II. Pre-requisites

1. Relativistic kinematics
2. Wave description of free particles
3. Scattering matrix and transition amplitudes
4. Cross section and phase space
5. Decay width, lifetimes and Dalitz plots

Literature: *F. Halzen, A.D. Martin, "Quarks and Leptons"*
O. Nachtmann, "Elementarteilchenphysik"

1. Relativistic kinematics

1.1 Notations

- 4-vector
  - contra-variant form $x^\mu = \left(x^0, \vec{x}\right) = (t, \vec{x}) \quad p^\mu = \left(p^0, \vec{p}\right) = (E, \vec{p})$
  - covariant form $x_\mu = \left(x^0, -\vec{x}\right) = (t, -\vec{x}) \quad p_\mu = \left(p^0, -\vec{p}\right) = (E, -\vec{p})$
- Metric tensor

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad x_\mu = g_{\mu\nu} x^\nu \qquad x^\mu = g^{\mu\nu} x_\nu$$

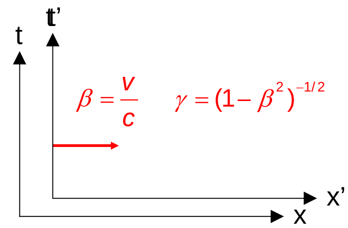
- Derivative operator

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right) \qquad \partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

- Scalar product

$$ab = a_\mu b^\mu = g_{\mu\nu} a^\nu b^\mu = (a^0 b^0 - \vec{a} \cdot \vec{b})$$

1.2 Lorentz invariants



Lorentz transformation:

moving particle with $p = (E, \vec{p})$

$$p' = \begin{cases} \begin{pmatrix} E' \\ \vec{p}_l' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E \\ \vec{p}_l \end{pmatrix} \\ \vec{p}_t' = \vec{p}_t \end{cases}$$

w/r to rest frame: $\beta = \frac{|\vec{p}|}{E}$ $\gamma = \frac{E}{m}$

Scalar products are invariant under Lorentz transformations: $a'b' = ab$

Example 1: invariant mass

$$p^2 = p_\mu p^\mu = E^2 - \vec{p}^2 = m^2$$

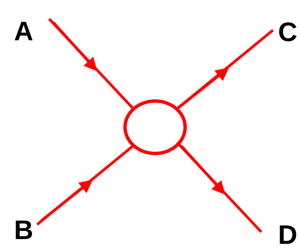
Example 2: CMS energy of 2 particle collision calculated in any frame

$$s = (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$$

Remark: Lorentz invariants (e.g. cross sections) can only depend on scalars

1.3 Mandelstam variables

$$A + B \to C + D$$



(unpolarized particles)

What are the Lorentz scalars the cross section can depend on ?

$p_i p_k$ with $p_{i,k \geq i} = p_A, p_B, p_C, p_D$

10 combinations

$p_i^2 = m_i^2$ 4 constraints

4-mom. conservation: 4 constraints

➡ **2 independent products**

Instead of $p_i p_k$ use 2 out of the 3 **Mandelstam variables**

$$s = (p_A + p_B)^2$$
$$t = (p_A - p_c)^2$$
$$u = (p_A - p_D)^2$$

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$$

2. Wave description of free particles

2.1 Schrödinger Equation for non-relativistic free particles

$$i\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\nabla^2\psi$$

Solution for energy $E = \frac{p^2}{2m}$

$$\psi(\vec{r},t) = \frac{1}{\sqrt{V}}\exp\left[i(\vec{p}\vec{x} - Et)\right]$$

Continuity equation:

$$\rho = |\psi|^2$$

$$\vec{j} = \frac{1}{2im}\left(\psi^*(\nabla\psi) - (\nabla\psi^*)\psi\right)$$

$$\frac{\partial\rho}{\partial t} + \nabla\vec{j} = 0$$

Schrödinger Eq uses classical E-p relation $E^2=p^2/2m$ and the replacement $E \to i\frac{\partial}{\partial t}$ and $\vec{p} = -i\vec{\nabla}$

2.2 Klein-Gordon Equation

Starts from relativistic energy relation $E^2=p^2+m^2$:

Describes relativistic Spin 0 particles

$$\frac{\partial^2}{\partial t^2}\phi - \nabla^2\phi + m^2\phi = 0$$

Solutions for energy values:

$$E_\pm = \pm\sqrt{p^2+m^2} \gtrless 0$$

$$\phi(\vec{r},t) = N\exp\left[i(\vec{p}\vec{x} - E_\pm t)\right]$$

negative E values cannot be ignored as otherwise solutions are incomplete

with $\rho = \left(i\phi^*\frac{\partial}{\partial t}\phi - i\phi\frac{\partial}{\partial t}\phi^*\right)$ and $\vec{j} = \left(-i\phi^*(\nabla\phi) - i\phi(\nabla\phi^*)\right)$

Continuity equation: $\frac{\partial\rho}{\partial t} + \nabla\vec{j} = 0$

For the solution: $\phi(\vec{r},t) = N \exp\left[i(\vec{p}\vec{x} - E_{\pm}t)\right]$

$$\vec{j} = \left(-i\phi^*(\nabla\phi) - i\phi(\nabla\phi^*)\right) \qquad \vec{j} = 2\vec{p}|N|^2$$

$$\rho = \left(i\phi^* \frac{\partial}{\partial t}\phi - i\phi\frac{\partial}{\partial t}\phi^*\right) \qquad \rho = 2E|N|^2$$

What are negative probabilities for the E < 0 solutions ?

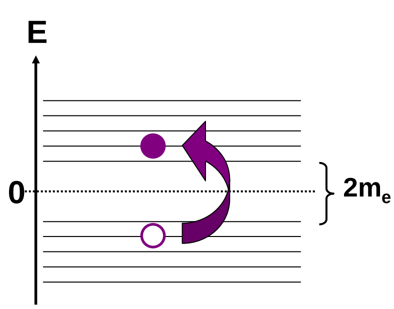
Normalization schemes:

N = 1/√(2EV) ⇒ 1 particle per unit volume V

N = 1/√V ⇒ 2E particles per unit volume V

2.3 Anti-particles

Dirac interpretation for fermions: **Vacuum = sea of occupied neg. E levels**



For fermions the negative energy levels are w/o influence as long as they are fully occupied

Missing e⁻ w/ negative energy corresponds to to a positron w/ E>0

<u>e^+e^- annihilation:</u>

Free energy level in the sea. e^- drops into the hole and releases energy by photon emission: $E_\gamma > 2m_e$

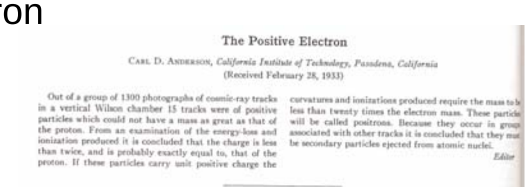
<u>Photon conversion for $E_\gamma > 2m_e$</u>

Excitation of e^- from neg. energy level to pos. level: $\gamma \to e^+e^-$

Model predicts anti-particles (Discovery of positron by Anderson in 1933)

Discovery of the positron

Anderson, 1933



The Positive Electron

Carl D. Anderson, *California Institute of Technology, Pasadena, California*

(Received February 28, 1933)

Out of a group of 1300 photographs of cosmic-ray tracks in a vertical Wilson chamber 15 tracks were of positive particles which could not have a mass as great as that of the proton. From an examination of the energy-loss and ionization produced it is concluded that the charge is less than twice, and is probably exactly equal to, that of the proton. If these particles carry unit positive charge the curvatures and ionizations produced require the mass to b less than twenty times the electron mass. These particle will be called positrons. Because they occur in group associated with other tracks it is concluded that they mu be secondary particles ejected from atomic nuclei.

Edito

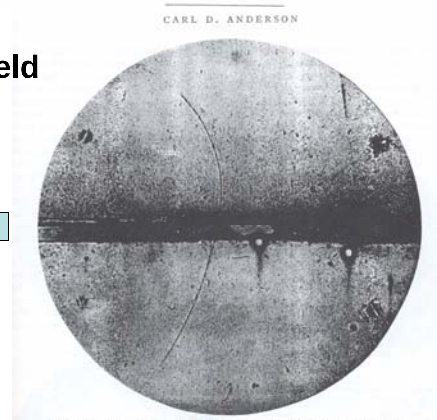
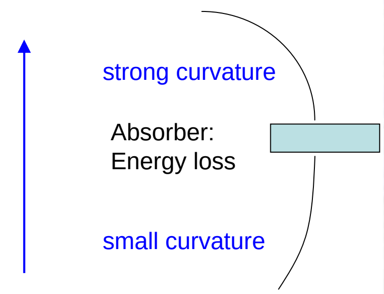


Fig. 1. A 63 million volt positron ($H\rho = 2.1\times10^5$ gauss-cm) passing through a 6 mm lead plate and emerging as a 23 million volt positron ($H\rho = 7.5\times10^4$ gauss-cm). The length of this latter path is at least ten times greater than the possible length of a proton path of this curvature.

⊗ B field



$$\vec{F}_L = q\vec{v} \times \vec{B}$$

$$\Rightarrow \quad q = +e$$

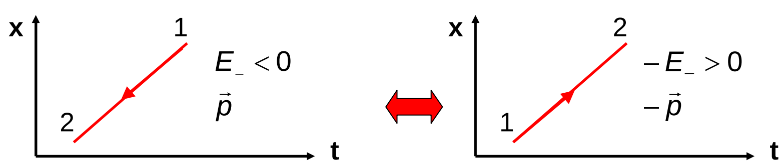
Feynman Stückelberg interpretation

$$E_+ = E: \quad \phi_+ = N\exp(i\vec{p}\vec{x} - iEt)$$

Solutions with neg. energy propagate backwards in time: $\Rightarrow$

$$E_- = -E: \quad \phi_- = N\exp(i\vec{p}\vec{x} + iEt)$$

Solutions describe anti-particles propagating forward in time:



$E_- < 0$, $\vec{p}$ $\Leftrightarrow$ $-E_- > 0$, $-\vec{p}$

Neg. probability density

$$\left.\begin{array}{l} \rho = 2E|N|^2 \\ \vec{j} = 2\vec{p}|N|^2 \end{array}\right\} \times q \Rightarrow \begin{array}{l} J^0 = q \cdot 2E|N|^2 \\ \vec{J} = q \cdot 2\vec{p}|N|^2 \end{array}$$

Charge density / currents

Example

Particle T^- with q= –e and energy E_- = –E < 0

$$J^0(T^-) = (-e)\cdot 2(-E)|N|^2 = (+e)\cdot 2(+E)|N|^2 = J^0(T^+)$$

$$\vec{J}(T^-) = (-e)\cdot 2\vec{p}|N|^2 = (+e)\cdot 2(-\vec{p})|N|^2 = \underbrace{\vec{J}(T^+)}$$

$$T^+ \text{ with } E(T^+) > 0,\ \vec{p}_{T^+} = -\vec{p}_{T^-}$$

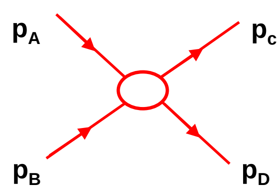
Description of creation and annihilation:

- Emission of anti-particle $\bar{T}$ with p^μ = (E, **p**) ⇔ absorption of particle T with p^μ = (-E, **-p**)
- Absorption of anti-particle $\bar{T}$ with p^μ = (E, **p**) ⇔ emission of T with p^μ = (-E, **-p**)

3. Scattering matrix and transition amplitude

Scattering process:

$$\pi p \rightarrow \pi p$$



Described through quantum numbers of initial and final state: $|i\rangle \rightarrow |i'\rangle$

Scattering operator (S matrix): $|i'\rangle = \mathbf{S}|i\rangle$

Measurement selects a specific state f. Probability to find f: $\langle f|i'\rangle = \langle f|\mathbf{S}|i\rangle = \mathbf{S}_{fi}$

As there is the probability that $|i'\rangle = |i\rangle$ it is useful to introduce the transition operator T

$$\mathbf{S} = \mathbf{1} + \mathbf{T} \quad \text{with} \quad \mathrm{T}_{fi} = \langle f|\mathbf{T}|i\rangle$$

Instead of T_{fi} , conventionally one uses the transition or scattering amplitude M_{fi}

$$T_{fi} = -i \cdot (2\pi)^4 \underbrace{N_A N_B N_C N_D}_{\text{normalization}} \underbrace{\delta^4(p_A + p_B - p_C - p_D)}_{\text{4-momentum conservation}} \cdot M_{fi}$$

normalization:
$N_k = 1/\sqrt{V} \rightarrow 2E$ particles/V

4-momentum conservation

M_{fi}: Feynman rules for calculation

Transition probability:

$$w_{fi} = |T_{fi}|^2 = (2\pi)^8 (N_A N_B N_C N_D)^2 [\delta^4(p_A + p_B - p_C - p_D)]^2 |M_{fi}|^2$$

In this convention the transition probability is given for a single „possible final state“. It turns out that the final state particles $C(p_C)$ and $D(p_D)$ can be in more than one state. The number of possible final states is described by the **phase space factor** and will be considered when calculating observable quantites.

Transition rate per unit volume:

$$W_{fi} = \frac{|T_{fi}|^2}{T \cdot V} = \frac{1}{T \cdot V} (2\pi)^8 (N_A N_B N_C N_D)^2 [\delta^4(p_A + p_B - p_C - p_D)]^2 |M_{fi}|^2$$

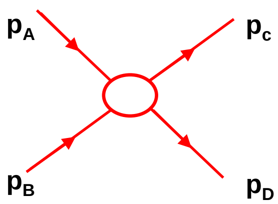
$$= (2\pi)^4 \frac{1}{V^4} \delta^4(p_A + p_B - p_C - p_D) |M_{fi}|^2$$

Fermi‘s Trick:

$$[\delta^4(p_A + p_B - p_C - p_D)]^2 = \frac{VT}{(2\pi)^4} \delta^4(p_A + p_B - p_C - p_D)$$

4. Cross section and phase space

$$A + B \rightarrow C + D$$



Transition rate:

$$W_{fi} = \frac{(2\pi)^4}{V^4} \delta^4(p_A + p_B - p_C - p_D) \cdot |M_{fi}|^2$$

$$\Rightarrow \text{Cross section} = \frac{W_{fi}}{\text{(initial flux)}} \text{(number of final states)}$$

Cross section: $$\sigma = \frac{W_{fi}}{F} \rho_f(C,D)$$

ρ_f number of final states for given configuration
F incident particle flux of A and B

4.1 Number of final states (phase space)

Quantum mechanics restricts the number of final states $d\rho_f$ of a single particle in a volume V with momentum $\in [\vec{p}, \vec{p} + d\vec{p}]$

$$d\rho_f = \frac{Vd^3p}{2E\,h^3} = \frac{Vd^3p}{2E\,\hbar^3 (2\pi)^3} \overset{\hbar = 1}{=} \frac{Vd^3p}{2E\,(2\pi)^3}$$

Factor 2E is the result of normalization of the wave function: 2E particles / V

For particle C and D scattered into momentum elements d^3p_C and d^3p_D

$$d\rho_f(C,D) = \frac{Vd^3p_C}{2E_C(2\pi)^3} \frac{Vd^3p_D}{2E_D(2\pi)^3}$$

4.2 Incident particle flux F

Choose rest frame of particle B to calculate F (simplest)

F = (flux density A) × (density B)

$A \xrightarrow{\vec{v}_A} \bullet\ B$

$$F = |\vec{v}_A| \frac{2E_A}{V} \cdot \frac{2E_B}{V}$$

with $\vec{v}_A = \frac{\vec{p}_A}{E_A}$

<u>CMS frame:</u>

$A \xrightarrow{\vec{p}_A} \xleftarrow{\vec{p}_B} B$

$\vec{p}_A = -\vec{p}_B = \vec{p}_i$

$$F = \frac{4}{V^2}|\vec{p}_i| \cdot (E_A + E_B) = \frac{4}{V^2}|\vec{p}_i|\sqrt{s}$$

<u>General form:</u>

$$F = \frac{2w(s,m_1^2,m_2^2)}{V^2}$$

(see Nachtmann)

with $w(s,m_1,m_2) = \left((s-(m_1+m_2)^2)(s-(m_1-m_2)^2)\right)^{1/2}$

4.3 Lorentz invariant phase space factor

Putting everything together $d\sigma = \frac{W_{fi}}{F} d\rho_f$

$$d\sigma = \frac{(2\pi)^4}{V^4}\delta^4(p_A + p_B - p_C - p_D) \cdot |M_{fi}|^2 \frac{V^2}{|\vec{v}_A| 2E_A 2E_B} \cdot \frac{V d^3 p_c}{2E_C (2\pi)^3} \cdot \frac{V d^3 p_D}{2E_D (2\pi)^3}$$

$$= \frac{|M_{fi}|^2}{|\vec{v}_A| 2E_A 2E_B} \cdot (2\pi)^4 \delta^4(p_A + p_B - p_C - p_D) \cdot \frac{d^3 p_c}{2E_C (2\pi)^3} \cdot \frac{d^3 p_D}{2E_D (2\pi)^3}$$

$= 4|\vec{p}_i|\sqrt{s}$ CMS

Particle flux ***F***

Lorentz invariant 2-particle phase space factor $d\Phi_2$

Remark: volume V drops out !

Phase space factor for n particles in the final state:

$$d\Phi_n(\underbrace{P, p_1, p_2, \ldots, p_n}_{\text{Final state}}) = (2\pi)^4 \delta^4(P - (p_1 + p_2 + \ldots + p_n)) \prod_{final} \frac{d^3 p_f}{(2\pi)^3 2E_f}$$

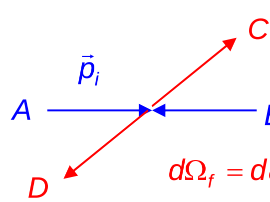
See also PDG
http://pdg.lbl.gov/2007/reviews/kinemarpp.pdf

Phase space integration for two-particles final-state (CMS)

CM System:

$\vec{p}_i = \vec{p}_A = -\vec{p}_B \quad \vec{p}_f = \vec{p}_C = -\vec{p}_D$

$s = (E_A + E_B)^2$



$\vec{p}_f$, $\vec{p}_i$

$d\Omega_f = d\varphi_C d\cos\theta_C$

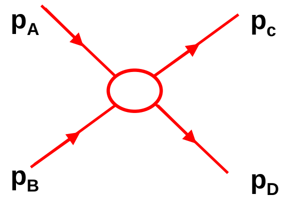
$$d\Phi_2 \xrightarrow{\int}$$

$$\int d\Phi_2 = \frac{1}{4\pi^2} \int \delta^3(\vec{p}_C + \vec{p}_D)\delta(E_A + E_B - E_C - E_D) \frac{d^3 p_C}{2E_C} \frac{d^3 p_D}{2E_D}$$

$$\int d\Phi_2 = \frac{1}{16\pi^2} \int \frac{|\vec{p}_f|}{\sqrt{s}} d\Omega_f$$

4.4 Differential cross section …putting everything together

$$A + B \to C + D$$



CMS

$$d\sigma = \frac{|M_{fi}|^2}{F} d\Phi_2 = \frac{1}{64\pi^2} \cdot \frac{1}{s} \cdot \frac{|\vec{p}_f|}{|\vec{p}_i|} \cdot |M_{fi}|^2 d\Omega_f$$

$$\frac{d\sigma}{d\Omega_f} = \frac{1}{64\pi^2} \cdot \frac{1}{s} \cdot \frac{|\vec{p}_f|}{|\vec{p}_i|} \cdot |M_{fi}|^2$$

- The dynamics of the scattering process is contained in the matrix element M_{fi} which can be calculated using Feynman rules
- 1/s dependence of the cross section because of initial/final state kinematics