III. QED for "pedestrians"

1. Dirac equation for spin ½ particles
2. Quantum-Electrodynamics and Feynman rules
3. Fermion-fermion scattering
4. Higher orders

<u>*Literature:*</u> *F. Halzen, A.D. Martin, "Quarks and Leptons"*
O. Nachtmann, "Elementarteilchenphysik"

1. Dirac Equation for spin ½ particles

Idea: Linear ansatz to obtain a relativistic wave equation w/ linear time derivatives (remove negative energy solutions).

" E = p + m "

$E \rightarrow i\frac{\partial}{\partial t}$

$\vec{p} = -i\vec{\nabla}$

$$E\psi = (\vec{\alpha}\cdot\vec{p} + \beta\cdot m)\psi$$

$$i\frac{\partial}{\partial t}\psi = -i\left(\alpha_1\frac{\partial}{\partial x_1}\psi + \alpha_2\frac{\partial}{\partial x_2}\psi + \alpha_3\frac{\partial}{\partial x_3}\psi\right) + \beta m\psi$$

Solutions should also satisfy the relativistic energy momentum relation:

$$E^2\psi = (\vec{p}^{\,2} + m^2)\psi \qquad \text{(Klein-Gordon Eq.)}$$

This is only the case if coefficients fulfill the relations:

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij}$$
$$\alpha_i \beta + \beta \alpha_j = 0$$
$$\beta^2 = 1$$

➡ Cannot be satisfied by scalar coefficients: Dirac proposed α_i and β being 4×4 matrices working on 4 dim. vectors:

4×4 martices: $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ σ_i are Pauli matrices

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Dirac Equation:

$$i\left(\beta \frac{\partial}{\partial t}\psi + \beta\vec{\alpha}\cdot\vec{\nabla}\psi\right) - m\cdot 1\cdot\psi = 0$$

$$i\left(\gamma^0 \frac{\partial}{\partial t}\psi + \vec{\gamma}\cdot\vec{\nabla}\psi\right) - m\cdot 1\cdot\psi = 0$$

where $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha_i$, i = 1,2,3

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

Solutions ψ describe spin ½ (anti) particles: $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

Extremely compressed description

$$j = 1...4: \quad \sum_{k=1}^{4}\left(\sum_\mu i\cdot(\gamma^\mu)_{jk}\frac{\partial}{\partial x^\mu} - m\delta_{jk}\right)\psi_k$$

1.1 γ Matrices

$$\gamma^0 = \beta \qquad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \ \mathrm{i} = 1...3$$

$$\gamma^i = \beta\alpha_i \ , \ \mathrm{i} = 1,2,3$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 \qquad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Rules

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

$$(\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0, \quad (\gamma^0)^+ = \gamma^0, \quad (\gamma^k)^+ = -\gamma^k, \ k = 1..3$$

$$\gamma^0\gamma^0 = \mathbf{I}, \quad \gamma^k\gamma^k = -\mathbf{I}, \quad k = 1..3 \qquad \gamma^0\gamma^k = -\gamma^k\gamma^0$$

$$\gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0, \quad (\gamma^5)^+ = \gamma^5$$

1.2 Adjoint Equation (hermitian conjugated form)

Dirac eq.: $i\left(\gamma^0 \frac{\partial}{\partial t}\psi + \vec{\gamma}\cdot\vec{\nabla}\psi\right) - m\cdot 1\cdot\psi = 0$

(Dirac eq)$^+$: $i\left(\frac{\partial}{\partial t}\psi^+\gamma^0 + \vec{\nabla}\cdot\psi^+(-\vec{\gamma})\right) + m\cdot 1\cdot\psi^+ = 0$

hermitian conjugate

Introducing the adjoint spinor $\overline{\psi} = \psi^+\gamma^0$ allows to write the hermitian conjugate of the Dirac-Eq. in the covariant form:

$$(i\partial_\mu\gamma^\mu + m)\overline{\psi} = 0$$

→ can be used to derive a continuity equation for a 4-vector current

1.3 Fermion currents and continuity equation

Define fermion current

$$j^\mu = (\overline{\psi}\gamma^\mu\psi)$$

$$\overline{\psi}\gamma^\mu\partial_\mu\psi + (\partial_\mu\overline{\psi})\gamma^\mu\psi = 0$$

$$\Rightarrow \quad \partial_\mu(\overline{\psi}\gamma^\mu\psi) = 0$$

here:
$\rho =$
$j^0 = \overline{\psi}\gamma^0\psi = \psi^+\gamma^0\gamma^0\psi$
$= \psi^+\psi > 0$

Instead of the probability current the charge current is often used:

Electron current: $j_e^\mu = (-e)\cdot(\overline{\psi}\gamma^\mu\psi)$

Boson current: $j^\mu = (-e)\cdot 2p^\mu$ (reminder: Kl-G-Eq)

1.4 Free particle solutions for Dirac Eq.

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0$$

Ansatz: $\psi(x) = u(p)\cdot\exp(\mp ipx)$ for $E = \pm\sqrt{p^2 + m^2}$

4-comp. spinor: $u(p) = \begin{pmatrix}\varphi(p)\\ \chi(p)\end{pmatrix}$ and φ, χ 2-comp. spinors

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = (\pm\underbrace{\gamma^\mu p_\mu} - m)\begin{pmatrix}\varphi\\ \chi\end{pmatrix}\exp(\mp ipx) = 0$$

$$\gamma^\mu p_\mu = \gamma^0 p_0 - \vec{\gamma}\,\vec{p} = E\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix} - \begin{pmatrix}0 & \vec{\sigma}\vec{p}\\ -\vec{\sigma}\vec{p} & 0\end{pmatrix}$$

One obtains two coupled equations for the spinors φ and χ:

$$(E \mp m)\varphi - (\vec{\sigma}\,\vec{p})\chi = 0 \quad (*)$$
$$(E \pm m)\chi - (\vec{\sigma}\,\vec{p})\varphi = 0 \quad (**)$$

Solutions for positive energy: $E = +\sqrt{p^2 + m^2}$ (upper sign)

In this case E+m ≥ 2m, while (E-m)→0 for non relativistic case ⇒ use eq. (∗∗)

$$\chi = \frac{\vec{\sigma}\,\vec{p}}{E+m}\varphi \quad \text{with} \quad \vec{\sigma}\,\vec{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

The spinor φ can be freely selected: $\varphi_1 = N \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\varphi_2 = N \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Eq. (∗) is automatically fulfilled: $(\vec{\sigma}\,\vec{p})\chi = \frac{(\vec{\sigma}\,\vec{p})^2}{E+m}\varphi = \frac{\vec{p}^2}{E+m}\varphi = (E-m)\varphi$

Solutions for positive energy: $E = +\sqrt{p^2 + m^2}$

solution spin ↑

$$u_1(p) = N \cdot \begin{pmatrix} \varphi_1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\varphi_1 \end{pmatrix} = N \cdot \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}$$

solution spin ↓

$$u_2(p) = N \cdot \begin{pmatrix} \varphi_2 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\varphi_2 \end{pmatrix} = N \cdot \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$N = \sqrt{E+m}$ (norm.)

$$u^+ u = 2E$$

1.5 Spin and helicity

- u_1 and u_2 are both solutions to the same energy value
- operator to distinguish the 2 solutions:

Helicity $$\frac{1}{2}\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|} = \frac{1}{2}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} & 0 \\ 0 & \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} \end{pmatrix}$$

$$\left[\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}, H\right] = 0$$

$$H = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & m \end{pmatrix}$$

- u_1 and u_2 are eigenstates of $\frac{1}{2}\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}$ with eigenvalues ±½

- u_1 and u_2 for highly relativistic particles $u_1 = \sqrt{E}\cdot\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$ and $u_2 = \sqrt{E}\cdot\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$

  $E \approx p_z$

  $p_x, p_y \approx 0$