Addendum to Chapter III

3.4 Chirality and helicity

3.5 Summay fermion scattering

4. Higher Orders

$e^+e^- \rightarrow \mu^+\mu^-$ $\boxed{\left.\frac{d\sigma}{d\Omega}\right|_{CMS} = \frac{\alpha^2}{4s}\cdot(1+\cos^2\theta)}$

3.4 Chirality, Helicity and angular distribution

Chirality operator:

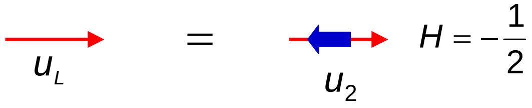
Projection of left- and right-handed components of spinor u

$$\boxed{\begin{aligned} u_L &= \frac{1}{2}(1-\gamma^5)u \\ u_R &= \frac{1}{2}(1+\gamma^5)u \end{aligned}} \qquad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Helicity operator:

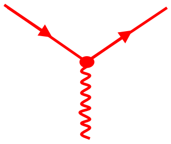
$$\boxed{H = \frac{1}{2}\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}} \qquad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

In the **relativistic limit** (or for massless particles) the eigenstates of the helicity operator correspond to the chirality states.



u_L $=$ u_2 $H = -\frac{1}{2}$

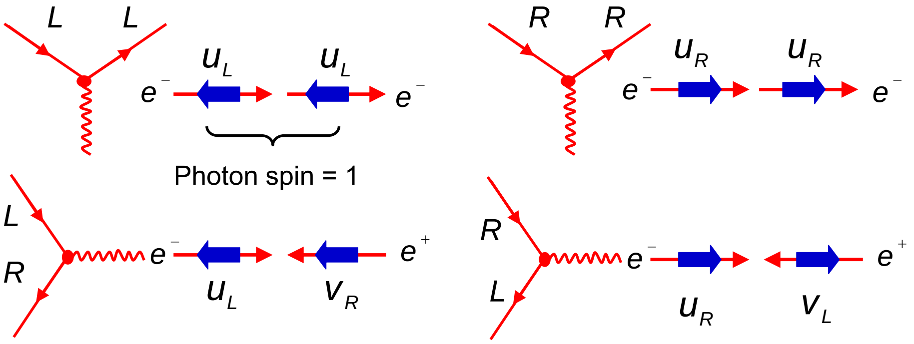
Decomposition of the fermion current:



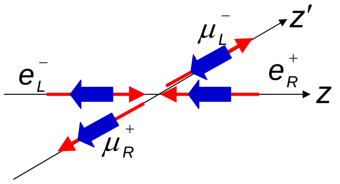
$$\bar{u}\gamma^\mu u = (\bar{u}_R + \bar{u}_L)\gamma^\mu (u_R + u_L)$$
$$= \bar{u}_R\gamma^\mu u_R + \bar{u}_L\gamma^\mu u_L$$

Photon (vector ieγ^μ) coupling:

Attention “arrows” correct
only for massless electrons



Angular distribution $e^+e^- \to \mu^+\mu^-$



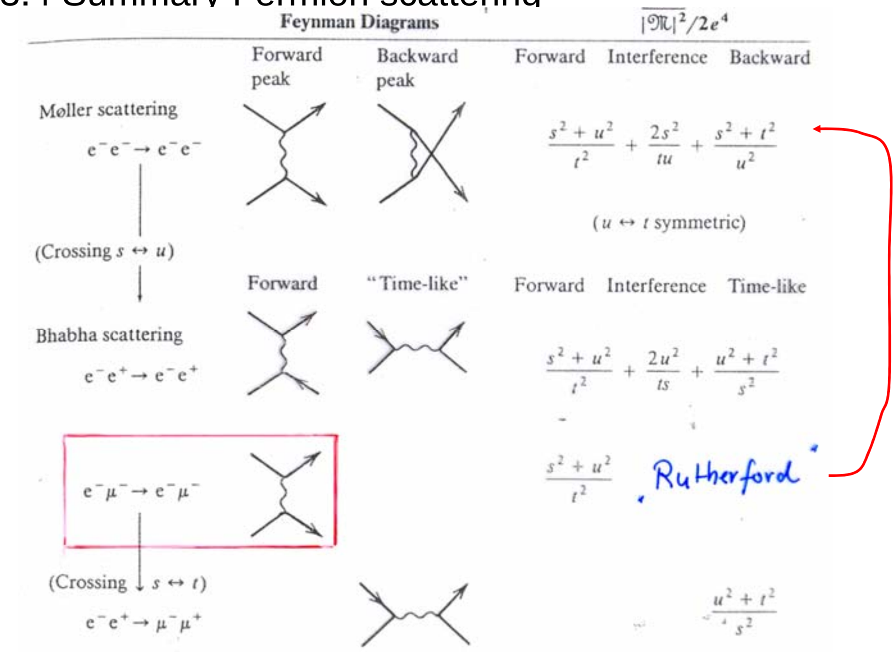
$$\boxed{d^J_{m'm}}$$

$$\text{Axis } z \xrightarrow{\text{rotation}} \text{Axis } z'$$
$$\left.\begin{matrix}J=1\\ m_z=-1\end{matrix}\right\} \xrightarrow{d^1_{-1-1}} \left\{\begin{matrix}J=1\\ m_{z'}=-1\end{matrix}\right.$$
$$\left.\begin{matrix}J=1\\ m_z=-1\end{matrix}\right\} \xrightarrow{d^1_{+1-1}} \left\{\begin{matrix}J=1\\ m_{z'}=+1\end{matrix}\right.$$

Change of quantization axis

$$\boxed{\frac{d\sigma}{d\Omega} \sim \left(d^1_{-1-1}\right)^2 + \left(d^1_{+1-1}\right)^2 \sim \frac{1}{4}(1+\cos\theta)^2 + \frac{1}{4}(1-\cos\theta)^2 \sim 1+\cos^2\theta}$$

3.4 Summary Fermion scattering

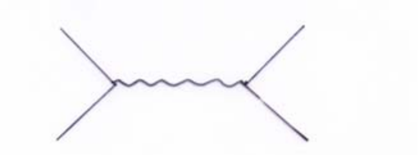


	Feynman Diagrams		$\overline{\lvert\mathcal{M}\rvert^2}/2e^4$		
	Forward peak	Backward peak	Forward	Interference	Backward
Møller scattering $e^-e^- \to e^-e^-$			$\frac{s^2+u^2}{t^2}$ +	$\frac{2s^2}{tu}$ +	$\frac{s^2+t^2}{u^2}$
(Crossing $s \leftrightarrow u$)			($u \leftrightarrow t$ symmetric)		
	Forward	"Time-like"	Forward	Interference	Time-like
Bhabha scattering $e^-e^+ \to e^-e^+$			$\frac{s^2+u^2}{t^2}$ +	$\frac{2u^2}{ts}$ +	$\frac{u^2+t^2}{s^2}$
$e^-\mu^- \to e^-\mu^-$			$\frac{s^2+u^2}{t^2}$	Rutherford	
(Crossing $s \leftrightarrow t$) $e^-e^+ \to \mu^-\mu^+$					$\frac{u^2+t^2}{s^2}$

4. Higher orders

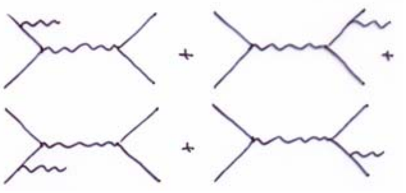
- Important to compare QED predictions to measurements: Test of QED

4.1 Radiative corrections to "Born" / "tree" level predictions



Born diagram

$$\frac{d\sigma^0_{f\bar{f}}}{d\Omega}$$

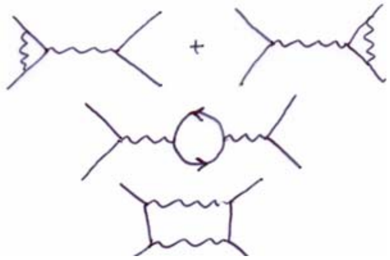


Different final state cannot be distinguished for soft photons

Bremsstrahlung corrections

$$\frac{d\sigma^{Brems}_{f\bar{f}\gamma}}{d\Omega}$$

Soft and **hard** photons



Virtual corrections:

$$\frac{d\sigma_{f\bar{f}}^{Virtual}}{d\Omega}$$

- Vertex corrections
- Propagator corrections
- Box corrections

Experimental cross section has to be compared to "**Born+corrections**":

$$\boxed{\left(\frac{d\sigma_{ff(\gamma)}}{d\Omega}\right)_{\exp} \Leftrightarrow \frac{d\sigma_{ff}^{0}}{d\Omega}\left(1+\delta_{Brems}+\delta_{Virtual}\right)}$$

Remark: at higher energies also electro-weak corrections are important

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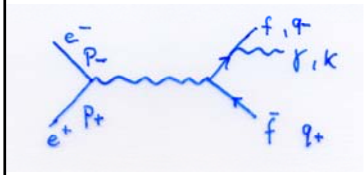
4.2 Bremsstrahlung

a.) soft photon radiation: $E_\gamma < \Delta E << \sqrt{s}$

No influence on ff prod./kinematics



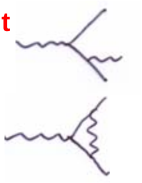
$$\frac{d\sigma_{f\bar{f}\gamma}}{d\Omega} = \frac{d\sigma_{f\bar{f}}^{0}}{d\Omega}\cdot \underbrace{R(p_+,p_-,q_+,q_-,k)\cdot k^0 dk^0\, d\Omega_\gamma}$$

Radiative corrections factorize:

$$\sim \frac{2\alpha}{\pi}\log(\frac{s}{m_f^2})\frac{dk^0}{k^0}$$

Problems:

- Corrections are divergent for $k^0 \to 0$ ($E_\gamma \to 0$): **infra-red divergent**
- If $E_\gamma < \Delta E$=detection threshold: $e^+e^- \to f\bar{f}\gamma \quad \Leftrightarrow \quad e^+e^- \to f\bar{f}$
- Vertex corrections to $e^+e^- \to f\bar{f}$ are also divergent



⇒Treat vertex + bremsstrahlung corrections at the same time:

$$\left.\frac{d\sigma_{f\bar{f}}}{d\Omega}\right|^{1st\ order} = \frac{d\sigma_{f\bar{f}}^{0}}{d\Omega}\cdot\left[\beta(s,m_e,m_f)\cdot\log\frac{\Delta E}{\sqrt{s}}+\dots\right]$$

⬅ Divergences cancel in sum

b.) hard photon radiation: $E_\gamma > \Delta E$

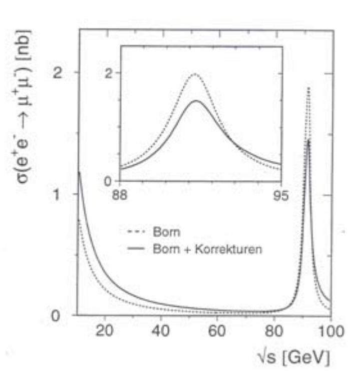
Final state with a detectable photon: $f\,\bar{f}\,\gamma$

$\Rightarrow$ Photon changes the kinematics and also production cross sections:

Initial state radiation (ISR): $\Rightarrow$ reduced effective CMS energy s’=zs

$$\boxed{\sigma_{ff(\gamma)} = \int_{4m_f^2/s}^{1} G(z)\sigma_{ff}^0(zs)\,dz} \qquad z = 1 - \frac{2E_\gamma}{\sqrt{s}}$$

Radiator function G(z) describes photon radiation $\Rightarrow$ large effects if σ_{ff}^0 has large s dependence.



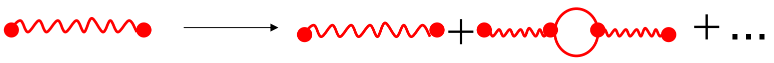
Final state radiation (FSR):

$$\sigma_{ff(\gamma)} = \sigma_{ff}^0 (1 + \frac{3\alpha}{4\pi} + ...) \approx 1.0017 \cdot \sigma_{ff}^0$$

4.3 Propagator corrections and running couplings

$\rightarrow$ Leads to a change of the effective coupling constant

Propagator



$$\underbrace{\frac{e^2}{q^2}}_{} \longrightarrow \frac{e^2}{q^2}\left[1 - \Pi^\gamma(q^2)\right]$$

divergent

$$\underbrace{e}_{\text{finite results}} \longrightarrow \underbrace{e\left[1 - \frac{1}{2}\Pi^\gamma(q^2)\right]}_{\text{divergent (2}^{\text{nd}}\text{ order)}}$$

keeping terms in same O(α)

finite results

divergent (2nd order)

Thomson limit $q^2 \rightarrow 0$: effective charge should be equal to “e”
Measurement includes of course all orders !

Is the “charge e” in the Feynman diagrams equal to the Thomson value ?

⇒ Redefine the charge used in Feynman rules as “bare” charge **e_0** which is not measurable. The relation between **e_0** and the physical charge **e** depends on the order of the calculation:

$$e_0 = e + \delta e \quad \text{with renormalization condition} \quad \delta e = \frac{1}{2} e\,\Pi^\gamma(0)$$

2nd order

1st order

$$\frac{e_0^{\,2}}{q^2} \longrightarrow$$

$$e_0 = e$$

2nd order

$$\frac{e_0^{\,2}}{q^2}\left[1 - \Pi^\gamma(q^2)\right]$$

Renormalization scale

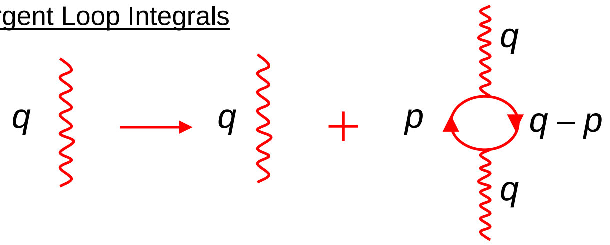
$$e_0 = e\,(1 + \frac{1}{2}\Pi^\gamma(0))$$

Physical charge at renormalization q=0

$$\frac{e^{\,2}}{q^2}\left[1 + \Pi^\gamma(0) - \Pi^\gamma(q^2)\right]$$

(to 1st order in Π)

Remark: Divergent Loop Integrals



$$-i\frac{g_{\mu\nu}}{q^2} \rightarrow -i\frac{g_{\mu\nu}}{q^2} + \frac{(-i)}{q^2} I_{\mu\nu} \frac{(-i)}{q^2}$$

$$I_{\mu\nu}(q^2) = (-1)^n \int \frac{d^4p}{(2\pi)^4}\,\mathrm{Trace}\left\{(ie\gamma^\mu)\frac{i(\not p + m)}{p^2 - m^2}(ie\gamma^\mu)\frac{i(\not p - \not q + m)}{(p-q)^2 - m^2}\right\}$$

Integral over all possible momenta p:

Logarithmically divergent

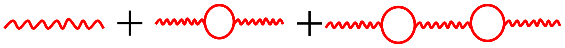
$$\frac{e_0^2}{q^2} \longrightarrow \frac{e^2}{q^2}\left[1+\Pi^\gamma(0)-\Pi^\gamma(q^2)\right]$$

(to 1st order in $\Pi(q^2)$)

$$= \frac{e^2}{q^2}\left[1-\hat{\Pi}^\gamma(q^2)\right] \quad \text{with} \quad \underset{\substack{\uparrow \\ \text{finite}}}{\hat{\Pi}^\gamma(q^2)} = \underset{\substack{\uparrow \\ \text{divergent}}}{\Pi^\gamma(q^2)} - \underset{\substack{\uparrow \\ }}{\Pi^\gamma(0)}$$

For $q^2 >> m^2$ (fermion masses)

$$\hat{\Pi}^\gamma(q^2) = -\frac{\alpha}{3\pi}\sum_f Q_f^2 \log\frac{q^2}{m_f^2}$$



$$\left[1-\hat{\Pi}^\gamma(q^2)+[\hat{\Pi}^\gamma(q^2)]^2-...\right]=\left[\frac{1}{1+\hat{\Pi}^\gamma(q^2)}\right]$$

(geometric progression)

Physical charge at $q^2=0$

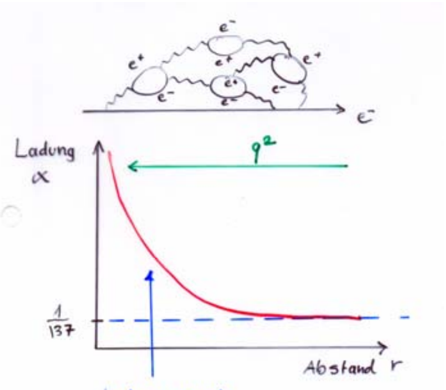
$$\boxed{e^2(q^2) = \frac{e^2}{1+\hat{\Pi}^\gamma(q^2)}}$$

(including higher orders)

Running coupling α

"Screening" at large distances

$$\boxed{\alpha(q^2) = \frac{\alpha}{1+\hat{\Pi}^\gamma(q^2)} = \frac{\alpha}{1-\underbrace{\frac{\alpha}{3\pi}\cdot\sum_f Q_f^2\cdot\log\frac{q^2}{m_f^2}}_{\text{all possible fermions (including quarks)}}}}$$



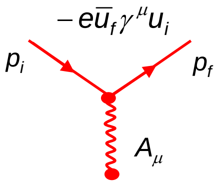
hochenergetische Testladung (→ testet kleine Abstände)

$\alpha(q^2=0) = \frac{1}{137}$ (Thomson limit)

$\alpha(q^2=45^2 GeV^2) = \frac{1}{129}$ (PETRA)

$\alpha(q^2=91^2 GeV^2) = \frac{1}{128}$ (LEP)

4.4 Vertex corrections and anomalous magnetic moment



Interaction due to spin

$$= \frac{e}{2m}\bar{u}_f\left((p_f+p_i)^\mu + \underbrace{i\sigma^{\mu\nu}(p_f-p_i)^\nu\Big)u_i\;A_\mu}\right.$$

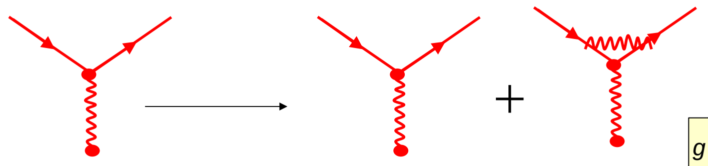
spinless charge

$$\sigma^{\mu\nu}=\frac{i}{2}\left(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu\right)$$

Non-relativistic limit: $\varphi_f^+\left(\frac{e}{2m}\vec{\sigma}\vec{B}\right)\varphi_i$ $\quad u=\begin{pmatrix}\varphi\\ \cancel{\chi}\end{pmatrix}$

Magnetic moment $\vec{\mu}=-2\cdot\mu_B\cdot\vec{s}\quad g=2$

Vertex corrections:



$\vec{\mu}=-2\cdot\mu_B\cdot\vec{s}$

$\vec{\mu}=-(2+\frac{\alpha}{\pi})\cdot\mu_B\cdot\vec{s}$

$g=2+\frac{\alpha}{\pi}$

$a=\frac{g-2}{2}=\frac{\alpha}{2\pi}$

Higher order corrections to g-2

Radiative corrections g-2 are calculated to the 4-loop level:

Feynman Graphs		
O(α)		1
O(α²)		7
O(α³)	analytically	72
O(α⁴)	numerically	891
til O(α⁴)		971

Fig. 8.2 The Feynman graphs which have to be evaluated in computing the a³ corrections the lepton magnetic moments (after Lautrup *et al.* 1972).

Most precise QED prediction.

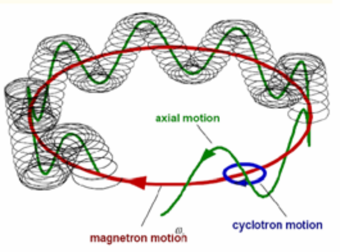
T. Kinoshita et al.

Kinoshita 2006 $a_e = \frac{\alpha}{2\pi} - 0.328...\left(\frac{\alpha}{\pi}\right)^2 + 1.182....\left(\frac{\alpha}{\pi}\right)^3 - 1.505....\left(\frac{\alpha}{\pi}\right)^4$

Kinoshita 2007 $a_e = \frac{\alpha}{2\pi} - 0.328...\left(\frac{\alpha}{\pi}\right)^2 + 1.182....\left(\frac{\alpha}{\pi}\right)^3 - 1.9144....\left(\frac{\alpha}{\pi}\right)^4$

Electron g-2 measurement

H. Dehmelt et al., 1987
G. Gabrielse et al., 2006



Cyclotron frequency $\omega_C = 2\frac{eB}{2mc}$

Spin precession frequency $\omega_S = g\frac{eB}{2mc}$

Experimental method:

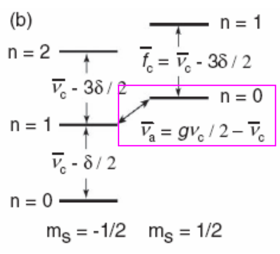
Storage of **single** electrons in a Penning trap (electrical quadrupole + axial B field)

⇒ complicated electron movement (cyclotron and magnetron precessions).

Idea: bound electron:

$$E(n, m_s) = \frac{g}{2}h\nu_c m_s + \left(n + \frac{1}{2}\right)h\bar{\nu}_c - \frac{1}{2}h\delta\left(n + \frac{1}{2} + m_s\right)^2$$

Energy levels of single electron



Trigger RF induced transitions (ω_a)
between different n states or spin flips:

$$\omega_a = \omega_s - \omega_c = (g-2)\mu_B B$$
$$a = \frac{g-2}{2} = \frac{\omega_s - \omega_c}{\omega_c}$$

$$a_{e^-} = 0.001159\,652\,188\,4\,(43)$$
$$a_{e^+} = 0.001159\,652\,187\,9\,(43)$$

H. Dehmelt et al. 1987

$$a_e = 0.001159\,652\,180\,85\,(76)$$

G. Gabrielse et al. 2006

⇒ most precise value of α:

$$\alpha^{-1}(a_e) = 137.035\,999\,710\,(96)$$

For comparison α from Quanten Hall

$$\alpha^{-1}(qH) = 137.036\,003\,00\,(270)$$

$$a_e = \frac{\alpha}{2\pi} - 0.328...\left(\frac{\alpha}{\pi}\right)^2 + 1.182....\left(\frac{\alpha}{\pi}\right)^3 - 1.505....\left(\frac{\alpha}{\pi}\right)^4$$

Theory

$$a_e = 0.001159\,652\,133\,(290)$$

Phys. Rev. Lett. ***97****, 030801 (2006)*
Phys. Rev. Lett. ***97****, 030802 (2006)*

$$a_e = 0.001159\,652\,180\,85\,(76)$$

Muon g-2 experiment

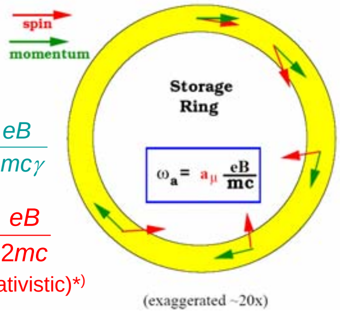
Principle:

- store polarized muons in a storage ring; revolution with cyclotron frequency ω_c
- measure spin precession around the magnetic dipole field relative to the direction of cyclotron motion

$$\omega_c = 2\frac{eB}{2mc\gamma}$$

$$\omega_s = g\frac{eB}{2mc}$$

(non relativistic)*)



Precession:

$$\vec{\omega}_a = -\frac{e}{m_\mu c}\left[\underbrace{a_\mu \vec{B}}_{} - \underbrace{(a_\mu - \frac{1}{\gamma^2-1})\vec{\beta}\times\vec{E}}_{}\right]$$

Difference between Lamor and cyclotron frequency: = spin precession

Effect of electrical focussing fields (relativistic effect).
$= 0$ for $\gamma = 29.3$
⇔ $p_\mu = 3.094$ GeV/c

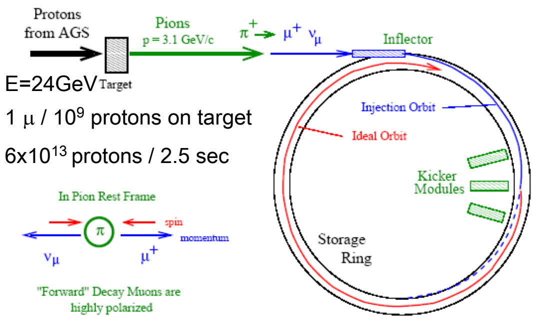
First measurements:

CERN 70s

$$a_{\mu^-} = 0.001165\,937\,(12)$$
$$a_{\mu^+} = 0.001165\,911(11)$$

*) For a relativistic treatment: see Jackson, Classical E-Dynamics

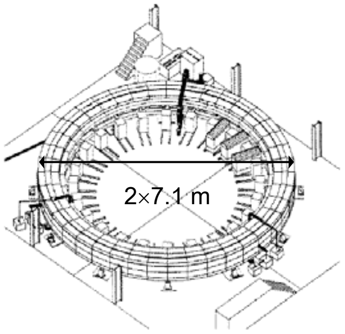
$(g\text{-}2)_\mu$ Experiment at BNL

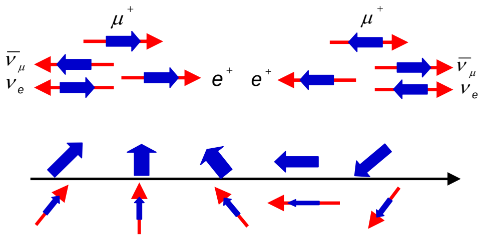


E=24GeV

1 μ / 10^9 protons on target

$6x10^{13}$ protons / 2.5 sec

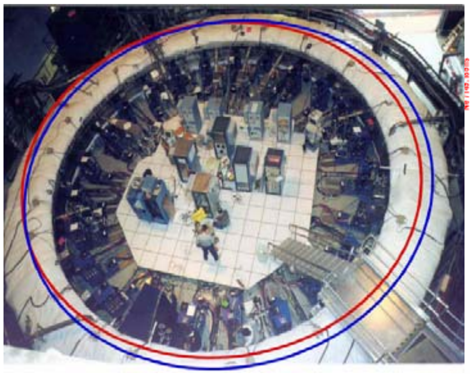




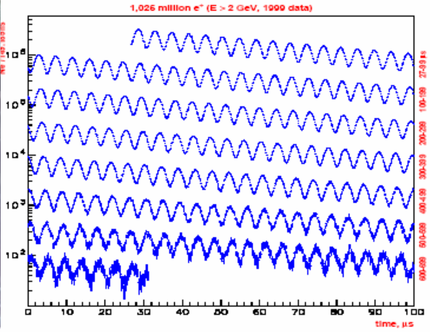
“V-A” structure of weak decay:

Use high-energy e^+ from muon decay to measure the muon polarization

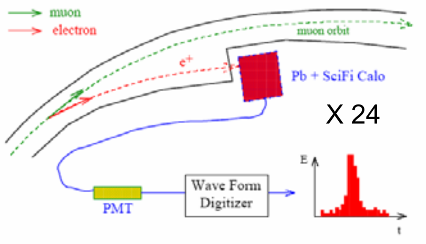
Weak charged current couples to LH fermions (RH anti-fermions)



Measure electron rate:



$$N(t) = N_0\, e^{-\lambda t}\left[1 + A\cos(\omega_a t + \varphi)\right]$$



X 24

$$\frac{\omega_a}{2\pi} = 229023.59(16)\,\mathrm{Hz}$$

(0.7ppm)

$$a_\mu = \frac{\omega_a}{\frac{e}{m_\mu c}\boxed{\langle B\rangle}}\ ?$$

From ω_a to a_μ - How to measure the B field

$<B>$ is determined by measuring the proton nuclear magnetic resonance (NMR) frequency ω_p in the magnetic field.

$$a_\mu = \frac{\omega_a}{\frac{e}{m_\mu c}\langle B\rangle} = \frac{\omega_a}{\frac{e}{m_\mu c}\frac{\hbar\tilde{\omega}_p}{2\mu_p}} = \frac{\omega_a}{\frac{4\mu_\mu}{\hbar g_\mu}\frac{\hbar\tilde{\omega}_p}{2\mu_p}} = \frac{\omega_a/\tilde{\omega}_p}{\mu_\mu/\mu_p}(1+a_\mu)$$

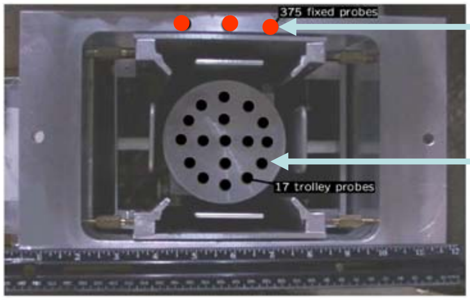
$$\Downarrow$$

$$a_\mu = \frac{\omega_a/\omega_p}{\mu_\mu/\mu_p - \omega_a/\omega_p}$$

$\mu_{\mu^+}/\mu_p = 3.183\,345\,39(10)$

W. Liu *et al.*, Phys. Rev. Lett. **82**, 711 (1999).

NMR trolley

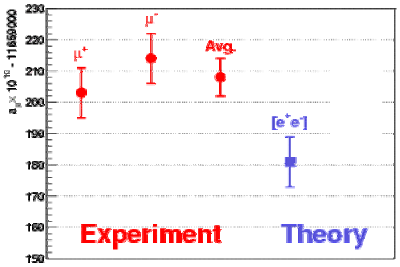


375 fixed NMR probes around the ring

17 trolley NMR probes

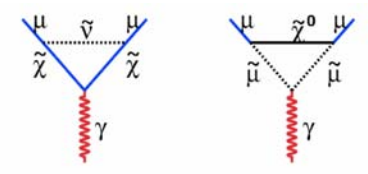
$\tilde{\omega}_p/2\pi$ = 61 791 400(11) Hz (0.2ppm)

$$a_{\mu^+} = 11659\,203(8) \times 10^{-10} (0.7 ppm)$$
$$a_{\mu^-} = 11659\,214(8) \times 10^{-10} (0.7 ppm)$$
$$a_{\mu} = 11659\,208(6) \times 10^{-10} (0.5 ppm)$$



About 2.6σ deviation:

- Often interpreted as sign of New Physics:
  SUSY contributions
- careful:
  "Theory" has uncertainties!



Potential SUSY contributions:

Remarks: Theoretical prediction of a_μ

Beside pure QED corrections there are weak corrections (W, Z) exchange and „hadronic corrections“

$$a_\mu = a_\mu^{QED} + a_\mu^{Had} + a_\mu^{EW}$$

(For the electron with much lower mass the hadronic and weak corrections are suppressed, and can be neglected.)

→ Determination of hadronic corrections is difficult and is in addition based on data: hot discussion amongst theoreticians how to correctly use the data.

Hadronic corrections