cross-section for $e^- e^+ \to \mu^- \mu^+$

- differential cross-section $d\sigma$ per unit volume

$$d\sigma = \frac{1}{T\,F} \frac{d^3p_3}{(2\pi)^3} \frac{1}{2p_3^0} \frac{d^3p_4}{(2\pi)^3} \frac{1}{2p_4^0}$$

$$\times \sum_{\text{spins}} \left| \langle \mu^+(p_4)\mu^-(p_3) \right| S \left| e^+(p_2)e^-(p_1) \rangle \right|^2$$

with incident particle flux F, and unit volume $V = 1$.

cross-section for $e^- e^+ \rightarrow \mu^- \mu^+$

- differential cross-section $d\sigma$ per unit volume

$$d\sigma = \frac{1}{T\,F} \frac{d^3 p_3}{(2\pi)^3} \frac{1}{2p_3^0} \frac{d^3 p_4}{(2\pi)^3} \frac{1}{2p_4^0}$$

$$\times \sum_{\text{spins}} \left| \langle \mu^+(p_4) \mu^-(p_3) | \, S \, | e^+(p_2) e^-(p_1) \rangle \right|^2$$

with incident particle flux F, and unit volume $V = 1$.

S-Matrix element for $e^- e^+ \longrightarrow \mu^- \mu^+$

- S-Matrix:

$$S = T\, e^{ie \int d^4x\, A_\nu(x)\, :\bar{\psi}\gamma^\nu \psi(x):}$$

with

$$\bar{\psi}\gamma^\nu \psi(x) = \bar{\psi}_e \gamma^\nu \psi_e(x) + \bar{\psi}_\mu \gamma^\nu \psi_\mu(x)$$

S-Matrix element for $e^- e^+ \to \mu^- \mu^+$

- S-Matrix:

$$S = T\, e^{ie \int d^4x\, A_\nu(x) :\bar{\psi}\gamma^\nu \psi(x):}$$

- Expansion of S-matrix element for $e^- e^+ \to \mu^- \mu^+$:

$$\langle \mu^+(p_4)\mu^-(p_3)|\, S\, |e^+(p_2)e^-(p_1)\rangle$$

$$= \frac{(ie)^2}{2} \langle \mu^+(p_4)\mu^-(p_3)|\, T \int d^4x\, d^4x' A_\nu(x) A_\mu(x')$$

$$\times :\bar{\psi}\gamma^\nu\psi(x)::\bar{\psi}\gamma^\mu\psi(x'):\ |e^+(p_2)e^-(p_1)\rangle + O(e^4)$$

S-Matrix element for $e^-e^+ \to \mu^-\mu^+$

- S-Matrix:

$$S = T\, e^{ie\int d^4x\, A_\nu(x)\, :\bar\psi\gamma^\nu\psi(x):}$$

- Expansion of S-matrix element for $e^-e^+ \to \mu^-\mu^+$:

$$\langle \mu^+(p_4)\mu^-(p_3)|\, S\, |e^+(p_2)e^-(p_1)\rangle$$

$$= \frac{(ie)^2}{2}\langle \mu^+(p_4)\mu^-(p_3)|\, T\int d^4x\, d^4x' A_\nu(x) A_\mu(x')$$

$$\times :\bar\psi\gamma^\nu\psi(x)::\bar\psi\gamma^\mu\psi(x'): |e^+(p_2)e^-(p_1)\rangle + O(e^4)$$

- states

$$|e^+(p_2)e^-(p_1)\rangle = b_e^\dagger(\vec p_2)a_e^\dagger(\vec p_1)|0\rangle$$

- fermionic field operator

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\sum_{s=\pm 1/2}\left\{e^{ipx}v_s(p)b^\dagger(\vec p) + e^{-ipx}u_s(p)a(\vec p)\right\}$$

S-Matrix element for $e^-e^+ \to \mu^-\mu^+$

- A_μ commutes with $\psi, \bar{\psi}$:

$$\langle \mu^+(p_4)\mu^-(p_3)|\, S\, |e^+(p_2)e^-(p_1)\rangle$$

$$= \frac{(ie)^2}{2} \int d^4x\, d^4x' \langle 0\,|T\, A_\nu(x) A_\mu(x')\,|0\rangle$$

$$\times \langle \mu^+(p_4)\mu^-(p_3)|\, T\, :\bar{\psi}\gamma^\nu\psi(x)::\bar{\psi}\gamma^\mu\psi(x'):\, |e^+(p_2)e^-(p_1)\rangle + O(e^4)$$

S-Matrix element for $e^- e^+ \to \mu^- \mu^+$

- A_μ commutes with $\psi, \bar{\psi}$:

$$\langle \mu^+(p_4)\mu^-(p_3)|\, S\, |e^+(p_2)e^-(p_1)\rangle$$

$$= \frac{(ie)^2}{2} \int d^4x\, d^4x' \langle 0\,|T\, A_\nu(x) A_\mu(x')\,|0\rangle$$

$$\times \langle \mu^+(p_4)\mu^-(p_3)|\ :\bar{\psi}\gamma^\nu\psi(x)::\bar{\psi}\gamma^\mu\psi(x'):\ |e^+(p_2)e^-(p_1)\rangle + O(e^4)$$

S-Matrix element for $e^-e^+ \to \mu^-\mu^+$

- A_μ commutes with $\psi, \bar{\psi}$:

$$\langle \mu^+(p_4)\mu^-(p_3)| \, S \, |e^+(p_2)e^-(p_1)\rangle$$

$$= (ie)^2 \int d^4x \, d^4x' \langle 0 | T \, A_\nu(x) A_\mu(x') \, |0\rangle$$

$$\times \langle \mu^+(p_4)\mu^-(p_3)| \; : \bar{\psi}_\mu \gamma^\nu \psi_\mu(x) : : \bar{\psi}_e \gamma^\mu \psi_e(x') : \; |e^+(p_2)e^-(p_1)\rangle + O(e^4)$$

S-Matrix element for $e^-e^+ \longrightarrow \mu^-\mu^+$

- A_μ commutes with $\psi, \bar{\psi}$:

$$\langle \mu^+(p_4)\mu^-(p_3)|\, S\, |e^+(p_2)e^-(p_1)\rangle$$

$$= (ie)^2 \int d^4x\, d^4x' \langle 0\,| T\, A_\nu(x) A_\mu(x')\,|0\rangle$$

$$\times \langle \mu^+(p_4)\mu^-(p_3)|\ :\bar{\psi}_\mu \gamma^\nu \psi_\mu(x) :: \bar{\psi}_e \gamma^\mu \psi_e(x') :\ |e^+(p_2)e^-(p_1)\rangle + O(e^4)$$

- counting annihilation/creation operators $a^{(\dagger)}, b^{(\dagger)}$

$$\langle \mu^+(p_4)\mu^-(p_3)|\ :\bar{\psi}_\mu \gamma^\nu \psi_\mu(x) :: \bar{\psi}_e \gamma^\mu \psi_e(x') :\ |e^+(p_2)e^-(p_1)\rangle$$

$$= \langle \mu^+(p_4)\mu^-(p_3)|\ :\bar{\psi}_\mu \gamma^\nu \psi_\mu(x) :\ |0\rangle\langle 0|\ :\bar{\psi}_e \gamma^\mu \psi_e(x') :\ |e^+(p_2)e^-(p_1)\rangle$$

S-Matrix element for $e^- e^+ \to \mu^- \mu^+$

- further reduction

$$\langle 0| \, : \bar{\psi}_e \gamma^\mu \psi_e(x') : \, |e^+(p_2) e^-(p_1)\rangle$$

$$= \langle 0| \, : \bar{\psi}_e \gamma^\mu \psi_e(x') : \, b_e^\dagger(\vec{p}_2) a_e^\dagger(\vec{p}_1) |0\rangle$$

- reminder

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \sum_{s=\pm 1/2} \{ e^{ipx} v_s(p) b^\dagger(\vec{p}) + e^{-ipx} u_s(p) a(\vec{p}) \}$$

S-Matrix element for $e^-e^+ \to \mu^-\mu^+$

- further reduction

$$\langle 0| \; : \bar{\psi}_e \gamma^\mu \psi_e(x') : \; b_e^\dagger(\vec{p}_2) a_e^\dagger(\vec{p}_1)|0\rangle$$

$$= -\langle 0| \, \bar{\psi}_e(x') \, b_e^\dagger(\vec{p}_2)|0\rangle \gamma^\mu \langle 0|\psi_e(x') a_e^\dagger(\vec{p}_1)|0\rangle$$

S-Matrix element for $e^- e^+ \to \mu^- \mu^+$

- further reduction

$$\langle 0| \; : \bar{\psi}_e \gamma^\mu \psi_e(x') : \; b_e^\dagger(\vec{p}_2) a_e^\dagger(\vec{p}_1)|0\rangle$$

$$= -\langle 0| \, \bar{\psi}_e(x') \, b_e^\dagger(\vec{p}_2)|0\rangle \gamma^\mu \langle 0|\psi_e(x') a_e^\dagger(\vec{p}_1)|0\rangle$$

- expectation value $\langle 0|\psi|e^-\rangle$

$$\langle 0|\psi_e(x') a_e^\dagger(\vec{p}_1)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} e^{-ipx'} \sum_{s=\pm 1/2} u_s(p) \langle 0| \, a(\vec{p}) a_e^\dagger(\vec{p}_1)|0\rangle$$

S-Matrix element for $e^- e^+ \to \mu^- \mu^+$

- further reduction

$$\langle 0| \; : \bar{\psi}_e \gamma^\mu \psi_e(x') : \; b_e^\dagger(\vec{p}_2) a_e^\dagger(\vec{p}_1)|0\rangle$$

$$= -\langle 0| \, \bar{\psi}_e(x') \, b_e^\dagger(\vec{p}_2)|0\rangle \gamma^\mu \langle 0|\psi_e(x') a_e^\dagger(\vec{p}_1)|0\rangle$$

- expectation value $\langle 0|\psi|e^-\rangle$

$$\langle 0|\psi_e(x') a_e^\dagger(\vec{p}_1)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} e^{-ipx'} \sum_{s=\pm 1/2} u_s(p) \langle 0| \, \{a_e(\vec{p}) \, , a_e^\dagger(\vec{p}_1)\} |0\rangle$$

S-Matrix element for $e^- e^+ \to \mu^- \mu^+$

- further reduction

$$\langle 0| :\bar{\psi}_e \gamma^\mu \psi_e(x'): b_e^\dagger(\vec{p}_2) a_e^\dagger(\vec{p}_1)|0\rangle$$

$$= -\langle 0|\bar{\psi}_e(x')\, b_e^\dagger(\vec{p}_2)|0\rangle \gamma^\mu \langle 0|\psi_e(x') a_e^\dagger(\vec{p}_1)|0\rangle$$

- expectation value $\langle 0|\psi|e^-\rangle$

$$\langle 0|\psi_e(x') a_e^\dagger(\vec{p}_1)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} e^{-ipx'} \sum_{s=\pm 1/2} u_s(p) \langle 0| \{a_e(\vec{p}),\, a_e^\dagger(\vec{p}_1)\}|0\rangle$$

- reminder

$$\{a_s(\vec{p}),\, a_r^\dagger(\vec{p}_1)\} = (2\pi)^3 2p^0 \delta_{rs} \delta(\vec{p} - \vec{p}_1)$$

S-Matrix element for $e^-e^+ \to \mu^-\mu^+$

- further reduction

$$\langle 0| \; : \bar{\psi}_e \gamma^\mu \psi_e(x') : \; b_e^\dagger(\vec{p}_2) a_e^\dagger(\vec{p}_1)|0\rangle$$

$$= -\langle 0| \, \bar{\psi}_e(x') \, b_e^\dagger(\vec{p}_2)|0\rangle \gamma^\mu \langle 0|\psi_e(x') a_e^\dagger(\vec{p}_1)|0\rangle$$

- expectation value $\langle 0|\psi|e^-\rangle$

$$\langle 0|\psi_e(x') a_e^\dagger(\vec{p}_1)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} e^{-ipx'} \sum_{s=\pm 1/2} u_s(p) \langle 0| \left\{ a_e(\vec{p}) \, , a_e^\dagger(\vec{p}_1) \right\} |0\rangle$$

$$= e^{-ip_1x'} u_e(p_1)$$

S-Matrix element for $e^- e^+ \to \mu^- \mu^+$

- further reduction

$$\langle 0| \; : \bar{\psi}_e \gamma^\mu \psi_e(x') : \; b_e^\dagger(\vec{p}_2) a_e^\dagger(\vec{p}_1)|0\rangle$$

$$= -\langle 0| \, \bar{\psi}_e(x') \, b_e^\dagger(\vec{p}_2)|0\rangle \gamma^\mu \langle 0|\psi_e(x') a_e^\dagger(\vec{p}_1)|0\rangle$$

- expectation values $\langle 0|\psi|e^-\rangle, \langle 0|\bar{\psi}|e^+\rangle$

$$\langle 0|\psi_e(x') a_e^\dagger(\vec{p}_1)|0\rangle = e^{-ip_1 x'} u_e(p_1)$$

$$\langle 0|\bar{\psi}_e(x') b_e^\dagger(\vec{p}_2)|0\rangle = e^{-ip_2 x'} \bar{v}_e(p_2)$$

S-Matrix element for $e^-e^+ \rightarrow \mu^-\mu^+$

- further reduction

$$\langle 0| \; : \bar{\psi}_e(x')\gamma^\mu\psi_e(x') : \; b_e^\dagger(\vec{p}_2)a_e^\dagger(\vec{p}_1)|0\rangle$$

$$= -\bar{v}_e(p_2)\gamma^\mu u_e(p_1)e^{-i(p_1+p_2)x'}$$

S-Matrix element for $e^- e^+ \to \mu^- \mu^+$

- further reduction

$$\langle 0| \; : \bar{\psi}_e(x')\gamma^\mu \psi_e(x') : \; b_e^\dagger(\vec{p}_2) a_e^\dagger(\vec{p}_1)|0\rangle$$

$$= -\bar{v}_e(p_2)\gamma^\mu u_e(p_1) e^{-i(p_1+p_2)x'}$$

- similarily

$$\langle 0| b_\mu(\vec{p}_4) a_\mu(\vec{p}_3) \; : \bar{\psi}_\mu(x)\gamma^\nu \psi_\mu(x) : \; |0\rangle$$

$$= \bar{u}_\mu(p_3)\gamma^\nu v_\mu(p_4) e^{i(p_3+p_4)x}$$

S-Matrix element for $e^- e^+ \longrightarrow \mu^- \mu^+$

- in summary

$$\langle \mu^+(p_4)\mu^-(p_3)| \, S \, |e^+(p_2)e^-(p_1)\rangle$$

$$\simeq \quad -(ie)^2 \int d^4x \, d^4x' \langle 0 \,| T \, A_\nu(x) A_\mu(x') \,|0\rangle e^{i(p_3+p_4)x} e^{-i(p_1+p_2)x'}$$

$$\times \bar{u}_\mu(p_3)\gamma^\nu v_\mu(p_4) \; \bar{v}_e(p_2)\gamma^\mu u_e(p_1)$$

S-Matrix element for $e^- e^+ \longrightarrow \mu^- \mu^+$

- in summary

$$\langle \mu^+(p_4)\mu^-(p_3)|\, S\, |e^+(p_2)e^-(p_1)\rangle$$

$$\simeq \quad -(ie)^2 \int d^4x\, d^4x' \langle 0\,|T\, A_\nu(x) A_\mu(x')\,|0\rangle e^{i(p_3+p_4)x} e^{-i(p_1+p_2)x'}$$

$$\times \bar{u}_\mu(p_3)\gamma^\nu v_\mu(p_4)\ \bar{v}_e(p_2)\gamma^\mu u_e(p_1)$$

- photon propagator

$$\langle 0\,|T\, A_\nu(x) A_\mu(x')\,|0\rangle = -ig_{\mu\nu} \lim_{\epsilon \to 0_+} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k^2 + i\epsilon}$$

S-Matrix element for $e^- e^+ \to \mu^- \mu^+$

- in summary

$$\langle \mu^+(p_4)\mu^-(p_3)|\, S\, |e^+(p_2)e^-(p_1)\rangle$$

$$\simeq \quad -(ie)^2 \int d^4x\, d^4x' \langle 0\,|T\, A_\nu(x)A_\mu(x')\,|0\rangle e^{i(p_3+p_4)x} e^{-i(p_1+p_2)x'}$$

$$\times \bar{u}_\mu(p_3)\gamma^\nu v_\mu(p_4)\; \bar{v}_e(p_2)\gamma^\mu u_e(p_1)$$

- momentum conservation

$$\int d^4x\, d^4x' \int \frac{d^4k}{(2\pi)^4} \frac{e^{-i(k-p_1-p_2)x}\, e^{i(k-p_3-p_4)x'}}{k^2+i\epsilon}$$

$$= \quad \frac{1}{s}(2\pi)^4 \delta(p_1+p_2-p_3-p_4)$$

with $s=(p_1+p_2)^2$, the square of the total energy

S-Matrix element for $e^-e^+ \longrightarrow \mu^-\mu^+$

- in summary

$$\langle \mu^+(p_4)\mu^-(p_3)|\, S \,|e^+(p_2)e^-(p_1)\rangle$$

$$\simeq \frac{ig_{\mu\nu}}{s}(ie)^2(2\pi)^4\delta(p_1+p_2-p_3-p_4)\bar{u}_\mu(p_3)\gamma^\nu v_\mu(p_4)\ \bar{v}_e(p_2)\gamma^\mu u_e(p_1)$$

S-Matrix element for $e^- e^+ \to \mu^- \mu^+$

- in summary

$$\langle \mu^+(p_4)\mu^-(p_3)|\, S \,|e^+(p_2)e^-(p_1)\rangle$$

$$\simeq \frac{i}{s}(ie)^2(2\pi)^4\delta(p_1 + p_2 - p_3 - p_4)\bar{u}_\mu(p_3)\gamma^\nu v_\mu(p_4)\ \bar{v}_e(p_2)\gamma_\nu u_e(p_1)$$

S-Matrix element for $e^- e^+ \to \mu^- \mu^+$

- in summary

$$\langle \mu^+(p_4)\mu^-(p_3)|\, S\, |e^+(p_2)e^-(p_1)\rangle$$

$$\simeq \frac{i}{s}(ie)^2(2\pi)^4\delta(p_1+p_2-p_3-p_4)\bar{u}_\mu(p_3)\gamma^\nu v_\mu(p_4)\ \bar{v}_e(p_2)\gamma_\nu u_e(p_1)$$

- reminder: differential cross-section $d\sigma$ per unit volume

$$d\sigma = \frac{1}{T\,F}\frac{d^3p_3}{(2\pi)^3}\frac{1}{2p_3^0}\frac{d^3p_4}{(2\pi)^3}\frac{1}{2p_4^0}$$

$$\times \sum_{\text{spins}} |\langle \mu^+(p_4)\mu^-(p_3)|\, S\, |e^+(p_2)e^-(p_1)\rangle|^2$$

S-Matrix element for $e^- e^+ \to \mu^- \mu^+$

- averaging/summation over spins in initial state/final state

$$\frac{1}{4} \sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2$$

S-Matrix element for $e^- e^+ \to \mu^- \mu^+$

- averaging/summation over spins in initial state/final state

$$\frac{1}{4} \sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2$$

$$= \frac{1}{2} \sum_{s,s'} \bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\bar{v}_{\mu,s'}(p_4)\gamma^\rho u_{\mu,s}(p_3)$$

$$\times \frac{1}{2} \sum_{r,r'} \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)\bar{u}_{e,r'}(p_1)\gamma_\rho v_{e,r}(p_2)$$

with

$$\left[\bar{v}_r(p)\gamma_\nu u_{r'}(q)\right]^* = u^\dagger_{r'}(q)\gamma^\dagger_\nu \gamma^{0\dagger} v_r(p) = \bar{u}_{r'}(q)\gamma_\nu v_r(p)$$

S-Matrix element for $e^-e^+ \to \mu^-\mu^+$

- averaging/summation over spins in initial state/final state

$$\frac{1}{4}\sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2$$

$$= \frac{1}{2}\sum_{s,s'} \bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\bar{v}_{\mu,s'}(p_4)\gamma^\rho u_{\mu,s}(p_3)$$

$$\times\frac{1}{2}\sum_{r,r'} \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)\bar{u}_{e,r'}(p_1)\gamma_\rho v_{e,r}(p_2)$$

S-Matrix element for $e^-e^+ \to \mu^-\mu^+$

- averaging/summation over spins in initial state/final state

$$\frac{1}{4}\sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2$$

- single sum

$$\frac{1}{2}\sum_{s,s'} \bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\bar{v}_{\mu,s'}(p_4)\gamma^\rho u_{\mu,s}(p_3)$$

- reminder: spin sums

$$\sum_{s=\pm 1/2} u_s(p)\bar{u}_s(p) = \not{p} + m\,, \qquad \sum_{s=\pm 1/2} v_s(p)\bar{v}_s(p) = \not{p} - m$$

S-Matrix element for $e^-e^+ \to \mu^-\mu^+$

- averaging/summation over spins in initial state/final state

$$\frac{1}{4}\sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2$$

- single sum

$$\frac{1}{2}\sum_{s,s'} \bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\bar{v}_{\mu,s'}(p_4)\gamma^\rho u_{\mu,s}(p_3)$$

- reminder: spin sums

$$\sum_{s=\pm 1/2} u_s(p)\bar{u}_s(p) = \not{p} + m\,, \qquad \sum_{s=\pm 1/2} v_s(p)\bar{v}_s(p) = \not{p} - m$$

S-Matrix element for $e^-e^+ \to \mu^-\mu^+$

- averaging/summation over spins in initial state/final state

$$\frac{1}{4}\sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2$$

- single sum

$$\frac{1}{2}\sum_{s,s'} \bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\bar{v}_{\mu,s'}(p_4)\gamma^\rho u_{\mu,s}(p_3)$$

- reminder: spin sums

$$\sum_{s=\pm 1/2} u_s(p)\bar{u}_s(p) = \not{p} + m\,, \qquad \sum_{s=\pm 1/2} v_s(p)\bar{v}_s(p) = \not{p} - m$$

S-Matrix element for $e^- e^+ \to \mu^- \mu^+$

- averaging/summation over spins in initial state/final state

$$\frac{1}{4} \sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2$$

- single sum

$$\frac{1}{2} \sum_{s,s'} \bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\bar{v}_{\mu,s'}(p_4)\gamma^\rho u_{\mu,s}(p_3) = \frac{1}{2}\mathrm{Tr}(\not{p}_3 + m_\mu)\gamma^\nu(\not{p}_4 - m_\mu)\gamma^\rho$$

- reminder: spin sums

$$\sum_{s=\pm 1/2} u_s(p)\bar{u}_s(p) = \not{p} + m\,, \qquad \sum_{s=\pm 1/2} v_s(p)\bar{v}_s(p) = \not{p} - m$$

S-Matrix element for $e^-e^+ \to \mu^-\mu^+$

- averaging/summation over spins in initial state/final state

$$\frac{1}{4}\sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2$$

- single sum

$$\frac{1}{2}\sum_{s,s'} \bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\bar{v}_{\mu,s'}(p_4)\gamma^\rho u_{\mu,s}(p_3) = \frac{1}{2}\mathrm{Tr}(\not{p}_3 + m_\mu)\gamma^\nu(\not{p}_4 - m_\mu)\gamma^\rho$$

- similarily

$$\frac{1}{2}\sum_{s,s'} \bar{v}_{e,s}(p_2)\gamma_\nu u_{e,s'}(p_1)\bar{u}_{e,s'}(p_1)\gamma_\rho v_{e,s}(p_2) = \frac{1}{2}\mathrm{Tr}(\not{p}_2 - m_e)\gamma_\nu(\not{p}_1 + m_e)\gamma_\rho$$

S-Matrix element for $e^- e^+ \to \mu^- \mu^+$

- averaging/summation over spins in initial state/final state

$$\frac{1}{4} \sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2$$

$$= \frac{1}{4}\mathrm{Tr}\left[(\not{p}_3 + m_\mu)\gamma^\nu(\not{p}_4 - m_\mu)\gamma^\rho\right]\mathrm{Tr}\left[(\not{p}_2 - m_e)\gamma_\nu(\not{p}_1 + m_e)\gamma_\rho\right]$$

S-Matrix element for $e^-e^+ \to \mu^-\mu^+$

- averaging/summation over spins in initial state/final state

$$\frac{1}{4}\sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2$$

$$= \frac{1}{4}\mathrm{Tr}\Big[(\not{p}_3 + m_\mu)\gamma^\nu(\not{p}_4 - m_\mu)\gamma^\rho\Big]\ \mathrm{Tr}\Big[(\not{p}_2 - m_e)\gamma_\nu(\not{p}_1 + m_e)\gamma_\rho\Big]$$

- high energy limit

$$s \gg m_\mu^2, m_e^2$$

$\to$ drop m_e, m_μ in traces

S-Matrix element for $e^- e^+ \rightarrow \mu^- \mu^+$

- averaging/summation over spins in initial state/final state

$$\frac{1}{4} \sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2$$

$$= \quad \frac{1}{4}\mathrm{Tr}\left[\not{p}_3\gamma^\nu \not{p}_4\gamma^\rho\right] \mathrm{Tr}\left[\not{p}_2\gamma_\nu \not{p}_1\gamma_\rho\right]$$

- traces

$$\mathrm{Tr}\ \gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\rho = 4\left(g^{\alpha\nu}g^{\beta\rho} + g^{\rho\alpha}g^{\nu\beta} - g^{\alpha\beta}g^{\nu\rho}\right)$$

S-Matrix element for $e^-e^+ \to \mu^-\mu^+$

- averaging/summation over spins in initial state/final state

$$
\begin{aligned}
& \frac{1}{4}\sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2 \\
= \quad & \frac{1}{4}\mathrm{Tr}\left[\not{p}_3\gamma^\nu\not{p}_4\gamma^\rho\right]\ \mathrm{Tr}\left[\not{p}_2\gamma_\nu\not{p}_1\gamma_\rho\right] \\
= \quad & 4\Big[(p_1p_4)(p_2p_3) + (p_2p_4)(p_1p_3)\Big]
\end{aligned}
$$

- traces

$$
\mathrm{Tr}\ \gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\rho = 4\left(g^{\alpha\nu}g^{\beta\rho} + g^{\rho\alpha}g^{\nu\beta} - g^{\alpha\beta}g^{\nu\rho}\right)
$$

S-Matrix element for $e^- e^+ \to \mu^- \mu^+$

- averaging/summation over spins in initial state/final state

$$\frac{1}{4} \sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2$$

$$= \quad 4\Big[(p_1 p_4)(p_2 p_3) + (p_2 p_4)(p_1 p_3)\Big]$$

- high energy limit revisited

$$p_1 p_3 = p_2 p_4 = \frac{s}{4}(1 - \cos\vartheta)\,, \qquad p_1 p_4 = p_2 p_3 = \frac{s}{4}(1 + \cos\vartheta)$$

with scattering angle

$$\cos\vartheta = \frac{\vec{p}_1 \vec{p}_3}{|\vec{p}_1|\,|\vec{p}_3|}$$

S-Matrix element for $e^-e^+ \to \mu^-\mu^+$

- averaging/summation over spins in initial state/final state

$$\frac{1}{4}\sum_{s,s',r,r'} |\bar{u}_{\mu,s}(p_3)\gamma^\nu v_{\mu,s'}(p_4)\ \bar{v}_{e,r}(p_2)\gamma_\nu u_{e,r'}(p_1)|^2$$

$$= \quad \frac{s^2}{2}(1+\cos^2\vartheta)$$

- high energy limit revisited

$$p_1p_3 = p_2p_4 = \frac{s}{4}(1-\cos\vartheta)\,, \qquad p_1p_4 = p_2p_3 = \frac{s}{4}(1+\cos\vartheta)$$

with scattering angle

$$\cos\vartheta = \frac{\vec{p}_1\vec{p}_3}{|\vec{p}_1|\,|\vec{p}_3|}$$

cross-section for $e^-e^+ \to \mu^-\mu^+$

- differential cross-section $d\sigma$ per unit volume

$$d\sigma = \frac{1}{T\,F}\frac{d^3p_3}{(2\pi)^3}\frac{1}{2p_3^0}\frac{d^3p_4}{(2\pi)^3}\frac{1}{2p_4^0}$$

$$\times \sum_{\text{spins}} \left|\langle \mu^+(p_4)\mu^-(p_3)|\, S\, |e^+(p_2)e^-(p_1)\rangle\right|^2$$

with particle flux F, and unit volume $V = 1$.

- reminder: S-Matrix element

$$\langle \mu^+(p_4)\mu^-(p_3)|\, S\, |e^+(p_2)e^-(p_1)\rangle$$

$$\simeq \frac{i}{s}(ie)^2(2\pi)^4\delta(p_1+p_2-p_3-p_4)\bar{u}_\mu(p_3)\gamma^\nu v_\mu(p_4)\ \bar{v}_e(p_2)\gamma_\nu u_e(p_1)$$

cross-section for $e^- e^+ \to \mu^- \mu^+$

- differential cross-section $d\sigma$ per unit volume

$$d\sigma = \frac{1}{T\,F}\frac{d^3p_3}{(2\pi)^3}\frac{1}{2p_3^0}\frac{d^3p_4}{(2\pi)^3}\frac{1}{2p_4^0}$$

$$\times[(2\pi)^4\delta(p_1+p_2-p_3-p_4)]^2\,\frac{e^4}{s^2}\,\frac{s^2}{2}(1+\cos^2\vartheta)$$

- Fermi's trick

$$[(2\pi)^4\delta(p_1+p_2-p_3-p_4)]^2$$

$$= (2\pi)^4\delta(p_1+p_2-p_3-p_4)\int_{VT} d^4x\, e^{ix((p_1+p_2-p_3-p_4)}$$

$$= VT\,(2\pi)^4\delta(p_1+p_2-p_3-p_4)$$

cross-section for $e^- e^+ \to \mu^- \mu^+$

- differential cross-section $d\sigma$ per unit volume

$$d\sigma = \frac{1}{F} \frac{d^3 p_3}{(2\pi)^3} \frac{1}{2p_3^0} \frac{d^3 p_4}{(2\pi)^3} \frac{1}{2p_4^0}$$

$$\times (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \frac{e^4}{2} (1 + \cos^2 \vartheta)$$

- $\alpha = \frac{e^2}{4\pi}$

cross-section for $e^- e^+ \to \mu^- \mu^+$

- differential cross-section $d\sigma$ per unit volume

$$d\sigma = 2\alpha^2 \frac{1}{F} \frac{d^3 p_3}{2p_3^0} \frac{d^3 p_4}{2p_4^0} \delta(p_1 + p_2 - p_3 - p_4)\,(1 + \cos^2\vartheta)$$

- high energy limit + CMS-system: $\vec{p}_1 + \vec{p}_2 = 0, \qquad p_1^0 + p_2^0 \simeq \sqrt{s}$

cross-section for $e^- e^+ \to \mu^- \mu^+$

- differential cross-section $d\sigma$ per unit volume

$$d\sigma = 2\alpha^2 \frac{1}{F} \frac{d^3 p_3}{2p_3^0} \frac{d^3 p_4}{2p_4^0} \delta(p_1 + p_2 - p_3 - p_4)\,(1 + \cos^2 \vartheta)$$

- high energy limit + CMS-system: $\vec{p}_1 + \vec{p}_2 = 0, \qquad p_1^0 + p_2^0 \simeq \sqrt{s}$
- differential cross-section $\dfrac{d\sigma}{d\Omega_3}$

$$\frac{d\sigma}{d\Omega_3} = \alpha^2 \frac{1}{2F} \int_0^\infty d|\vec{p}_3|\,|\vec{p}_3| \int \frac{d^3 p_4}{|\vec{p}_4|} \delta(\sqrt{s} - |\vec{p}_3| - |\vec{p}_4|)\delta(\vec{p}_3 + \vec{p}_4)\,(1 + \cos^2 \vartheta)$$

- Flux: $F = 2p_1^0\, 2p_2^0 \dfrac{|\vec{p}_1|}{p_1^0}$ $\qquad (= |\vec{v}_A| 2E_A 2E_B)$

cross-section for $e^-e^+ \to \mu^-\mu^+$

- differential cross-section $d\sigma$ per unit volume

$$d\sigma = 2\alpha^2 \frac{1}{F} \frac{d^3p_3}{2p_3^0} \frac{d^3p_4}{2p_4^0} \delta(p_1 + p_2 - p_3 - p_4)\,(1 + \cos^2\vartheta)$$

- high energy limit + CMS-system: $\vec{p}_1 + \vec{p}_2 = 0, \qquad p_1^0 + p_2^0 \simeq \sqrt{s}$
- differential cross-section $\dfrac{d\sigma}{d\Omega_3}$

$$\frac{d\sigma}{d\Omega_3} = \frac{\alpha^2}{4}(1 + \cos^2\vartheta)$$

cross-section for $e^- e^+ \to \mu^- \mu^+$

- differential cross-section $d\sigma$ per unit volume

$$d\sigma = 2\alpha^2 \frac{1}{F} \frac{d^3 p_3}{2p_3^0} \frac{d^3 p_4}{2p_4^0} \delta(p_1 + p_2 - p_3 - p_4)\,(1 + \cos^2\vartheta)$$

- high energy limit + CMS-system: $\vec{p}_1 + \vec{p}_2 = 0, \qquad p_1^0 + p_2^0 \simeq \sqrt{s}$
- differential cross-section $\frac{d\sigma}{d\Omega_3}$

$$\frac{d\sigma}{d\Omega_3} = \frac{\alpha^2}{4}(1 + \cos^2\vartheta)$$

- total cross-section $\sigma = \int d\Omega_3 \frac{d\sigma}{d\Omega_3}$

$$\sigma_{\text{total}}(e^- e^+ \to \mu^- \mu^+) = \frac{4\pi\alpha^2}{3}$$