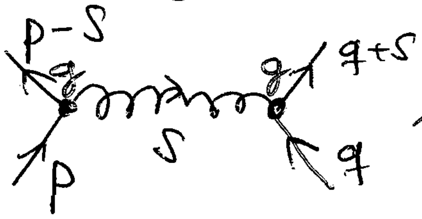
5.2 Running coupling

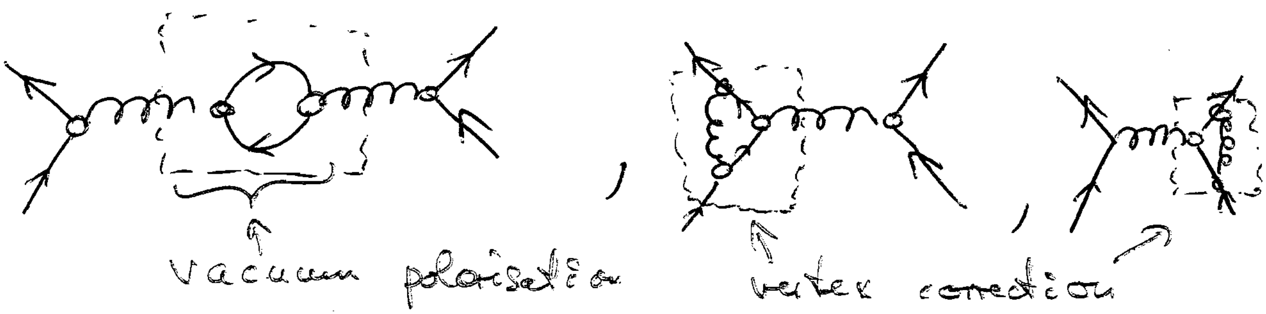
The strength of the coupling α_s depends on the energy/momentum of the process, in which it is measured

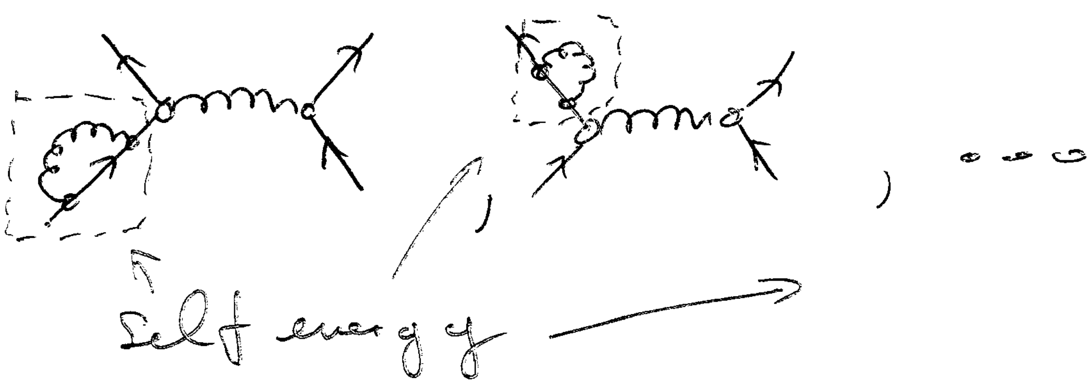
$q\bar{q}$ - scattering:



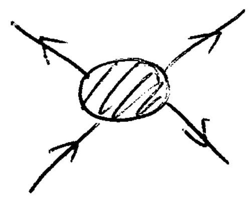
$\sim \alpha(s^2)$

α^2 - corrections:





full $q\bar{q}$-scattering:

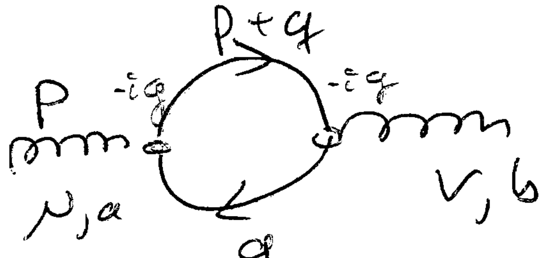


can be expressed in terms of $\alpha(s^2)$

[and other running parameters]

Computation of $\alpha(s^2)$ via the integration of loop integrals, such as

from Feynman rules: $t^a = \lambda^a/2$, $\mathrm{Tr}\, t^a t^b = \frac{1}{2}\delta^{ab}$

$\Pi^{ab}_{\mu\nu}(p) =$ 

spin and SU(3)

$$= -(ig)^2 \int \frac{d^4q}{(2\pi)^4}\, \mathrm{Tr}\, \frac{i}{\not{q} - m}\gamma^\mu t^a \frac{i}{(\not{p}+\not{q} - m)}\gamma^\nu t^b$$

$$\bar\psi\, \psi\, \bar\psi\, \psi \sim \bar\psi\, (\psi \bar\psi)\, \psi = -(\psi\bar\psi)(\psi\bar\psi)$$

$$\Rightarrow \Pi^{ab}_{\mu\nu}(p) \sim -\frac{\delta^{ab}}{2}(ig)^2 \int \frac{d^4q}{(2\pi)^4} \, \mathrm{tr}\, \frac{\not{q}}{q^2}\gamma^\mu \frac{\not{p}+\not{q}}{(p+q)^2}\gamma^\nu$$

$$= \left(p^2 g_{\mu\nu} - p_\mu p_\nu\right) \delta^{ab}/2 \; \Pi(p^2)$$

gauge symmetry

2 Problems:

(1) q-integral is divergent $\sim \int d^4q \, \frac{1}{q^2}$

(2) integrand divergent: $q^2 = 0$, $(p+q)^2 = 0$

Remark: vacuum polarisation in QCD is identical to that in QED, only the group factor $\delta^{ab}/2$ is different

Remedy: (Renormalisation)

(1) Regularisation: $\Pi(p^2, m^2) \to \Pi_\Lambda(p^2, m^2)$

$$\int \frac{d^4q}{(2\pi)^4} \to \int_{q^2 \leq \Lambda^2} \frac{d^4q}{(2\pi)^4}$$

(2) Wick rotation: $t_M \to i t_E$, $p_M^0 \to i p_E^0$

$$p_\nu p^\nu \to - p_\nu^E p_\nu^E$$

One-to-one relation between

Euklidean Correl. fcts. G_E
Minkowski " " G_M

Euclidean:

$$\int \frac{d^4q}{(2\pi)^4} \to \int_{q_\nu^E q_\nu^E \leq \Lambda^2} \frac{d^4q^E}{(2\pi)^4}$$

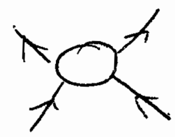
with cut-off parameter Λ.

In gauge theories: dimensional reg, $d^4q \to d^{4-2\varepsilon}q$; or Pauli-Villars

But: Results should not depend on Λ.

$\Rightarrow$ Renormalisation

(i) Parameter in classical action/Lagrangian are $\boxed{\text{not}}$ measurable!

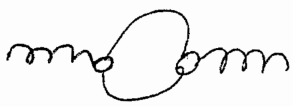
Example: coupling  vs 

$\Rightarrow$ have to be adjusted such that Observables ($>\!\!\alpha$) fit physics.

e.g. $\Rightarrow$ $A_\nu(\partial_\rho \partial^\rho g^{\nu\rho} - \partial^\nu \partial^\rho) A_\rho$

$$\xrightarrow{A_\nu = Z_A A_{ren\,\nu}} Z_A \, A_{ren\,\nu}(\partial_\rho \partial^\rho g^{\nu\rho} - \partial^\nu \partial^\rho) A_{ren\,\nu}$$

$\Rightarrow$  $^{-1} \longrightarrow Z_A$  $^{-1} \leftarrow \dfrac{\delta^2 S_{cl}}{\delta A^2_{ren}}$

$\Rightarrow Z_A$ ~~~$^{-1}$ +  $= \left(Z_A + \Pi_2(p^2, m^2)\right)$

amputated

$$\cdot \left(p^2 \delta_{\mu\nu} - p_\mu p_\nu\right) \frac{\delta^{ab}}{2}$$

Demand $Z_A^{-1} + \Pi_\Lambda(p^2, m^2)$ Λ-indep

or

$$\boxed{\left(Z_A^{-1} + \Pi_\Lambda(p^2, m^2) \Big|_{O(\alpha_s)} \right) = 0}$$

Structure of Π_Λ: $(m=0)$

$$\Pi_\Lambda(p^2, m^2) = \alpha_s \left(Z_1 \ln p^2/\Lambda^2 + f_0 + O(p^2/\Lambda^2) + \dots \right)$$

$$\Rightarrow \quad Z_A = 1 + \alpha_s Z_1 \left(\ln \Lambda^2/\mu^2 + \text{finite} \right) + O(\alpha_s^2)$$

RG-scale

Hence

$$Z_A + \Pi_\Lambda(p^2, 0) = 1 + \alpha_s Z_1 \ln p^2/\mu^2 + \text{finite} + O(\alpha_s^2)$$

In summary we demand:

$$\boxed{\mu \frac{d}{d\mu} \text{ Observables} = 0}$$

Computation:

$$\Pi^{\mu\nu}(p,m) = \left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \frac{\delta^{ab}}{2} \cdot \Pi(p^2, m^2)$$

$Z_A \ln \Lambda^2/\mu^2$ $\quad \boxed{\mu = m}$

with renormalised Π : $\Pi_\Lambda(p^2, m^2) - \Pi_\Lambda(0, m^2)$

$$\Pi(p^2) = -\frac{2\alpha}{\pi} i \int_0^1 dx\; x(1-x) \ln \frac{m^2}{m^2 - x(1-x)p^2}$$

$$= -\frac{2\alpha}{\pi} i \int_0^1 dx\; x(1-x) \ln \frac{m^2/p^2}{m^2/p^2 - x(1-x)}$$

UV-asymptotics:

$$\Pi(p^2) \simeq \frac{\alpha}{3\pi}\left[\ln\left(-p^2/m^2\right) \underbrace{-\,5/3}_{\text{finite} = c} + \mathcal{O}\left(m^2/p^2\right)\right]$$

QED

$$\Rightarrow \quad \alpha_{eff}(p^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi} \ln \frac{-p^2}{m^2 c}} \quad \xrightarrow{-p^2 \to m^2 c\, e^{3\pi/\alpha}} \infty, \qquad \xrightarrow{-p^2 \to 0} 0$$

with (here)

$$c = e^{5/3}, \quad \boxed{\alpha_{eff}(c\, m^2) = \alpha}$$

1-loop β-fct

RG-eq:

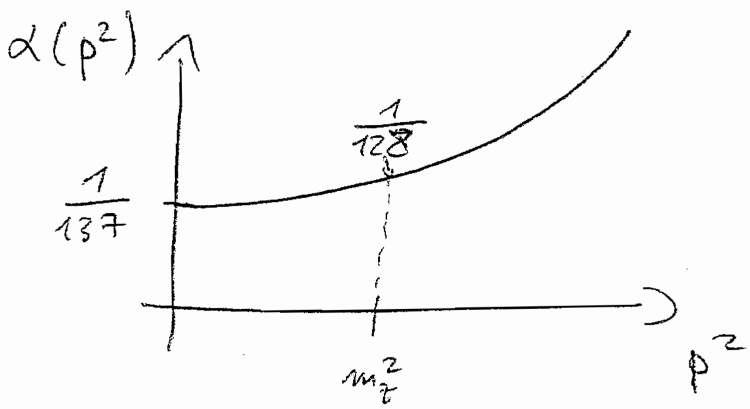
$$\left(\mu \frac{d}{d\mu} Z_A\right)/Z_A = \frac{2}{3\pi}\alpha = \boxed{\frac{1}{6\pi^2}\alpha}$$

5.2 Running coupling

As in QED we compute the running coupling

p. 5-13a: QED (Euclidean)

$p^2 \gg m^2_{\text{lept.}}$ $$\alpha(p^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi} \ln p^2/\mu^2}$$



computed from 

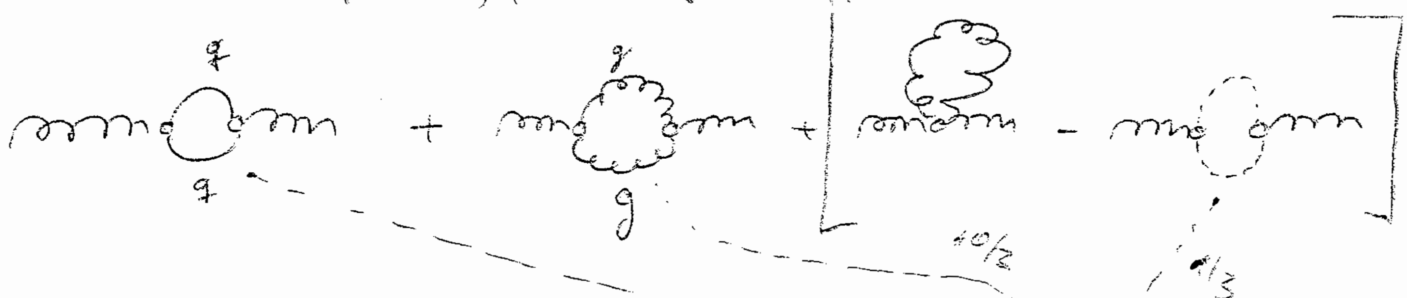
β-function: $$\beta_{QED}(\alpha) \simeq p^2 \partial_{p^2} \alpha(p^2) = \frac{\alpha^2}{3\pi} + \mathcal{O}(\alpha^3)$$

$$> 0$$

$$\beta_{QED} = -\beta_0 \alpha^2 - \beta_1 \alpha^3 + \ldots$$

$$\beta_0 = -\boxed{\frac{1}{3\pi}}$$

QCD: (background field formalism)



$$\beta_{QCD} \simeq p^2 \partial_{p^2} \alpha_s(p^2) = -\frac{1}{12\pi}(33 - 2N_f)\,\alpha_s^2$$

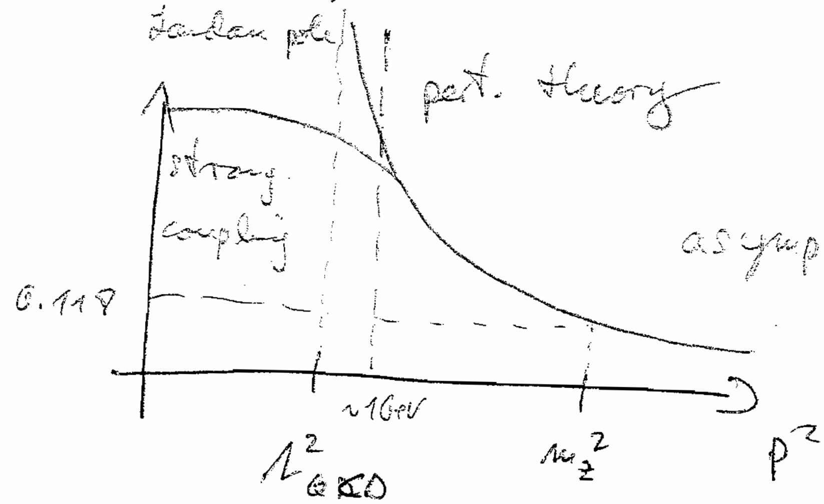
$$+\,\mathcal{O}(\alpha_s^3)$$

$$\Rightarrow \boxed{\beta_0 = \frac{1}{12\pi}(33 - 2N_f)}$$

$$\Rightarrow \quad \alpha_s(p^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)\,\beta_0 \ln p^2/\mu^2}$$

$$= \frac{1}{\beta_0 \ln p^2/\Lambda^2_{QCD}}$$

with $(\mu^2 = \Lambda^2_{QCD} \cdot e^{-\beta_0/\alpha_s(\mu)})$



Gross, Wilczek, Politzer '04

asympt. freedom

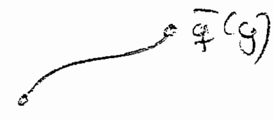
$\alpha_s(p^2) \sim \frac{1}{\beta_0 \ln p^2/\Lambda^2_{QCD}} \to 0$

$\Lambda_{QCD} \simeq 217\left(^{+25}_{-23}\right)^{?}$ MeV

5.3 Confinement

no coloured asymptotic states

Example: $q\bar{q}$-pair

(1) Definition of $q\bar{q}$-state: 

$\bar{q}(y)\, q(x)$ not gauge invariant

but

$$\bar{q}(y)\, \underbrace{P e^{+i g_s \int_y^x dz^\mu\, t^a A^a_\mu(z)}}_{\prod_{z=y}^{x} \left(1 - i g_s\, dz^\mu\, t^a A^a_\mu(z)\right)}\, q(x)$$

gauge trafo: $A_\mu(x) \to U(x)\, A_\mu(x)\, U^\dagger(x) - \frac{i}{g_s} U(x)\, \partial_\mu U^\dagger(x)$ p. 4-5

$$\Rightarrow \left(1 + i g_s\, dz^\mu t^a A^a_\mu(z)\right)' = U(z)\left(1 + i g_s\, dz^\mu t^a A^a_\mu(z)\right) \cdot \underbrace{\left(U^\dagger(z) + \partial_\mu U^\dagger(z)\, dz^\mu\right)}_{U^\dagger(z+dz)}$$

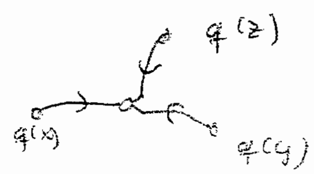
$$\Rightarrow \quad P e^{i g_s \int_y^x dz^\mu A_\mu(z)} \to U(y)\, P e^{i \int_y^x g_s\, dz^\mu A_\mu(z)}\, U^\dagger(x)$$

(2) $$\lim_{|x-y|\to\infty} \left\langle \bar{q}(y)\, P e^{i g_s \int_y^x dz^\mu A_\mu(z)}\, q(x) \right\rangle \to 0$$

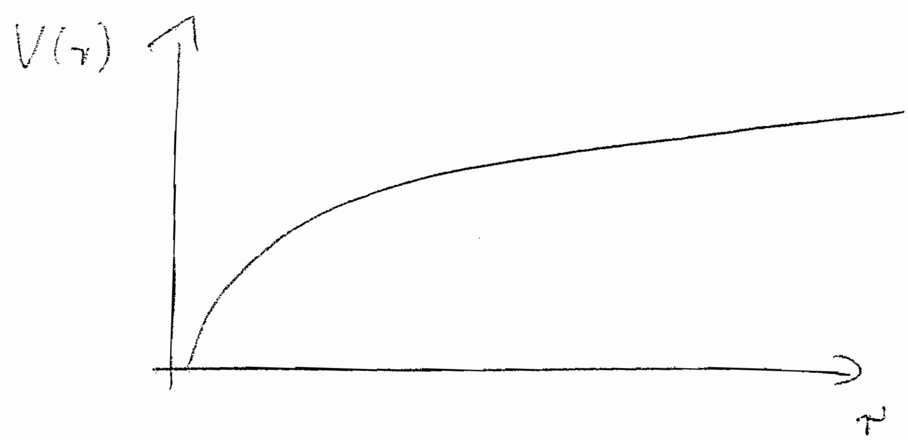
in quenched QCD
(no dyn. quarks)

confinement

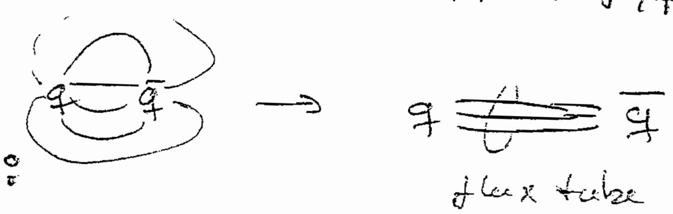
[three quarks:



]

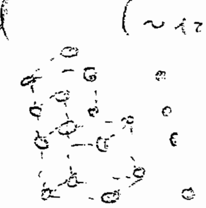


$$V(r) = V_0 + k\cdot r - e/r + f/r^2$$

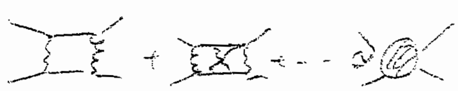


Remarks:

- strong coupling is not enough!!
- mass gap in Yang-Mills (pure glue [Millenium Prize (Jaffe, Witten)]
- perturbation theory fails ⇒ non-pert. methods
  - lattice: space-time grid (~ 128^4 lattices)



  - operator product expansions / sum rules ...

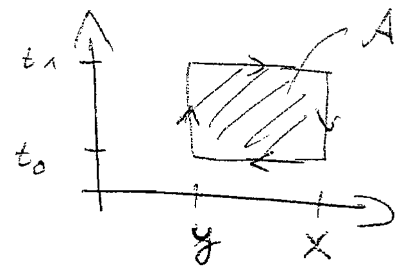


  - renormalisation group methods
    solve theory via relations between correlation fcts.

leave out?

) Area law of Wilson Loop

$$W(\mathcal{C}_{x,y}) = \mathrm{tr}\, P\, e^{ig_s \int_{\mathcal{C}_{x,y}} dz_\mu A^\mu(z)}$$



[QED: $e^{-ie\int d^4x\, j_\mu(x)\, A^\mu(x)}$

with $j_\mu = \int_{\mathcal{C}_{x,y}} dz_\mu\, \delta(x-z)$

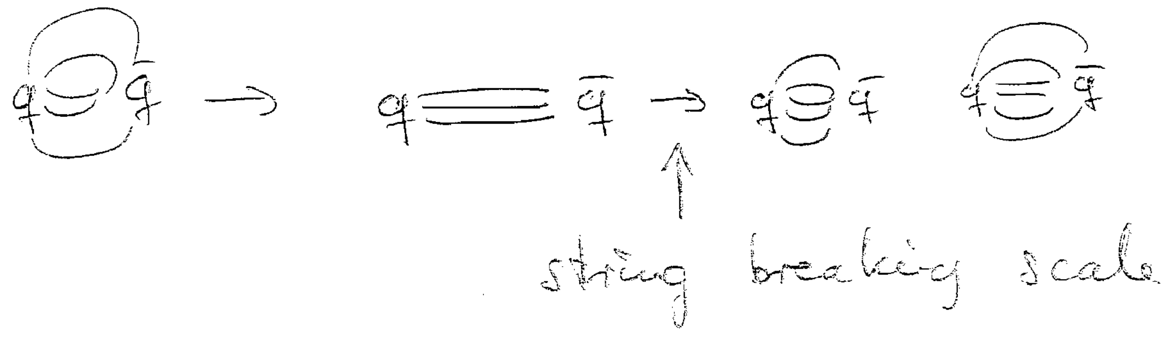
word line of an electron]

$$\Rightarrow \langle W(\mathcal{C}_{x,y}) \rangle \sim e^{-F_{q_x \bar{q}_y}} \longrightarrow 0$$

$$\sim e^{-\sigma A}$$

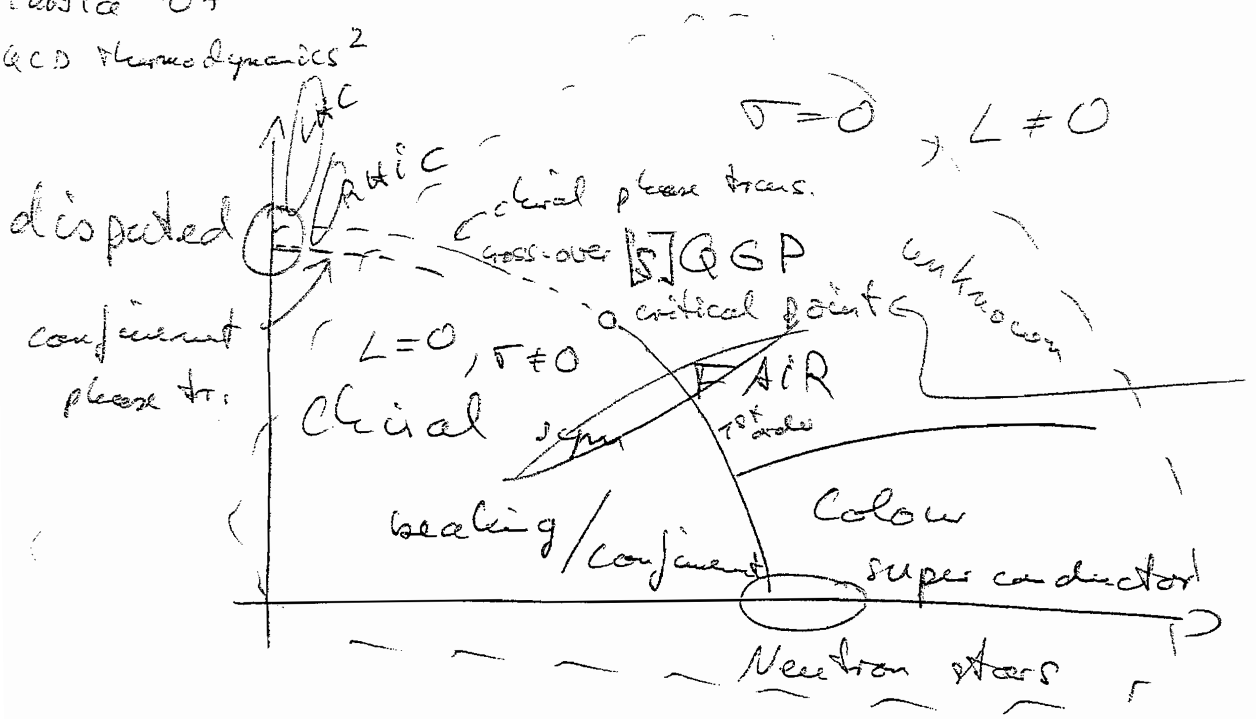
---

(4) dynamical quarks



5.4 Phase diagram of QCD

Lattice '07
QCD thermodynamics²



Order parameter:

- chiral condensate: $\langle \bar{q} q \rangle = \sigma$
- Polyakov loop $L \sim e^{-F_q}$

Remark on phase transitions