Fockspace: Construction as for scalar field, but ACR → Anti-symmetric states

$|0\rangle$: vacuum state, normalised $\langle 0|0\rangle = 1$

$$a_s(\vec{p})|0\rangle = 0$$

$$b_s(\vec{p})|0\rangle = 0$$

One-particle states:

$$|e^-(p,s)\rangle = a_s^+(\vec{p})|0\rangle$$

$$|e^+(p,s)\rangle = b_s^+(\vec{p})|0\rangle$$

$|e^-\rangle / |e^+\rangle$ describe elektron/positron with momentum p and spin $s = \pm 1/2$ ($S_z$ in rest-frame)

---

Remark: prediction of $e^+, e^-$ with identical mass is triumph of Dirac theory.

---

orthogonality:
$$\langle e^-(p',s')|e^-(p,s)\rangle$$
$$= \langle 0| a_{s'}(\vec{p}')\, a_s^+(\vec{p})|0\rangle$$
$$= \langle 0| \{a_{s'}(\vec{p}')\, a_s^+(\vec{p})\}|0\rangle = (2\pi)^3\, 2p_0\, \delta_{s's}\, \delta^{(3)}(\vec{p}-\vec{p}')$$

two-particle states:

$$|e^-(p_1,s_1)\, e^-(p_2,s_2)\rangle = a^\dagger_{s_1}(\vec p_1)\, a^\dagger_{s_2}(\vec p_2)|0\rangle$$

Pauli principle

$$|e^-(p_1,s_1)\, e^-(p_2,s_2)\rangle = a^\dagger_{s_1}(\vec p_1)\, a^\dagger_{s_2}(\vec p_2)|0\rangle$$

$$= - a^\dagger_{s_2}(\vec p_2)\, a^\dagger_{s_1}(\vec p_1)|0\rangle$$

$$= -|e^-(p_2,s_2)\, e^-(p_1,s_1)\rangle$$

---

N-particle states:

$$a^\dagger_{s_1}(\vec p_1) \cdots a^\dagger_{s_n}(\vec p_n)\, b^\dagger_{r_1}(\vec q_1) \cdots b^\dagger_{r_m}(\vec q_m)|0\rangle$$

---

Finally, with $\psi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k^0} \sum_{r=\pm 1/2} \left\{ e^{ikx} v_r(k) b^\dagger_r(\vec k) + e^{-ikx} u_r(k)\, a_r(\vec k) \right\}$

Exercise:)

$$\langle 0|\psi(x)|e^-(p,s)\rangle = u_s(p)\, e^{-ipx}$$

$$\langle e^+(p,s)|\psi(x)|0\rangle = v_s(p)\, e^{ipx}$$

$\psi$: annihilation of an electron / creation of positron at $x$

$$\langle 0|\bar\psi(x)|e^+(p,s)\rangle = \bar v_s(p)\, e^{-ipx}$$

$$\langle e^-(p,s)\,\bar\psi(x)|0\rangle = \bar u_s(p)\, e^{ipx}$$

$\bar\psi$: annihilation of a positron / creation of electron at $x$

Symmetries:

$$S_D[\psi,\overline{\psi}] = \int d^4x \; \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi \qquad (3\text{-}94)$$

(1) Invariance of $S_D[\psi,\overline{\psi}]$ under orthochronous Poincaré trafos

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

p. 1-2 (1.4)

$$\mathcal{U}(\Lambda,a)\,\psi(x)\,\mathcal{U}^+(\Lambda,a) = S^{-1}(\Lambda)\,\psi(\Lambda x + a) \qquad (3\text{-}95)$$

where $S$ satisfies

$$S^{-1}(\Lambda)\,\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\rho \gamma^\rho \qquad (3\text{-}96)$$

and $\mathcal{U}$ is unitary.

Dirac adjoint spinor

$$\mathcal{U}(\Lambda,a)\,\overline{\psi}(x)\,\mathcal{U}^+(\Lambda,a) = \overline{\psi}(\Lambda x + a)\,S(\Lambda) \qquad (3\text{-}97)$$

---

$$\int d^4x \; \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi \to \int d^4x \; \overline{\psi}(\Lambda x + a)\, S\left(i\gamma^\mu\partial_\mu - m\right)\psi(\Lambda x + a)$$

$$= \int d^4x \; \overline{\psi}(x)\, S\left(i\gamma^\mu\partial_\nu(\Lambda^{-1})_\mu{}^\nu - m\right)S^{-1}\psi$$

$$= \int d^4x \; \overline{\psi}(x)\, S\left(i\Lambda^\nu{}_\mu\gamma^\mu\partial_\nu - m\right)S^{-1}\psi \qquad (3\text{-}98)$$

$$= \int d^4x\, \bar{\psi}(x)\, S\,(i S^{-1} \gamma^\nu S\, \partial_\nu - m)\, S^{-1} \psi(x)$$

$$= \int d^4x\, \bar{\psi}(x)\, (i\, \gamma^\mu \partial_\mu - m)\, \psi(x)$$

---

General bilinears:

(a) $\bar{\psi}\psi$ scalar: $m\bar{\psi}\psi$

pseudoscalar later

(b) $\bar{\psi}\gamma^\mu\psi$ vector

pseudo vector later

(c) $\bar{\psi}\,\sigma^{\mu\nu}\psi$ tensor, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

(2) Invariance of $S[\psi, \bar{\psi}]$ under Parity

$$\Lambda_P = \begin{pmatrix} 1 & \\ & -\mathbb{1}_3 \end{pmatrix} \qquad (3\text{-}99)$$

Unitary repres.:

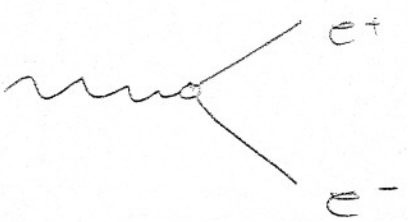
$$\mathcal{U}(P)\, \psi(\vec{x}, t)\, \mathcal{U}^{+}(P) = \gamma^0\, \psi(-\vec{x}, t)$$

$$\mathcal{U}(P)\, |e^-(\vec{p}, s)\rangle = |e^-(-\vec{p}, s)\rangle \qquad (3\text{-}100)$$

$$\mathcal{U}(P)\, |e^+(\vec{p}, s)\rangle = -|e^+(-\vec{p}, s)\rangle$$

$e^+, e^-$ are parity 'eigen states'

<u>Relative</u> intrinsic parity can be measured:




(more later)

(3) Invariance of $S[\Psi, \bar{\Psi}]$ under time reversal

$$\Lambda_T = \begin{pmatrix} -1 & \\ & 1_3 \end{pmatrix} \qquad (3\text{-}101)$$

Anti-unitary transformation $V$:

$$\left(V(T)\, \psi(\vec{x},t)\, V^{-1}(T)\right)^{+} = S(T)\, \bar{\Psi}^{T}(\vec{x},-t) \qquad (3.102)$$

with

$$S(T) = i\gamma^2 \gamma_5$$

$$\text{and} \quad \gamma_5 = \tfrac{i}{4!}\, \varepsilon_{\mu\nu\rho\sigma}\, \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \qquad (3.103)$$

$$[\varepsilon_{0123} = 1]$$

$$\{\gamma_5, \gamma^\mu\} = 0$$

We have

$$V(T)\, |e^-(\vec{p},s)\rangle = (-1)^{s-1/2}\, |e^-(-\vec{p},-s)\rangle$$

$$V(T)\, |e^+(\vec{p},s)\rangle = (-1)^{s-1/2}\, |e^+(-\vec{p},-s)\rangle \qquad (3.104)$$

(4) Charge conjugation

$$e^+ \overset{C}{\rightleftarrows} e^-$$

$$\mathcal{U}(C)\, \psi(x)\, \mathcal{U}^{-1}(C) = S(C)\, \bar{\psi}^T(x) \qquad (3.105)$$

with

$$S(C) = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & -\varepsilon \\ -\varepsilon & 0 \end{pmatrix}$$

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (3.106)$$

and

$$\mathcal{U}(C)\, |e^-(\vec{p},s)\rangle = |e^+(\vec{p},s)\rangle$$

$$\mathcal{U}(C)\, |e^+(\vec{p},s)\rangle = |e^-(\vec{p},s)\rangle \qquad (3.107)$$

Bilinears:

(1) scalar : $\bar{\psi}\psi(x)$ 1 gen

pseudo-scalar : $i\bar{\psi}\gamma_5\psi$ 1 gen

(2) vector : $\bar{\psi}\gamma^\mu\psi$ 4 gen

pseudo-vector : $\bar{\psi}\gamma_5\gamma^\mu\psi$ 4 gen

(3) tensor : $\bar{\psi}\sigma^{\mu\nu}\psi$ with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

6 gen

---

16 gen

of Lorentz group

---

Remark $(\gamma^\mu\gamma^\nu\gamma^\rho\,\varepsilon_{\mu\nu\rho\sigma}) \cdot \gamma^\sigma\gamma^\sigma \sim \gamma_5\gamma^\sigma$

# III-5 Lagrangian of QED

In eq (3.51), p 3-19 the Lagrangian density of the electromagnetic field coupled to an external current $j^\mu(x)$ was presented.

In QED, $j^\mu$ describes the coupling to the electron, muon, Tau-current with

$$j^\mu(x) = - e\,\overline{\Psi}\gamma^\mu \Psi(x) \qquad (3.108)$$

where $\Psi$ is given by

$$\Psi = \begin{pmatrix} \Psi_e \\ \Psi_\mu \\ \Psi_\tau \end{pmatrix} \qquad (3.109)$$

Together with the Dirac term we get

$$\boxed{\mathcal{L}_{QED}(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \overline{\Psi}\left(i\gamma^\mu D_\mu - m\right)\Psi} \qquad (3.110)$$

where $D_\mu$ is the covariant derivative:

$$D_\mu \Psi(x) = \left(\partial_\mu - ie A_\mu(x)\right)\Psi(x) \quad , \; m = \begin{pmatrix} m_e & & 0 \\ & m_\mu & \\ 0 & & m_\tau \end{pmatrix} \qquad (3.111)$$

$\mathcal{L}'$ couples $e^\pm$ to the electromagnetic field $A_\mu$, the photon.

Similarily we couple $\mu^\pm$, and $\tau^\pm$ to $A_\mu$:

$$\Psi(x) = \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix} \qquad \text{see } (3.109)$$

with

$$\mathcal{L}_0 = \bar{\psi}_e (i\gamma^\mu \partial_\mu - m_e)\psi_e + \bar{\psi}_\mu (i\gamma^\mu \partial_\mu - m_\mu)\psi_\mu$$
$$+ \bar{\psi}_\tau (i\not{\partial} - m_\tau)\psi_\tau \qquad \text{see } (3.110)$$

and $m_e = 0.511$ MeV

$m_\mu = 105.7$ MeV

$m_\tau = 1784$ MeV

$$\mathcal{L}(x) = e\left[\bar{\psi}_e \not{A}\, \psi_e + \bar{\psi}_\mu \not{A}\, \psi_\mu + \bar{\psi}_\tau \not{A}\, \psi_\tau\right]$$

see (3.110)

Gauge invariance : $g(x) = e^{ie\alpha(x)} \in U(1)$

coupling

$$\psi(x) \to g(x)\psi(x) = e^{ie\alpha(x)}\psi(x) = \psi^g$$

$$\Leftrightarrow \quad \bar{\psi}(x) \to \bar{\psi}(x)\, g^{\dagger}(x) = \bar{\psi}(x) e^{-ie\alpha(x)} = \bar{\psi}^g \qquad (3.112)$$

$$A_\mu(x) \to \frac{i}{e}\, g\, \mathcal{D}_\mu\, g^{\dagger} = A_\mu(x) + \partial_\mu \alpha(x) = A^g$$

$$\Rightarrow \quad \mathcal{D}_\mu(x) \to g\, \mathcal{D}_\mu\, g^{\dagger}(x)$$

$$(1) \quad -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \to -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$(2) \quad \bar{\psi}(i \not{\mathcal{D}} - m)\psi \to \bar{\psi}\, g^{\dagger}(i\, g \not{\mathcal{D}} g^{\dagger} - m)\, g\, \psi \qquad (3.113)$$

$$g g^{\dagger} = 1 \quad \to \quad = \bar{\psi}(i \not{\mathcal{D}} - m)\psi$$

with

$$\not{\mathcal{D}} = \gamma^\mu \mathcal{D}_\mu \qquad (3.114)$$

Remark : $\partial_\mu \to \partial_\mu - ieA_\mu$

is called 'minimal coupling'

$A_\mu$ is a connection (Zusammenhang)

Consequences of gauge invariance: (example)

action of QED:

$$S_{QED}[A,\psi,\bar\psi] = \int d^4x\, \mathcal{L}_{QED}(x)$$

with $\mathcal{L}_{QED}$ from eq. (3.110).

Gauge invariance:

$$S[A^g,\psi^g,\bar\psi^g] = S[A,\psi,\bar\psi]$$

Infinitesimal

$$S[A+\partial\alpha, (1+i\alpha e)\psi, \bar\psi(1-i\alpha e)] - S[A,\psi,\bar\psi] = 0$$

$$= \int d^4x \left( \partial_\nu\alpha(x) \frac{\delta}{\delta A_\nu(x)} + i e\alpha(x)\psi(x)\frac{\delta}{\delta\psi(x)} - \bar\psi(x)\frac{\delta}{\delta\bar\psi(x)} \right) S$$

e.g. $\psi=\bar\psi=0$:

$$\int d^4x\, \partial_\nu\alpha(x) \frac{\delta}{\delta A_\nu(x)} S[A,0,0] = 0$$

$$= -\int d^4x\, \alpha(x)\, \partial_\nu \frac{\delta S[A,0,0]}{\delta A_\nu(x)} = 0$$

$\alpha(x)$ arbitrary:

$$\partial_\nu \frac{\delta S[A]}{\delta A_\nu(x)} = 0$$

also

$$\partial^x_\nu \left.\frac{\delta^2 S}{\delta A_\nu(x)\,\delta A_\nu(y)}\right|_{A=0} = 0$$

$$\Rightarrow \partial^x_\nu \left(\partial_\rho\partial^\rho g^{\mu\nu} - \partial^\mu\partial^\nu\right)\delta(x-y) = 0 \quad ✓$$

# III-6 Perturbation theory

Lagrangian density:

$$\mathcal{L}_{QED}(x) = \underbrace{\mathcal{L}_{kin}(x)}_{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\Psi}(x)(i\gamma^\mu\partial_\mu - m)\Psi(x)} + \underbrace{\mathcal{L}_I(x)}_{e A_\mu \overline{\Psi}(x)\gamma^\mu \Psi(x)} \qquad (3.115)$$

with

$$\mathcal{L}_I(x) = e\, A_\mu(x)\, \overline{\Psi}(x)\, \gamma^\mu\, \Psi(x) \qquad (3.116)$$

---

Remarks:

(i) $\overline{\Psi}\gamma^\mu\Psi$, $A^\mu$ transform as vectors under Lorentz trafo's

$$A^\mu(x) \to \Lambda^\mu{}_\nu A^\nu(x)$$

$$\overline{\Psi}\gamma^\mu\Psi(x) \to \Lambda^\mu{}_\nu \overline{\Psi}\gamma^\nu\Psi(x)$$

$$\Rightarrow A_\mu \overline{\Psi}\gamma^\mu\Psi \to A_\mu\overline{\Psi}\gamma^\mu\Psi$$

(ii) $\mathcal{L}_{QED}(x)$ is gauge invariant

but $\mathcal{L}_I(x)$ is not!

Quantisation in interaction picture

$$i \frac{\partial}{\partial t} |t\rangle = H_I(t) |t\rangle \qquad (3.117$$

with

$$H_I(t) = - \int d^3x \, \mathcal{L}_{I_{op}}(x)$$

$$= - e \int d^3x \, A_\mu(x) :\bar{\psi}(x) \gamma^\mu \psi(x): \qquad (3.118)$$

$A_{\mu \, op}$ is bosonic (spin 1 particle) and commutes with $\psi, \bar{\psi}$

(3.117) is solved by

$$|t\rangle = \mathcal{U}(t, t_0) |t_0\rangle$$

with

$$\boxed{\begin{aligned} \mathcal{U}(t,t_0) &= T e^{-i \int_{t_0}^{t} H_I(t') dt'} \\ &= T e^{ie \int_{t_0}^{t} d^4x \, A_\mu :\bar{\psi} \gamma^\mu \psi:} \end{aligned}}$$