III Introduction to QED

Introductory remarks:

1.) Quantisation: (scalar field)

$(x_1, p_1) \quad \cdots \quad (x_N, p_N)$

a

$$\begin{array}{ccc} \text{Classical Mechanics} & \xrightarrow[a \to 0]{} & \text{Field Theory} \\ \text{particle with } x,\ p & & \text{field } \Phi(x), \text{ mom. } \Pi(x) \\ \Big\downarrow \text{Quant.} & & \\ \text{Quantum Mechanics} & \xrightarrow[a \to 0]{} & \text{Quantum Field Theory} \\ [x, p] = i\hbar & & [\phi(x_0, \vec{x}), \pi(x_0, \vec{y})] \\ & & = i\hbar\delta(\vec{x} - \vec{y}) \end{array}$$

2.) Particle content in QED

(i) $\underbrace{\text{electrons, muons, taus}}_{\text{Leptons}}$

spin 1/2 particles, fermions

(ii) photons gauge boson of QED

spin 1 particle, bosons

III-1.) Scalar field theory

see also chapter 2
in SM script

Action principle: (real scalar field)

free action $\qquad$ (3.1)

$$S[\varphi] = \frac{1}{2}\int d^4x \, \underbrace{\left(\partial_\nu \varphi \, \partial^\nu \varphi - m^2 \varphi(x)^2\right)}_{\mathcal{L}[\varphi, \partial_\nu \varphi]} \qquad \partial_\nu = \frac{\partial}{\partial x^\nu}$$

Field equations via minimum of action:

$$\boxed{\frac{\delta S[\varphi]}{\delta \varphi(y)} = 0} \qquad (3.2)$$

with

$$\frac{\delta \varphi(x)}{\delta \varphi(y)} = \delta(x-y) = \delta(x_0 - y_0)\,\delta(x_1 - y_1)\cdots\delta(x_3 - y_3)$$

$$\frac{\delta\left(\partial_\nu \varphi(x)\right)}{\delta \varphi(y)} = \frac{\partial}{\partial x^\nu}\delta(x-y) \qquad (3.3)$$

results in

$$\boxed{\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} = 0} \qquad (3.4)$$

$$\Rightarrow \boxed{\left(\partial_\nu \partial^\nu + m^2\right) \varphi(y) = 0} \qquad (3.5)$$

Here we have used that

a.) $$\frac{\delta}{\delta\varphi(y)} \frac{1}{2} \int d^4x \; \partial_\nu \varphi(x) \; \partial^\nu \varphi(x)$$

$$= \frac{1}{2} \int d^4x \left\{ \underbrace{\frac{\delta\, \partial_\nu \varphi(x)}{\delta\varphi(y)}}_{\partial^x_\nu \delta(x-y)} \partial^\nu \varphi(x) + \partial_\nu \varphi(x) \underbrace{\frac{\delta\, \partial^\nu \varphi(x)}{\delta\varphi(y)}}_{\partial^{x\nu} \delta(x-y)} \right\}$$

$$= -\frac{1}{2} \left\{ \partial_\nu \partial^\nu \varphi(y) - \partial^\nu \partial_\nu \varphi(y) \right\} = \partial_\nu \partial^\nu \varphi(y) \qquad (3.6)$$

b.) $$\frac{\delta}{\delta\varphi(y)} \frac{1}{2} \int d^4x \; m^2 \; \varphi(x)^2$$

$$= \int d^4x \; m^2 \; \varphi(x) \; \delta(x-y) = m^2 \varphi(y) \qquad (3.7)$$

Classical solutions of the KG equ.,

take e.g. $\varphi = \varphi(x^1)$: $\left(-\partial_1^2 + m^2\right)\varphi(x_1) = 0$ (3.8)

$\Rightarrow$ solutions are plane waves

$$\varphi(x) = e^{\pm i k x} \quad \text{with} \quad kx = k^\mu x_\mu$$

$$\text{with} \quad k^2 = m^2 \quad , \quad k^0 = \pm\omega = \pm\sqrt{\vec{k}^2 + m^2} \qquad (3.9)$$

general solution: linear superpos. of plane waves

$$\varphi(x) = \underbrace{\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega}}_{\int \frac{d^4k}{(2\pi)^4}\,\delta(k^2 - m^2)} \left(e^{ikx}\,\alpha^*(\vec{k}) + e^{-ikx}\,\alpha(\vec{k})\right) \qquad (3.10)$$

$$\text{with} \quad k = \begin{pmatrix} \omega \\ \vec{k} \end{pmatrix}$$

QFT:

$$\varphi(x) \to \phi(x) \quad \text{operator}$$

expectation value $\langle \phi(x) \rangle$ classical field

ϕ obeys canonical commutation relations:

$$[\phi(\vec{x},t), \underbrace{\dot{\phi}}_{\substack{\text{can. conj. mom. } \frac{\partial \mathcal{L}}{\partial \partial_0 \phi}}}(\vec{y},t)] = i\,\delta(\vec{x}-\vec{y}) \tag{3.11}$$

class. mech $\xrightarrow[a \to 0]{\text{...}}$ field theory

QM $\xrightarrow[a \to 0]{}$ QFT with

$[x,p] = i(\hbar)$ $\qquad$ $[\phi, \Pi_0] = i$

$\phi(x)$ still obeys Klein-Gordon eq $(\Box + m^2)\phi = 0$

$$\Rightarrow \phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \left\{ e^{ikx} a^+(\vec{k}) + e^{-ikx} a(\vec{k}) \right\} \tag{3.12}$$

Inserting (3) into (2) results in (see p.3-4a)

$$[a(\vec{k}), a^+(\vec{k}')] = (2\pi)^3 2\omega\, \delta(\vec{k}-\vec{k}')$$

$$[a(\vec{k}), a(\vec{k}')] = 0 = [a^+(\vec{k}), a^+(\vec{k}')] \tag{3.13}$$

$$[\phi(\vec{x},t), \dot{\phi}(\vec{y},t)]$$

$$= -i \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{2\omega} \frac{\omega'}{2\omega'} \Big\{ e^{i(kx - k'y)} [a^+(\vec{k}), a(\vec{k}')]$$

$$- e^{-i(kx - k'y)} [a(\vec{k}), a^+(\vec{k}')]$$

$$- e^{i(kx + k'y)} \underbrace{[a^+(\vec{k}), a^+(\vec{k}')]}_{0} + e^{-i(kx+k'y)} \underbrace{[a(\vec{k}), a(\vec{k}')]}_{0}$$

$$3.13) \rightarrow = i \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}(\vec{x}-\vec{y})} = \delta(\vec{x}-\vec{y})$$