings between gauge bosons.



Cross section: $\sigma_{WW} \sim E_{cm}$

$W_L W_L$ scattering probability becomes larger than unity for E_{cm} > 1.2 TeV.
Violation of unitarity if force remains weak at this scale.

To restore unitary it needs a scalar boson "*H*" with
$g_{HWW} \sim m_W$
$g_{Hff} \sim m_f$
M_H < 1TeV



Then σ_{WW} will remain constant for high energies

Standard Model Parameters

e (α_{QED})

(G_F and $\sin\theta_W$) or (M_W and M_Z) or …

α_s (strong coupling constant)

9 fermion masses, neutrinos are massless

4 quark mixing parameters (CKM matrix)

M_H

$\Big\}$ 18 parameters

too many?

+ 3 neutrino masses

+ 4 neutrino mixing parameters

Shortcomings of the electro-weak unification

- Not one group but two: $SU(2)_L \times U(1)_Y$
- Two couplings *g* and *g'* remain
- Mathematically, other unifications of weak and e.m. forces could be realised, e.g. the lagrangian **is** in fact invariant w.r.t. **2 global U(1) transformations**:

$$\begin{pmatrix} \nu(x) \\ e(x) \end{pmatrix}_L \rightarrow e^{i\phi'} \begin{pmatrix} \nu(x) \\ e(x) \end{pmatrix}_L$$

$$e(x)_R \rightarrow e^{i\phi} e(x)_R$$

The lagrangian could be invariant w.r.t. 2 independent **local** U(1) trasformations.
In this case 2 massless photon fields would be observed.

Full understanding is supposed to be obtained only in the Grand Unification Theory (GUT) which should have one common group for electro-weak and QCD fields.

Grand Unification

$$\alpha_1 = \frac{5}{3} g'^2 = \frac{5}{3} \frac{\alpha}{\cos^2\theta_W} \qquad \alpha_2 = g^2 = \frac{\alpha}{\sin^2\theta_W} \qquad \alpha_3 = \alpha_s$$