III - 4 Spin 1/2 Fields

Non-relativistic fermion:

$$\psi(\vec{x},t) = \begin{pmatrix} \psi_1(\vec{x},t) \\ \psi_2(\vec{x},t) \end{pmatrix} \qquad (3.65)$$

with probability $|\psi_i(\vec{x},t)|^2 d^3x$ for spin up, $i=1$, and down, $i=2$, respectively.

angular momentum op.:

$$\vec{J} = \underbrace{\vec{L}}_{\frac{1}{i}(\vec{x}\times\vec{\nabla})} + \frac{1}{2}\vec{\sigma} \qquad (3.66)$$

spin $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

[2dim representation of rotation group

$e^{i/2\,\sigma_i\,\omega^i}$

Relativistic fermion:

[2dim representation of Lorentz group $\cup$ rot. group

Consider $\hat{x} = x_\nu \sigma^\nu$, $\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}_2$

$$\boxed{x_\nu x^\nu = \det \hat{x}} \qquad (3.67)$$

Rotation:

$$\boxed{\hat{x}' = e^{-i\frac{1}{2}\sigma_i\omega^i}\,\hat{x}\,e^{i\frac{1}{2}\sigma_i\omega^i}} \quad (3.68)$$

transf. Lorentz group:

$$\hat{x}' = \Lambda^{-1}\,\hat{x}\,\Lambda$$

Left-handed:

$$\Lambda_L = e^{i\frac{1}{2}\sigma_i(\omega^i - i v^i)}$$

rotation boost (3.69)

Right-handed: P

$$\Lambda_R = e^{i\frac{1}{2}\sigma_i(\omega^i + i v^i)}$$

with Parity $\boxed{P:\ \vec{x} \to -\vec{x} \quad t \to t}$

obviously $x_\nu x^\nu \xrightarrow{P} x_\nu x^\nu$ (3.70)

Time-reversal $\boxed{T:\ t \to -t \quad \vec{x} \to \vec{x}}$

$x_\nu x^\nu \xrightarrow{T} x_\nu x^\nu$

Dirac equation: $\psi(x) = \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_4(x) \end{pmatrix}$ $\begin{matrix} \psi_L \\ \psi_R \end{matrix}$

$$\left(i \gamma^\mu \partial_\mu - m \right) \psi(x) = 0 \qquad (3.71)$$

with γ^μ are 4×4 matrices with

$$\boxed{\{ \gamma^\mu , \gamma^\nu \} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\, g^{\mu\nu} \cdot \mathbb{1}} \qquad (3.72)$$

Clifford algebra

Standard repres.:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad (3.73)$$

with Pauli Matrices, see p. 2-6a

Remarks: (1) $\psi(x)$ consists of a two-comp. left-handed and a two-comp right-handed spinor.

chiral repres.:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad (3.74)$$

(2) γ^μ transforms as vector under Lorentz trafos

EoM (2.73) is derived from Lagrange density

$$\mathcal{L}_D = \overline{\psi}\left(i\gamma^\nu \partial_\nu - m\right)\psi(x) \qquad (3.75)$$

with $\overline{\psi} = \psi^\dagger \gamma^0$, the Dirac conjugate.

$$\frac{\partial \mathcal{L}_D}{\partial \overline{\psi}} - \partial_\nu \frac{\partial \mathcal{L}_D}{\partial \partial_\nu \overline{\psi}} = \frac{\partial \mathcal{L}_D}{\partial \overline{\psi}} = 0 \rightarrow (3.72)$$

$$(3.76)$$

Also

$$\frac{\partial \mathcal{L}_D}{\partial \psi} - \partial_\nu \frac{\partial \mathcal{L}_D}{\partial \partial_\nu \psi} = 0$$

$$\Rightarrow -m\overline{\psi} - i\partial_\nu \overline{\psi}\gamma^\nu = 0 \qquad (3.77)$$

Classical solution:

$$(-i\gamma^\nu\partial_\nu - m)(i\gamma^\nu\partial_\nu - m)\psi(x)$$

$$= \left[\tfrac{1}{2}\{\gamma^\nu, \gamma^\nu\}\partial_\nu\partial_\nu + m^2\right]\psi(x) \qquad (3.78)$$

$$= \left[g^{\nu\nu}\partial_\nu\partial_\nu + m^2\right]\psi(x)$$

$$= \left[\Box + m^2\right]\psi(x)$$

$$\Rightarrow \quad \psi(x) \sim e^{\pm i p\cdot x} \qquad \text{plane wave} \quad (3.79)$$

We have $(i\gamma^\mu \partial_\mu - m)e^{\pm i p\cdot x} = (\mp \not{p} - m)\, e^{\pm i p x}$ (3.80)

with $\not{p} := \gamma^\mu p_\mu = \gamma^0 p_0 - \gamma^1 p^1 - \gamma^2 p^2 - \gamma^3 p^3$

A solution to the Dirac-equ. reads, $s = \pm 1/2$

$$\psi(x) \sim u_s(p)\, e^{-ipx} \quad , v_s(p) e^{ipx} \qquad (3.81)$$

with $(\not{p} - m)\, u_s(p) = 0 = (\not{p} + m)\, V_s(p)$ (3.82)

Eq. (2.84) is satisfied with, $p_0 = +\sqrt{\vec{p}^{\,2} + m^2}$

in Standard repes. p. 3-30

$$u_s = \sqrt{p^0 + m} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\vec{p}}{p_0 + m}\chi_s \end{pmatrix} , \quad \chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(3.83)

$$v_s = -\sqrt{p^0 + m} \begin{pmatrix} \frac{\vec{\sigma}\vec{p}}{p^0 + m}\, \varepsilon \chi_s \\ \varepsilon \chi_s \end{pmatrix} , \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(Metric in Spin space)

with additional identity

$$\sum_{s=\pm 1/2} u_s(p)\, \overline{u}_s(p) = \not{p} + m, \qquad \sum_{s=\pm 1/2} v_s(p)\, \overline{v}_s(p) = \not{p} - m \qquad (3.84)$$

As for the scalar field the general solution is given by the Fourier integral:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} \sum_{s=\pm 1/2} \Big\{ e^{ipx}\, v_s(p)\, \beta_s^*(\vec{p}) + e^{-ipx}\, u_s(p)\, \alpha_s(\vec{p}) \Big\}$$

(3.85)

Hamiltonian density: (Energy density)

$$\mathcal{H}_D = \pi \dot{\psi} - \mathcal{L}_D \quad \text{with} \quad \pi = \frac{\partial \mathcal{L}_D}{\partial \dot{\psi}}$$

$$\left[\bar{\pi} = \frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}} = 0 \right]$$

$$\pi = \bar{\psi} i \gamma^0 = i \psi^+$$

(3.86)

It follows

$$\mathcal{H}_D = \bar{\psi} i \gamma^0 \dot{\psi} - \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

$$= + \bar{\psi}(i \vec{\gamma} \vec{\partial} + m)\psi$$

(3.87)

Hamiltonian:

$$H_D = \int d^3x\, \bar{\Psi}\,(i\vec{\gamma}\vec{\partial} + m)\,\Psi$$

$$= \int d^3x\, \Psi^{+}\,(i\gamma^0\vec{\gamma}\vec{\partial} + \gamma^0 m)\,\Psi \qquad (3.88)$$

Inserting (3.85), p. 3-33 into (3.88) leads to

$$H_D = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2p_0}\sum_s \Big(\alpha_s^*(p)\,\alpha_s(p)\,p_0 - \beta_s(p)\,\beta_s^*(p)\,p_0 \Big) \qquad (3.89)$$

from

$$\gamma^0(i\vec{\gamma}\vec{\partial} + m)\,u_s(p) = p^0\,u_s(p)$$
$$\gamma^0(i\vec{\gamma}\vec{\partial} + m)\,v_s(p) = -p^0\,v_s(p) \qquad (3.90)$$

$$\Rightarrow \quad H_D = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2}\sum_s \Big(\alpha_s^*(p)\,\alpha_s(p) - \beta_s(p)\,\beta_s^*(p) \Big) \qquad (3.91)$$

$\Rightarrow$ negative Energy states

lead to unbounded Hamiltonian,

no classical interpretation!

Quantisation:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \sum_{s=\pm 1/2} \Big\{ e^{ipx}\, v_s(p)\, \overset{\text{op}}{b_s^\dagger(\vec{p})} + e^{-ipx}\, u_s(p)\, \underset{\text{op}}{a_s(\vec{p})} \Big\} \qquad (3.92)$$

with anti-commutation relations

$$\{ a_r(\vec{p}), a_s^\dagger(\vec{p}\,') \} = \delta_{rs} (2\pi)^3 2p_0\, \delta^{(3)}(\vec{p}-\vec{p}\,')$$

$$\{ b_r(\vec{p}), b_s^\dagger(\vec{p}\,') \} = \delta_{rs} (2\pi)^3 2p_0\, \delta^{(3)}(\vec{p}-\vec{p}\,') \qquad (3.93a)$$

and

$$\{ a^{(\dagger)}, a^{(\dagger)} \} = \{ b^{(\dagger)}, b^{(\dagger)} \} = \{ a^{(\dagger)}, b^{(\dagger)} \} = \{ a, b^\dagger \} = 0 \qquad (3.93b)$$

Remarks:

(1) The anti-commutation relations are a manifestation of the spin-statistics theorem: spin $2n+1/2$ particles have fermi-statistics (ACR, Pauli principle), spin n particles have Bose-statistics.

(2) electric charge: (Noether), $J^\mu = -e\bar{\psi}\gamma^\mu\psi$

$$Q = \int d^3x\, J^0 = -e\int d^3x\, \psi^\dagger\psi \qquad (\bar{\psi}\gamma^0\psi)$$

$$= -e\int \frac{d^3p}{(2\pi)^3}\frac{1}{2p_0}\sum_s \left(a_s^\dagger(p)\, a(p) + b_s(p)\, b_s^\dagger(p) \right)$$