II. Prerequisites

1. Relativistic kinematics
2. Wave description of free particles
3. Non relativistic perturbation theory
4. Scattering matrix and transition amplitudes
5. Cross section and phase space
6. Decay width, lifetimes and Dalitz plots

Literature: *F. Halzen, A.D. Martin, "Quarks and Leptons"*
O. Nachtmann, "Elementarteilchenphysik"

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1. Relativistic kinematics

1.1 Notations

- 4-vector
  - contra-variant form $x^\mu = \left(x^0, \vec{x}\right) = (t, \vec{x}) \quad p^\mu = \left(p^0, \vec{p}\right) = (E, \vec{p})$
  - covariant form $x_\mu = \left(x^0, -\vec{x}\right) = (t, -\vec{x}) \quad p_\mu = \left(p^0, -\vec{p}\right) = (E, -\vec{p})$
- Metric tensor

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad x_\mu = g_{\mu\nu} x^\nu \qquad x^\mu = g^{\mu\nu} x_\nu$$

- Derivative operator

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right) \qquad \partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

- Scalar product

$$ab = a_\mu b^\mu = g_{\mu\nu} a^\nu b^\mu = (a^0 b^0 - \vec{a} \cdot \vec{b})$$

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1.2 Lorentz and Poincaré transformations (formal)

Poincaré transformations $x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu$

Combined transformations $(\Lambda_1, a_1) \circ (\Lambda_2, a_2) = (\Lambda_1 \cdot \Lambda_2, \Lambda_1 a_2 + a_1)$

Leaves the scalar product of space-time difference invariant:

$$(x-y)^2 = (x-y)^\mu (x-y)_\mu = (x_0 - y_0)^2 - (\vec{x} - \vec{y})^2$$

$$\Rightarrow \quad \det \Lambda = \pm 1 \quad \Rightarrow \quad \left(\Lambda^{-1}\right)^\mu_\sigma = \Lambda^\sigma_\mu$$

Lorentz group $(\Lambda, 0)$ $\quad \det \Lambda = +1, \quad \Lambda^0_0 > 0$ (orthochron, reachable from unity)

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \left(e^{\omega K_1}\right)^\mu_\nu$$

Rapidity ω: $\omega = \arctan(\beta)$

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With generator $K_1 = \begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$ $\quad \left[K_i, K_j\right] = -i\,\varepsilon_{ijk} K_k$

Remember rotation: $\left[J_i, J_j\right] = +i\,\varepsilon_{ijk} J_k$

Rotation & Lorentz-Trf $\left[J_i, K_j\right] = +i\,\varepsilon_{ijk} K_k$

Discrete Poincare transformations:

Parity P: $\vec{x} \to -\vec{x}$

$$\Lambda_P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Time reversal: $x^0 \to -x^0$

$$\Lambda_T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

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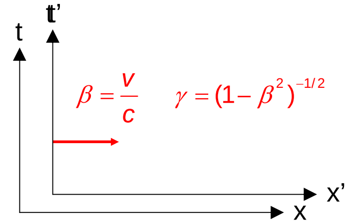
1.3 Lorentz invariants

Lorentz transformation:

moving particle with $p = (E, \vec{p})$

$$p' = \begin{cases} \begin{pmatrix} E' \\ \vec{p}_l' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E \\ \vec{p}_l \end{pmatrix} \\ \vec{p}_t' = \vec{p}_t \end{cases}$$



$\beta = \frac{v}{c}$ $\gamma = (1-\beta^2)^{-1/2}$

w/r to rest frame: $\beta = \frac{|\vec{p}|}{E}$ $\gamma = \frac{E}{m}$

Scalar products are invariant under Lorentz transformations: $a'b' = ab$

Example 1: invariant mass

$$p^2 = p_\mu p^\mu = E^2 - \vec{p}^2 = m^2$$

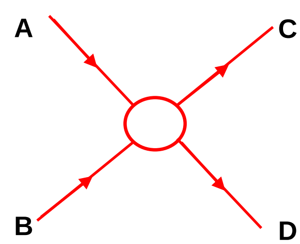
Example 2: CMS energy of 2 particle collision calculated in any frame

$$s = (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$$

Remark: Lorentz invariants (e.g. cross sections) can only depend on scalars 5

1.4 Mandelstam variables

$$A + B \rightarrow C + D$$



What are the Lorentz scalars the cross section can depend on ?

$p_i p_k$ with $p_{i,k\geq i} = p_A, p_B, p_C, p_D$

(unpolarized particles)

10 combinations

$p_i^2 = m_i^2$ — 4 constraints

4-mom. conservation: 4 constraints

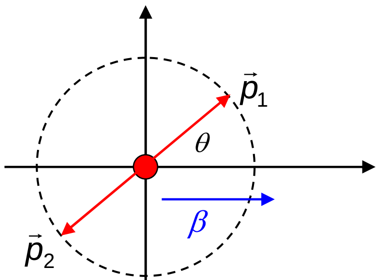
➡ **2 independent products**

Instead of $p_i p_k$ use 2 out of the 3 **Mandelstam variables**

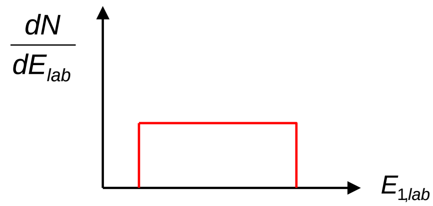
$$s = (p_A + p_B)^2$$
$$t = (p_A - p_c)^2$$
$$u = (p_A - p_D)^2$$

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$$

Example: 2-body decay of spinless particle



$\frac{dN}{d\Omega} = \text{const}$



$$E_{1,lab} = \gamma E_1 + \gamma\beta\,|\vec{p}_1|\cos\theta$$

$$dE_{1,lab} = \gamma\beta\,|\vec{p}_1|\,d\cos\theta$$

with $\frac{dN}{d\cos\theta} = \frac{dN}{dE_{1,lab}} \cdot \frac{dE_{1,lab}}{d\cos\theta}$ ➡ $\frac{dN}{dE_{1,lab}} = \text{const} \times \gamma\beta\,|\vec{p}_1|$ flat

$$\left(E_{1,lab}\right)_{\text{min, max}} = \gamma E_1 \pm \gamma\beta\,|\vec{p}_1|$$

In case of the decay of a particle with spin J angular momentum conservation leads to a deviation from the isotropic decay.

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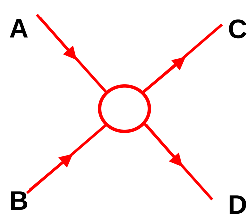
1.5 Crossing

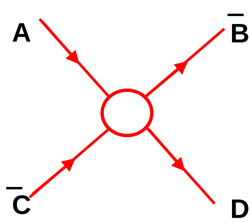
$$A + B \rightarrow C + D$$

$$s = (p_A + p_B)^2$$
$$t = (p_A - p_c)^2$$
$$u = (p_A - p_D)^2$$



$$A + \overline{C} \rightarrow \overline{B} + D$$

$$s' = (p_A + p_{\overline{C}})^2$$
$$t' = (p_A - p_{\overline{B}})^2$$
$$u' = (p_A - p_D)^2$$



Crossing:
$p_C \rightarrow -p_{\overline{C}}$
$p_B \rightarrow -p_{\overline{B}}$
$t \rightarrow s'$
$s \rightarrow t'$
$u \rightarrow u'$

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2. Wave description of free particles

2.1 Schrödinger Equation for non-relativistic free particles

$$i\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\nabla^2\psi$$

Solution for energy $E = \frac{p^2}{2m}$

$$\psi(\vec{r},t) = \frac{1}{\sqrt{V}}\exp\left[i(\vec{p}\vec{x} - Et)\right]$$

Continuity equation:

$$\rho = |\psi|^2$$

$$\vec{j} = \frac{1}{2im}\left(\psi^*(\nabla\psi) - (\nabla\psi^*)\psi\right)$$

$$\frac{\partial\rho}{\partial t} + \nabla\vec{j} = 0$$

Schrödinger Eq uses classical E-p relation $E^2=p^2/2m$ and the replacement $E \rightarrow i\frac{\partial}{\partial t}$ and $\vec{p} = -i\vec{\nabla}$

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2.2 Klein-Gordon Equation

Starts from relativistic energy relation $E^2=p^2+m^2$:

Describes relativistic Spin 0 particles

$$\frac{\partial^2}{\partial t^2}\phi - \nabla^2\phi + m^2\phi = 0$$

Solutions for energy values:

$$E_\pm = \pm\sqrt{p^2+m^2} \gtrless 0$$

$$\phi(\vec{r},t) = N\exp\left[i(\vec{k}\vec{x} - \omega_\pm t)\right]$$

negative E values cannot be ignored as otherwise solutions are incomplete

N = normalization (later)

Most general solution (superposition):

$$\phi(x) = N\int\frac{d^3k}{(2\pi)^3}\left(e^{ikx}\alpha^*(\vec{k}) + e^{-ikx}\alpha(\vec{k})\right) \quad \text{with} \quad k = (\omega,\vec{k}) = (E,\vec{p})$$

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with $\rho = \left(i\phi^* \frac{\partial}{\partial t}\phi - i\phi \frac{\partial}{\partial t}\phi^* \right)$ and $\vec{j} = \left(-i\phi^*(\nabla\phi) - i\phi(\nabla\phi^*) \right)$

➡ Continuity equation: $\frac{\partial \rho}{\partial t} + \nabla \vec{j} = 0$

For the solution: $\phi(\vec{r},t) = N \exp\left[i(\vec{p}\vec{x} - E_{\pm} t) \right]$

$\vec{j} = \left(-i\phi^*(\nabla\phi) - i\phi(\nabla\phi^*) \right)$ $\qquad \vec{j} = 2\vec{p}|N|^2$

$\rho = \left(i\phi^* \frac{\partial}{\partial t}\phi - i\phi \frac{\partial}{\partial t}\phi^* \right)$ $\qquad \rho = 2E|N|^2$

What are negative probabilities for the E < 0 solutions ?

Normalization schemes:

N = 1/√(2EV) ⇒ 1 particle per unit volume V

N = 1/√V ⇒ 2E particles per unit volume V

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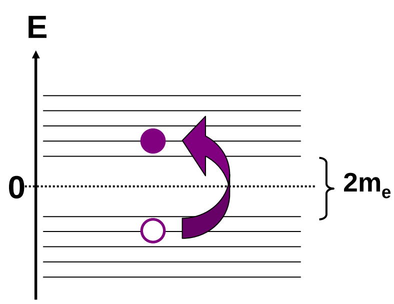
2.3 Anti-particles – Historical interlude

Dirac interpretation for fermions: **Vacuum = sea of occupied neg. E levels**



For fermions the negative energy levels are w/o influence as long as they are fully occupied

Missing e⁻ w/ negative energy corresponds to to a positron w/ E>0

e⁺e⁻ annihilation:

Free energy level in the sea. e⁻ drops into the hole and releases energy by photon emission: $E_\gamma > 2m_e$

Photon conversion for $E_\gamma > 2m_e$

Excitation of e⁻ from neg. energy level to pos. level: $\gamma \rightarrow e^+e^-$

Model predicts anti-particles (Discovery of positron by Anderson in 1933)

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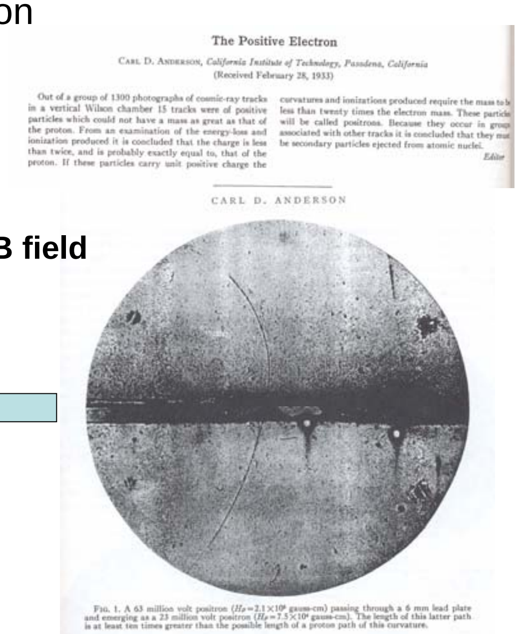
Discovery of the positron

Anderson, 1933



The Positive Electron

CARL D. ANDERSON, *California Institute of Technology, Pasadena, California*
(Received February 28, 1933)

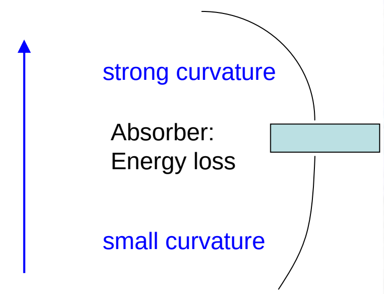
Out of a group of 1300 photographs of cosmic-ray tracks in a vertical Wilson chamber 15 tracks were of positive particles which could not have a mass as great as that of the proton. From an examination of the energy-loss and ionization produced it is concluded that the charge is less than twice, and is probably exactly equal to, that of the proton. If these particles carry unit positive charge the curvatures and ionizations produced require the mass to be less than twenty times the electron mass. These particles will be called positrons. Because they occur in groups associated with other tracks it is concluded that they must be secondary particles ejected from atomic nuclei.

Editor

CARL D. ANDERSON

FIG. 1. A 63 million volt positron ($H\rho = 2.1\times10^5$ gauss-cm) passing through a 6 mm lead plate and emerging as a 23 million volt positron ($H\rho = 7.5\times10^4$ gauss-cm). The length of this latter path is at least ten times greater than the possible length of a proton path of this curvature.

⊗ **B field**



$$\vec{F}_L = q\vec{v}\times\vec{B}$$

$$\Rightarrow \quad q = +e$$

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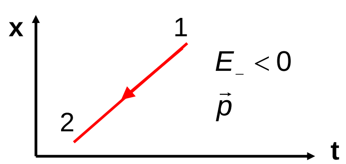
Feynman Stückelberg interpretation

$$E_+ = E: \quad \phi_+ = N\exp(i\vec{p}\vec{x} - iEt)$$

Solutions with neg. energy propagate backwards in time: $\Rightarrow$

$$E_- = -E: \quad \phi_- = N\exp(i\vec{p}\vec{x} + iEt)$$

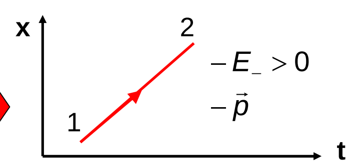
Solutions describe anti-particles propagating forward in time:



$E_- < 0$

$\vec{p}$

$\Leftrightarrow$



$-E_- > 0$

$-\vec{p}$

Neg. probability density

$$\left.\begin{aligned} \rho &= 2E|N|^2 \\ \vec{j} &= 2\vec{p}|N|^2 \end{aligned}\right\} \times q \quad \Rightarrow \quad \begin{aligned} J^0 &= q\cdot 2E|N|^2 \\ \vec{J} &= q\cdot 2\vec{p}|N|^2 \end{aligned}$$

Charge density / currents

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Example

Particle T^- with q= –e and energy $E\_ = –E < 0$

$$J^0(T^-) = (-e)\cdot 2(-E)|N|^2 = (+e)\cdot 2(+E)|N|^2 = J^0(T^+)$$
$$\vec{J}(T^-) = (-e)\cdot 2\vec{p}|N|^2 = (+e)\cdot 2(-\vec{p})|N|^2 = \underbrace{\vec{J}(T^+)}$$

T^+ with $E(T^+) > 0,\ \vec{p}_{T^+} = -\vec{p}_{T^-}$

Description of creation and annihilation:

- Emission of anti-particle $\bar{T}$ with $p^\mu = (E, \mathbf{p})$ ⇔ absorption of particle T with $p^\mu = (-E, -\mathbf{p})$
- Absorption of anti-particle $\bar{T}$ with $p^\mu = (E, \mathbf{p})$ ⇔ emission of T with $p^\mu = (-E, -\mathbf{p})$

The discussion of neg./pos. energy solutions can be avoided when using QFT, the correct relativistic quantum theory

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Interpretation in Quantum Field Theory

- In quantum mechanics observables become operators with an expectation value. The operators work on states (= vectors of a Hilbert space).
- An observable "classical field" (e.g. E or B) becomes a field operator with an expectation value.
- The classical observable of a meson wave is a scalar field function $\phi(x)$. Correspondingly there is an associated quanten-mechanical field operator $\Phi(x)$ with $\phi(x) = <|\Phi(x)|>$:

$$\Phi(x) = \frac{1}{2\omega}\int \frac{d^3k}{(2\pi)^3}\left(e^{ikx}a^+(\vec{k}) + e^{-ikx}a(\vec{k})\right)$$

- Again, possible states are vectors of a Hilbert space. The simplest state is the vacuum |0>. One finds that a(p) annihilates a particle with momentum k, while $b^+(k)$ creates a particle with momentum k:

$$a^+(k)|0\rangle = |k\rangle \quad a(k)|k\rangle = |0\rangle \quad a(k)|0\rangle = 0$$ ← In this way no neg. E

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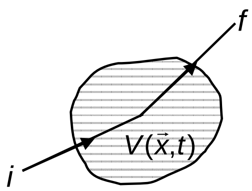
3. Non-relativistic perturbation theory

Free particle's "Schrödinger equation" with solutions ϕ_n:

$$H_0\,\phi_n = E_n\,\phi_n \qquad \text{with} \qquad \int_V \phi_m^*\,\phi_n\,d^3x = \delta_{mn}$$

For simplicity: normalization = 1 particle / V

➡ solve Schrödinger equation in presence of additional interaction potential



$$\left(H_0 + V(\vec{x},t)\right)\psi = i\frac{\partial\psi}{\partial t} \;\Rightarrow\; \psi = \sum_n a_n(t)\,\phi_n(\vec{x})\,e^{-iE_nt}$$

(any solution)

One finds for the coefficients $a_n(t)$:

$$\frac{da_f}{dt} = -i\sum_n a_n(t)\underbrace{\int \phi_f^* V \phi_n d^3x}_{\langle f|V|n\rangle}\, e^{i(E_f-E_n)t}$$

$$\cong -i\langle f|V|i\rangle e^{i(E_f-E_i)t} + O(V^2)$$

By using initial conditions: $a_i(-T/2) = 1 + O(V^2) \quad a_n(-T/2) = 0 \;\text{ for } n \neq i$

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For details, see Martin & Halzen, p. 79

$$\Rightarrow \quad a_f(t) = -i\int_{-T/2}^{t} dt'\,\langle f|V|i\rangle\, e^{i(E_f-E_i)t'}$$

Define transition amplitude:

$$A_{fi} \equiv a_f(T/2) = -i\int_{-T/2}^{T/2} dt'\,\langle f|V|i\rangle\, e^{i(E_f-E_i)t'}$$

For $t\to\infty$:

$$A_{fi} = -i\int d^4x\,\phi_f^*(x)V\phi_i(x)$$

Can the transition amplitude be interpreted as transition probability for $i\to f$?
Assume that V(x) is time independent:

$$\lim_{T\to\infty}|A_{fi}|^2 = \ldots = \left|\langle f|V|i\rangle\right|^2 \cdot 2\pi\,\delta(E_f - E_i)\cdot T \qquad \textit{(Fermi's trick)}$$

Transition probability per time

$$w_{fi} = \lim_{T\to\infty}\frac{|A_{fi}|^2}{2} = 2\pi\left|\langle f|V|i\rangle\right|^2\delta(E_f - E_i)$$

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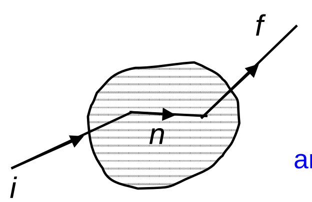
This equation can only be given a physical meaning after integrating over the set of possible final states: $\rho(E_f)$ is the density of final states, and $\rho(E_f)dE_f$ is the number of states with an energy in $[E_f ; E_f+dE_f]$.

Transition rate:

$$\Gamma_{fi} = 2\pi \int dE_f \rho(E_f) \left|\langle f|V|i\rangle\right|^2 \delta(E_f - E_i)$$
$$= 2\pi \left|\langle f|V|i\rangle\right|^2 \rho(E_i)$$

Fermi's golden rule



Higher orders

take $a_f(t) = -i \int_{-T/2}^{t} dt' \langle f|V|i\rangle\, e^{i(E_f - E_i)t'}$

and insert in $\frac{da_f}{dt} = -i \sum_n a_n(t) \int \phi_f^* V \phi_n d^3x\, e^{i(E_f-E_n)t}$

➡ 2nd order correction in Fermi's rule

$$V_{fi} = \left|\langle f|V|i\rangle\right|^2 \rightarrow V_{fi} + \sum_{n\neq i} V_{fn} \frac{1}{E_i - E_n + i\varepsilon} V_{ni}$$

Propagator for intermediate state

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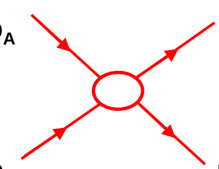
4. Scattering matrix and transition amplitude

Scattering process:

$$\pi p \rightarrow \pi p$$



Described through quantum numbers of initial and final state: $|i\rangle \rightarrow |i'\rangle$

Scattering operator (S matrix): $|i'\rangle = \mathbf{S}|i\rangle$

Measurement selects a specific state f. Probability to find f: $\langle f|i'\rangle = \langle f|\mathbf{S}|i\rangle = \mathbf{S}_{fi}$

As there is the probability that $|i'\rangle = |i\rangle$ it is useful to introduce the transition operator T $\quad \mathbf{S} = \mathbf{1} + \mathbf{T}$ with $T_{fi} = \langle f|\mathbf{T}|i\rangle$

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Instead of T_{fi} , conventionally one uses the transition or scattering amplitude M_{fi}

$$T_{fi} = -i \cdot (2\pi)^4 N_A N_B N_C N_D \delta^4 (p_A + p_B - p_C - p_D) \cdot M_{fi}$$

normalization:
$N_k = 1/\sqrt{V} \to$ 2E particles/V

4-momentum conservation

Feynman rules for calculation

Transition probability:

$$w_{fi} = |T_{fi}|^2 = (2\pi)^8 (N_A N_B N_C N_D)^2 [\delta^4 (p_A + p_B - p_C - p_D)]^2 |M_{fi}|^2$$

In this convention the transition probability is given for a single „possible final state“. It turns out that the final state particles $C(p_C)$ and $D(p_D)$ can be in more than one state. The number of possible final states is described by the **phase space factor** and will be considered when calculating observable quantites.

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Transition rate per unit volume:

$$W_{fi} = \frac{|T_{fi}|^2}{T \cdot V} = \frac{1}{T \cdot V} (2\pi)^8 (N_A N_B N_C N_D)^2 [\delta^4 (p_A + p_B - p_C - p_D)]^2 |M_{fi}|^2$$

$$= (2\pi)^4 \frac{1}{V^4} \delta^4 (p_A + p_B - p_C - p_D) |M_{fi}|^2$$

Fermi‘s Trick:

$$[\delta^4 (p_A + p_B - p_C - p_D)]^2 = \frac{VT}{(2\pi)^4} \delta^4 (p_A + p_B - p_C - p_D)$$

$$[2\pi\, \delta^4 (x - x')]^2 = \int_{-\infty}^{+\infty} dt\, e^{i(x-x')t} \cdot 2\pi\, \delta(x - x')$$

$$= \left(\int_{-\infty}^{+\infty} dt \right) \cdot 2\pi\, \delta(x - x')$$

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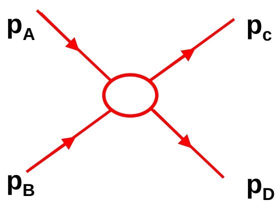
5. Cross section and phase space

$$A + B \to C + D$$



Transition rate:

$$W_{fi} = \frac{(2\pi)^4}{V^4}\delta^4(p_A + p_B - p_C - p_D)\cdot\left|M_{fi}\right|^2$$

$$\text{Cross section} = \frac{W_{fi}}{\text{(initial flux)}}\text{(number of final states)}$$

Cross section: $\sigma = \frac{W_{fi}}{F}\rho_f(C,D)$

ρ_f number of final states for given configuration
F incident particle flux of A and B

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5.1 Number of final states (phase space)

Quantum mechanics restricts the number of final states $d\rho_f$ of a single particle in a volume V with momentum $\in\left[\vec{p},\vec{p}+d\vec{p}\right]$

$$d\rho_f = \frac{Vd^3p}{2E\,h^3} = \frac{Vd^3p}{2E\,\hbar^3(2\pi)^3} = \frac{Vd^3p}{2E\,(2\pi)^3}$$

$\hbar = 1$

Factor 2E is the result of normalization of the wave function: 2E particles / V

For particle C and D scattered into momentum elements d^3p_C and d^3p_D

$$d\rho_f(C,D) = \frac{Vd^3p_C}{2E_C(2\pi)^3}\,\frac{Vd^3p_D}{2E_D(2\pi)^3}$$

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5.2 Incident particle flux F

Choose rest frame of particle B to calculate F (simplest)

F = (flux density A) × (density B)

$A \xrightarrow{\vec{v}_A} \bullet\ B$

$$F = |\vec{v}_A| \frac{2E_A}{V} \cdot \frac{2E_B}{V}$$

with $\vec{v}_A = \dfrac{\vec{p}_A}{E_A}$

CMS frame:

$A \xrightarrow{\vec{p}_A} \ \xleftarrow{\vec{p}_B} B$

$\vec{p}_A = -\vec{p}_B = \vec{p}_i$

$$F = \frac{4}{V^2}|\vec{p}_i| \cdot (E_A + E_B) = \frac{4}{V^2}|\vec{p}_i|\sqrt{s}$$

General form :
(in any frame)

$$F = \frac{2w(s, m_1^2, m_2^2)}{V^2}$$

(see Nachtmann, p. 19, 76)

with $w(s, m_1^2, m_2^2) = \left((s-(m_1+m_2)^2)(s-(m_1-m_2)^2)\right)^{1/2}$

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5.3 Lorentz invariant phase space factor

Putting everything together $\quad d\sigma = \dfrac{W_{fi}}{F} d\rho_f$

$$d\sigma = \frac{(2\pi)^4}{V^4}\delta^4(p_A + p_B - p_C - p_D) \cdot |M_{fi}|^2 \frac{V^2}{|\vec{v}_A| 2E_A 2E_B} \cdot \frac{Vd^3p_c}{2E_C(2\pi)^3} \cdot \frac{Vd^3p_D}{2E_D(2\pi)^3}$$

$$= \frac{|M_{fi}|^2}{|\vec{v}_A| 2E_A 2E_B} \cdot (2\pi)^4\delta^4(p_A + p_B - p_C - p_D) \cdot \frac{d^3p_c}{2E_C(2\pi)^3} \cdot \frac{d^3p_D}{2E_D(2\pi)^3}$$

$= 4|\vec{p}_i|\sqrt{s}$
CMS

Particle flux ***F***

Lorentz invariant 2-particle phase space factor $d\Phi_2$

Remark: volume V drops out !

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Phase space factor for n particles in the final state:

$$d\Phi_n(P,\underbrace{p_1,p_2,\ldots,p_n}_{\text{Final state}}) = (2\pi)^4\,\delta^4(P-(p_1+p_2+\ldots+p_n))\prod_{final}\frac{d^3p_f}{(2\pi)^3\,2E_f}$$

See also PDG
http://pdg.lbl.gov/2008/reviews/rpp2008-rev-kinematics.pdf

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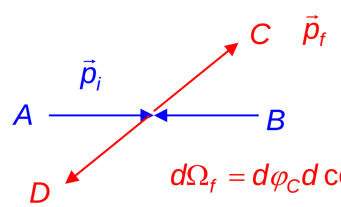
Phase space integration for two-particles final-state (CMS)

CM System:
$\vec{p}_i = \vec{p}_A = -\vec{p}_B \quad \vec{p}_f = \vec{p}_C = -\vec{p}_D$
$s = (E_A + E_B)^2$



$$d\Omega_f = d\varphi_C\, d\cos\theta_C$$

$$d\Phi_2 \xrightarrow{\int}$$

$$\int d\Phi_2 = \frac{1}{4\pi^2}\int \delta^3(\vec{p}_C+\vec{p}_D)\,\delta(E_A+E_B-E_C-E_D)\,\frac{d^3p_C}{2E_C}\frac{d^3p_D}{2E_D}$$

$$\int d\Phi_2 = \frac{1}{16\pi^2}\int\frac{|\vec{p}_f|}{\sqrt{s}}\,d\Omega_f$$

To perform the integration see e.g. also C. Berger.

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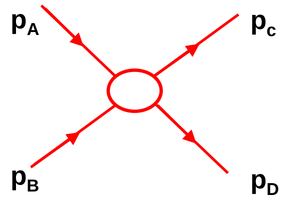
5.4 Differential cross section …putting everything together

$$A + B \rightarrow C + D$$



CMS

$$d\sigma = \frac{|M_{fi}|^2}{F} d\Phi_2 = \frac{1}{64\pi^2} \cdot \frac{1}{s} \cdot \frac{|\vec{p}_f|}{|\vec{p}_i|} \cdot |M_{fi}|^2 d\Omega_f$$

$$\frac{d\sigma}{d\Omega_f} = \frac{1}{64\pi^2} \cdot \frac{1}{s} \cdot \frac{|\vec{p}_f|}{|\vec{p}_i|} \cdot |M_{fi}|^2$$

- The dynamics of the scattering process is contained in the matrix element M_{fi} which can be calculated using Feynman rules
- 1/s dependence of the cross section because of initial/final state kinematics

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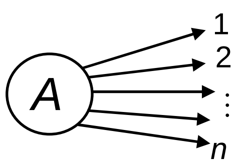
6. Decay width, lifetime and Dalitz plots

5.1 Decay width



$$\tau = \frac{1}{\Gamma} \qquad \Gamma = \sum \Gamma_i$$

Differential decay width (rate):

$$d\Gamma_i(A \rightarrow 1 + 2 + ... + n) = \frac{W_{fi}}{n_A} d\rho_f$$

$$d\Gamma_i = \frac{|M_{fi}|^2}{2E_A} \cdot (2\pi)^4 \delta^4(p_A - p_1 - p_2 - ... - p_n) \cdot \frac{d^3p_1}{2E_1(2\pi)^3} \cdot \frac{d^3p_2}{2E_2(2\pi)^3} \cdot ... \cdot \frac{d^3p_n}{2E_n(2\pi)^3}$$

Two-body decay:

$$A \rightarrow 1 + 2$$

$$d\Gamma_i(A \rightarrow 1 + 2) = \frac{|M_{fi}|^2}{2E_A} d\Phi_2 = \frac{|M_{fi}|^2}{2E_A} \frac{1}{16\pi^2} \frac{|\vec{p}_f|}{\sqrt{s}} d\Omega$$

CMS: $d\Phi_2 = \frac{1}{16\pi^2} \frac{|\vec{p}_f|}{\sqrt{s}} d\Omega_f$

$$\sqrt{s} = E_A = m_A$$

$$d\Gamma_i(A \rightarrow 1 + 2) = \frac{|\vec{p}_f|}{32\pi^2 m_A^2} |M_{fi}|^2 d\Omega$$

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Three-body decay:

flat in E_1 and E_2

$A \to 1 + 2 + 3$

$$\int d\Phi_3 = \frac{1}{8(2\pi)^5} \int dE_1 dE_2\, d\alpha\, d(\cos\beta)\, d\gamma$$

for scalar A or averaged over spins ➡

$$d\Gamma_i(E_1, E_2) = \frac{1}{64\pi^3} \frac{1}{m_A} |M_{fi}|^2 dE_1 dE_2$$

Remark: Instead of variables E_1 and E_2 one can use variables m_{12}^2 and m_{23}^2 = invariant mass of pairs (i,j)

$$m_{ij}^2 = (p_i + p_j)^2$$

$$dE_1 dE_1 = C \cdot dm_{12}^2 dm_{23}^2$$

$$d\Gamma_i(m_{12}{}^2, m_{23}{}^2) = \frac{1}{256\pi^3} \frac{1}{m_A^3} |M_{fi}|^2 dm_{12}{}^2 dm_{23}{}^2$$

If phase space is flat in E_i then it is also flat in m_{ij}

(for A being a scalar, or average over all spin states)

Experimental method to explore behavior of $\boldsymbol{M_{fi}}$: **Dalitz Analysis**

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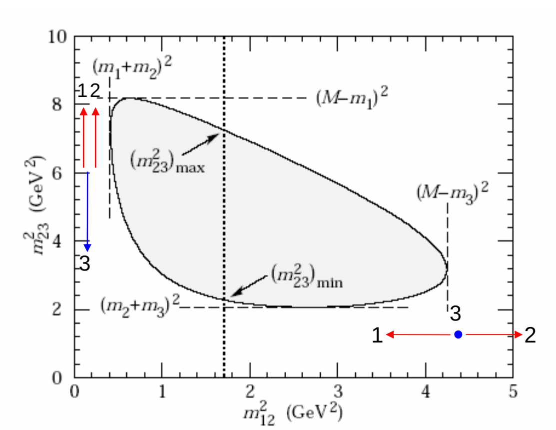
6.2 Dalitz Plots



Method:

Put every measured decay into a 2-dim,. (E_1,E_2) or (m_1^2 ,m_2^2) distribution. A flat distribution over the allowed region corresponds to a "flat matrix element". Structures in the distribution point to a varying matrix element

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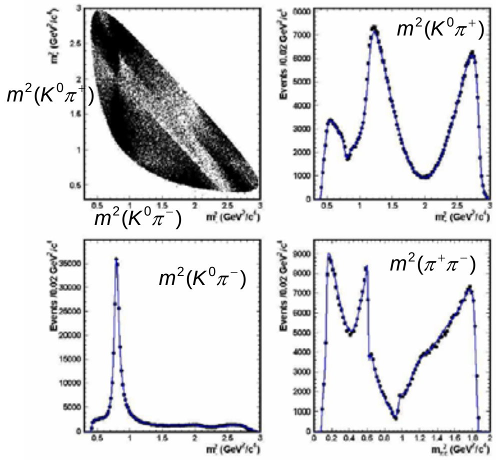
Dalitz-Plot at Work:

$D^0 \to K_s^0 \pi^+ \pi^-$



M. Staric, HEP 2007 (Manchester)

Resonance	Amplitude	Phase (deg)	Fit fraction
$K^*(892)^-$	1.629 ± 0.005	134.3 ± 0.3	0.6227
$K_0^*(1430)^-$	2.12 ± 0.02	-0.9 ± 0.5	0.0724
$K_2^*(1430)^-$	0.87 ± 0.01	-47.3 ± 0.7	0.0133
$K^*(1410)^-$	0.65 ± 0.02	111 ± 2	0.0048
$K^*(1680)^-$	0.60 ± 0.05	147 ± 5	0.0002
$K^*(892)^+$	0.152 ± 0.003	-37.5 ± 1.1	0.0054
$K_0^*(1430)^+$	0.541 ± 0.013	91.8 ± 1.5	0.0047
$K_2^*(1430)^+$	0.276 ± 0.010	-106 ± 3	0.0013
$K^*(1410)^+$	0.333 ± 0.016	-102 ± 2	0.0013
$K^*(1680)^+$	0.73 ± 0.10	103 ± 6	0.0004
$\rho(770)$	1 (fixed)	0 (fixed)	0.2111
$\omega(782)$	0.0380 ± 0.0006	115.1 ± 0.9	0.0063
$f_0(980)$	0.380 ± 0.002	-147.1 ± 0.9	0.0452
$f_0(1370)$	1.46 ± 0.04	98.6 ± 1.4	0.0162
$f_2(1270)$	1.43 ± 0.02	-13.6 ± 1.1	0.0180
$\rho(1450)$	0.72 ± 0.02	40.9 ± 1.9	0.0024
σ_1	1.387 ± 0.018	-147 ± 1	0.0914
σ_2	0.267 ± 0.009	-157 ± 3	0.0088
NR	2.36 ± 0.05	155 ± 2	0.0615

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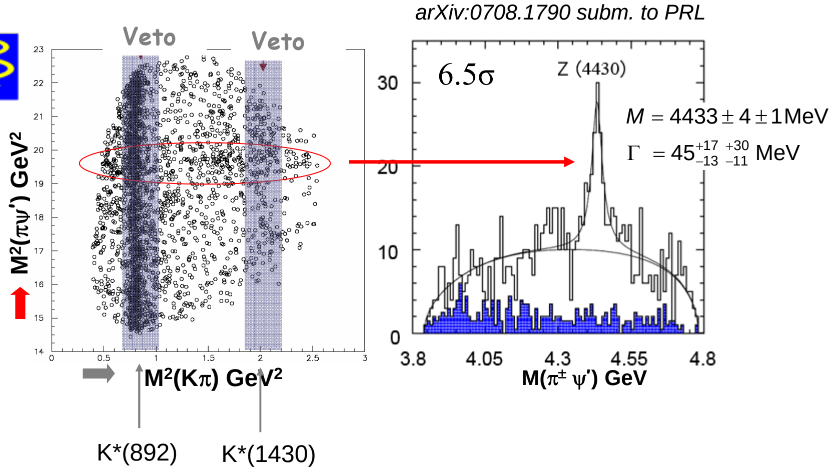
Recent discovery of a new exotic particle Z(4430)

$B \to K \pi^{\pm} \psi'$

arXiv:0708.1790 subm. to PRL



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