2. CKM-Matrix

The origine of the CKM-matrix are the non-diagonal Yukawa coupling matrices for u and d quarks. By diagonalizing these matrices the off-diagonal elements appear as CKM matrix in the charged currents. The 3x3 unitary CKM matrix has 3 real parameters and 1 phase which is the origine of CP violation in the quark sector.

→ Nobel price for Kobayashi and Maskawa in 2008.

2.1 Parametrizations of the CKM matrix

A convinient parametrization also adopted by the PDG uses 3 Euler angles and 1 phase parameter:

$$V_{CKM} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}$$

where $c_{ij} = \cos\theta_{ij}$ $s_{ij} = \sin\theta_{ij}$ → Fig, with multiplied matrix.

The phase in this convention is chosen to appear in the matrix to describe the mixing between 1st and 3rd generation. Size:

$$|V_{CKM}| = \begin{pmatrix} 0.97428 & 0.2253 & 0.00347 \\ 0.2252 & 0.97345 & 0.0410 \\ 0.00862 & 0.0403 & 0.99915 \end{pmatrix}$$

The strengths of the couplings V_{ij} exhibit a hierarchy:

$$|V_{CKM}| = \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}$$

This pattern motivated Wolfenstein to parametrize the CKM in powers of $\lambda \propto \sin\theta_{12}$ in the PDG convention.

Wolfenstein parametrization → usefull for quantitative discussion of many quark mixing effects

approximate parametrization: $\sin\theta_{12} \simeq \lambda$

$$\sin\theta_{23} \simeq A\lambda^2$$

$$\sin\theta_{13} e^{-i\delta_{13}} \simeq A\lambda^3(\rho - i\eta)$$

CKM matrix in $\mathcal{O}(\lambda^3)$:

$$V_{CKM} = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ \boxed{A\lambda^3(1-\rho-i\eta)} & -A\lambda^2 & 1 \end{pmatrix} + \delta V(\mathcal{O}(\lambda^4))$$

where A, ρ, η are of $\mathcal{O}(1)$.
Up to $\mathcal{O}(\lambda^3)$ only V_{td} and V_{ub} are complex!
Matrix element V_{ts} becomes complex only at $\mathcal{O}(\lambda^4)$

2.2 Unitarity relations and resulting constraints

From the unitarity conditions $V^\dagger V = 1$ or $VV^\dagger = 1$ one obtains the following 12 equations to be fullfilled:

(1) $$V^*_{ud} V_{ud} + V^*_{us} V_{us} + V^*_{ub} V_{ub} = 1$$

$\quad$ c c $\quad$ c c $\quad$ c c
$\quad$ t t $\quad$ t t $\quad$ t t

$$V^*_{ud} V_{ud} + V^*_{cd} V_{cd} + V^*_{td} V_{td} = 1$$

$\quad$ s s $\quad$ s s $\quad$ s s
$\quad$ b b $\quad$ b b $\quad$ b b

- Weak universality: u couples to the sum of $\sum_i d_i$ the same way as c and t as well as leptons!
- the sum of $u \to d + u \to s + u \to b$ add up to 1 → no space for another decay!

(2) orthogonality relations:

$$V_{ud}^* V_{cd} + V_{us}^* V_{cs} + V_{ub}^* V_{cb} = 0$$

B → ud td   us ts   ub tb = 0

cd td   cs ts   cb tb = 0

$$V_{ud} V_{us}^* + V_{cd} V_{cs}^* + V_{td} V_{ts}^* = 0$$

A → ud ub   cd cb   td tb =

us ub   cs cb   ts tb = 0

These relations describe triangular relations in the complex plane
It is very instructive to study 2 of these relations in more detail:

A: $V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0$

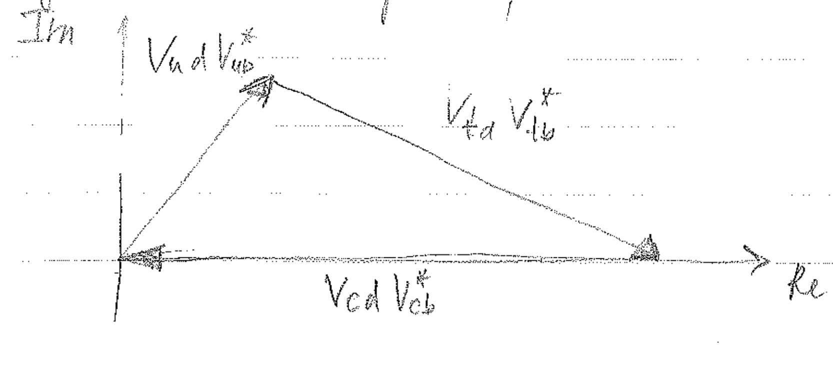
$\quad\mathcal{O}(\lambda^3) \quad \mathcal{O}(\lambda^3) \quad \mathcal{O}(\lambda^3)$

B: $V_{ud}^* V_{td} + V_{us}^* V_{ts} + V_{ub}^* V_{tb} = 0$

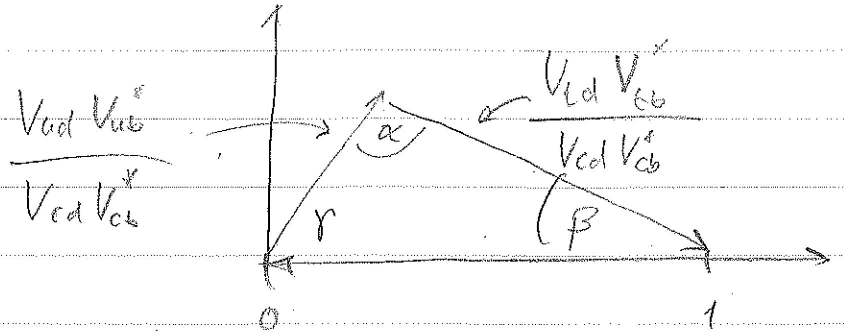
$\quad\mathcal{O}(\lambda^3) \quad \mathcal{O}(\lambda^3) \quad \mathcal{O}(\lambda^3)$

These are the only two „tri-angle" relations where all sides have approx. the same size $\mathcal{O}(\lambda^3)$. One can draw the relations as triangles in the complex plane:

A:



The other relations also describe triangles which are, however, squashed. By dividing in A all sides by $V_{cd}^{*}V_{cb}$ ($\simeq$ real) one obtains the Unitarity Triangle (renormalizing & rephasing)



The apex of the triangle $(\bar{\rho}, \bar{\eta})$ is given by:

$$\bar{\rho} + i\bar{\eta} = \frac{V_{ud}V_{ub}^{*}}{V_{cd}V_{cb}^{*}}$$

In term of the Wolfenstein Parameters ρ and η the apex is given by:

$$\bar{\rho} = \rho\left(1-\tfrac{1}{2}\lambda^2\right) + \mathcal{O}(\lambda^4) \qquad \bar{\eta} = \eta\left(1-\tfrac{1}{2}\lambda^2\right) + \mathcal{O}(\lambda^4)$$

The angles α, β, γ of the unitarity triangle are defined as:

$$(*) \quad \alpha \equiv \arg\left[-\frac{V_{td}V_{tb}^{*}}{V_{ud}V_{ub}^{*}}\right] \quad \beta \equiv \arg\left[-\frac{V_{cd}V_{cb}^{*}}{V_{td}V_{tb}^{*}}\right] \quad \gamma \equiv \arg\left[-\frac{V_{ud}V_{ub}^{*}}{V_{cd}V_{cb}^{*}}\right]$$

In the Wolfenstein Parametrization a phase convention is used such that V_{td}, V_{ub} and (V_{ts}) have an imaginary component (to order $\mathcal{O}(\lambda^4)$) and $V_{cd}V_{cb}^{*}$ is real!

Therefore one finds: $\beta \simeq -\arg(V_{td})$

$$\gamma = -\arg(V_{ub})$$

(for the unitarity triangle B $\quad \beta_s \simeq \arg(V_{ts}) + \pi$)

With these phases one can rewrite the CKM matrix using the Wolfenstein phase convention:

$$V_{CKM} = \begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}|e^{-i\gamma} \\ -|V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}|e^{-i\beta} & -|V_{ts}|e^{i\beta_s} & |V_{tb}| \end{pmatrix} + \mathcal{O}(\lambda^5)$$

While the specific appearance of the phases is a convention the parameters β, γ and β_s (as defined in (*)) are in principle rephasing invariant parameters, independent of the CKM parametrization

Another „rephasing" invariant variable is the Jarlskog Invariant

$$J = \pm \mathrm{Im}\left(V_{ij} V_{kl} V_{il}^* V_{kj}^* \right) \qquad i \neq k,\ j \neq l$$

$$\left[\ J = \mathrm{Im}\left(V_{ud} V_{cs} V_{us}^* V_{cd}^* \right) = -\mathrm{Im}\left(V_{ud} V_{cb} V_{ub}^* V_{cd}^* \right) = \dots \right]$$

Using the Wolfenstein parametrization:

$$J \approx A^2 \lambda^6 \eta \approx 3.03 \cdot 10^{-5} \qquad \text{(CKM-Fitter 2013)}$$

, or with the PDG parametrization

$$J = c_{12}\, c_{13}^2\, c_{23}\, s_{12}\, s_{13}\, s_{23} \sin\delta_{13}$$

Jarlskog Invariant describes the CP violation in the SM. It appears in all CP violating effects. It is zero if one of the mixing angle vanishes ($\theta_{ij} = 0$) or the phase vanishes ($\delta_{23} = 0$).

All the Jarlskog terms are products of the type

$$\mathrm{Im}\, A B^* = |A|\,|B|\, \mathrm{Im}\left(e^{i(\underbrace{\arg(A) - \arg(B)}_{\delta})} \right)$$

$$= |A|\,|B| \sin(\delta)$$

$\rightarrow$ = twice the area of a triangle in the $\mathbb{C}$-plane

Since the A and B are the sides of the unitarity triangle

$$\Rightarrow \quad \boxed{J = 2 \cdot A_{UT}}$$

Status of the Unitarity Triangle : Fig: CKM Triangle

$\lambda = 0.22457^{+0.00186}_{-0.00014}$

$A = 0.823^{+0.012}_{-0.033}$ CKM-Fitter 2013

$\bar{\rho} = 0.1289^{+0.0176}_{-0.0094}$

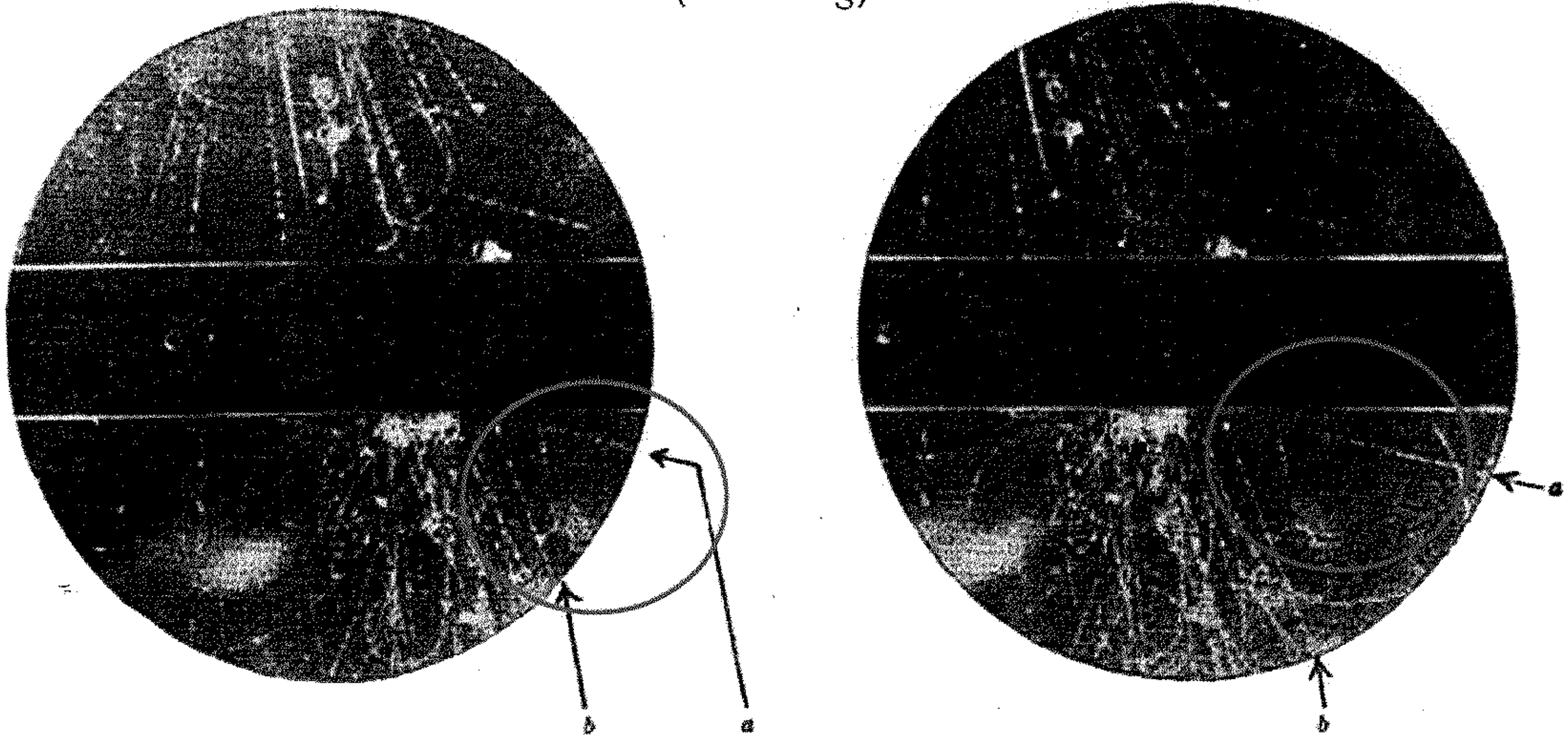
$\bar{\eta} = 0.348 \pm 0.012$

Discovery of the K_s

V^0 (now K^0_S)

Lead

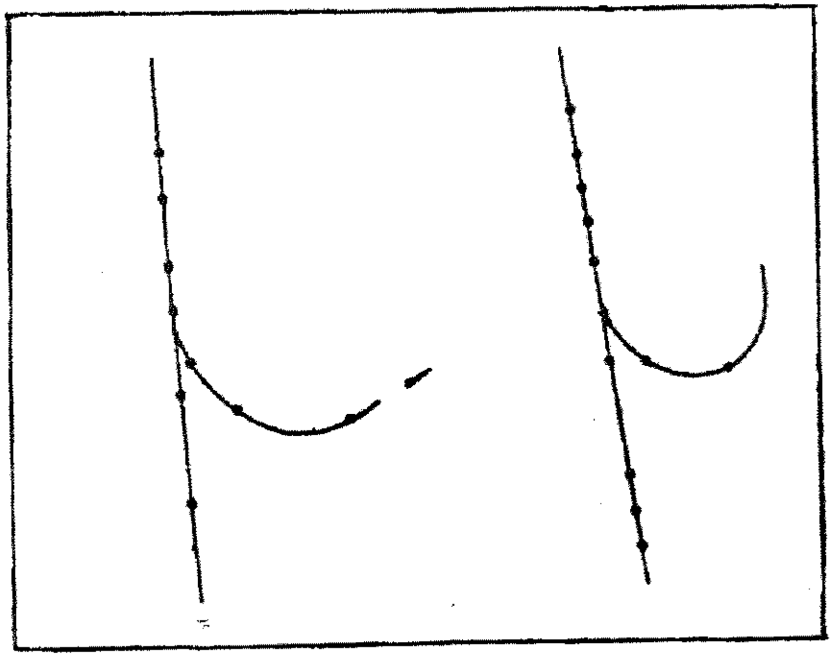
Gas



Rochester & Butler, 1947 in a cloud chamber exposed to cosmic rays

" **Forked tracks of a very striking character** "

Indications 500 MeV particle in 1943

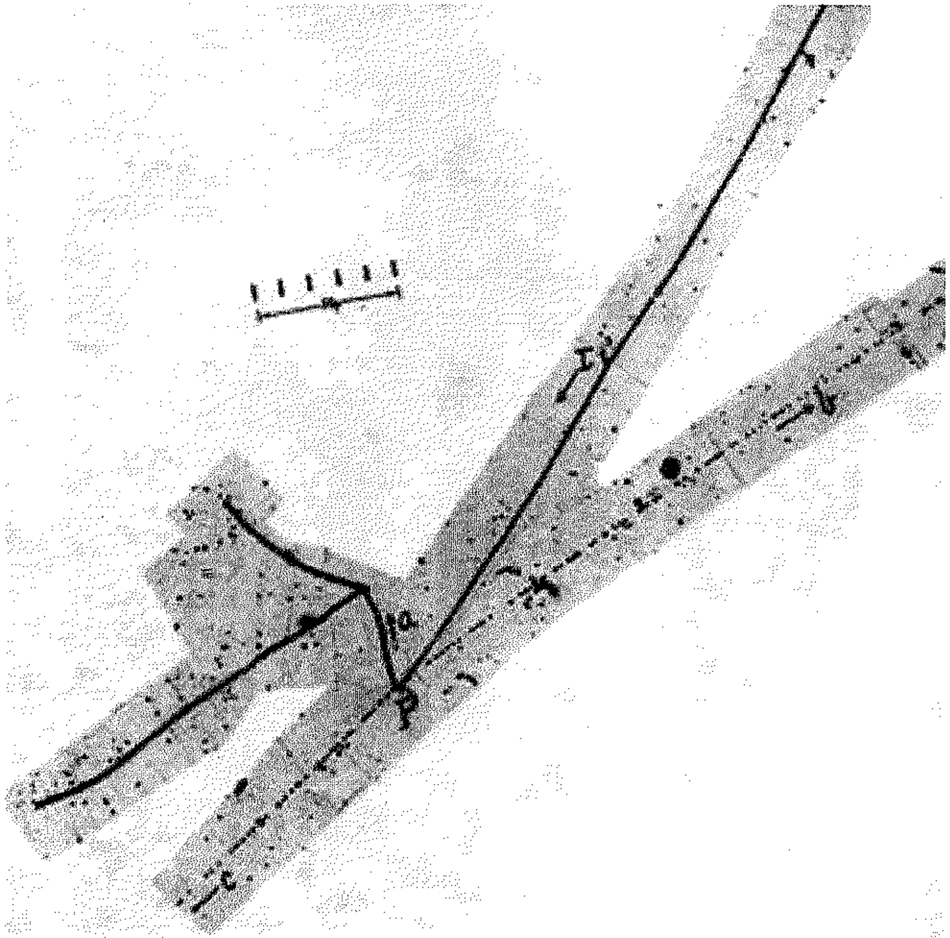


Dessin stéréoscopique de la collision.

Leprince-Ringuet and L'Héritiere, Wilson Chamber, 1943
Scattering of a positively charged particle on an atomic electron. If the scattering is assumed to be **elastic**, the implied mass of the particle is **990 m_e ~ 500 MeV!**

Discovery of $\boxed{K^+} \rightarrow \pi^+ \pi^- \pi^+$

$\boxed{\tau^+}$ historical name



Emulsion technique, Bristol group, 1949

Old Name	New Name
τ	$K_{\pi 3}$: $K^+ \rightarrow \pi^+\pi^+\pi^-$
V_1^0	$\Lambda^0 \rightarrow p\pi^-$
V_2^0 (θ^0)	$K^0_S \rightarrow \pi^+\pi^-$
κ	$K_{\mu 2}$: $K^+ \rightarrow \mu^+\nu$
	$K_{\mu 3}$: $K^+ \rightarrow \mu^+\pi^0\nu$
χ (θ^+)	$K_{\pi 2}$: $K^+ \rightarrow \pi^+\pi^0$
V^+, Λ^+	$\Sigma^+ \rightarrow p\pi^0$, $n\pi^+$

K^0 - $\overline{K^0}$ oscillation

Pure K^0 beam at t=0