# 1. Fermions, propagators and $e^-e^+ \to \mu^-\mu^+$ scattering

**1)** The Dirac action is given by

$$S[\phi] = \frac{1}{2}\int d^4x\, \bar\psi\, (i\partial\!\!\!/ - m)\, \psi(x)\,. \tag{1}$$

with the fermionic field operator

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\sum_{s=\pm 1/2}\left\{e^{ipx}v_s(p)b_s^\dagger(\vec p) + e^{-ipx}u_s(p)a_s(\vec p)\right\}\,. \tag{2}$$

Here $p^0 = \sqrt{\vec p^2 + m^2}$, and annihilation/creation operators of fermions and anti-fermions, e.g. electrons and positrons, $a, a^\dagger$ and $b, b^\dagger$, respectively with

$$\{a_s(\vec p)\,,\, a_r^\dagger(\vec p\,')\} = 2p^0(2\pi)^3\delta_{sr}\delta(\vec p - \vec p\,')\,, \qquad \{b_s(\vec p)\,,\, b_r^\dagger(\vec p\,')\} = 2p^0(2\pi)^3\delta_{sr}\delta(\vec p - \vec p\,')\,, \tag{3}$$

and

$$\sum_{s=\pm 1/2} u_s(p)\bar u_s(p) = p\!\!\!/ + m\,, \qquad \sum_{s=\pm 1/2} v_s(p)\bar v_s(p) = p\!\!\!/ - m\,. \tag{4}$$

**a)** Show with (3) that $\psi(x)$ and $\bar\psi(x)$ satisfy the anti-commutation relations

$$\{\psi(\vec x, t)\,,\, \bar\psi(\vec y, t)\} = \gamma^0\delta(\vec x - \vec y)\,, \qquad \{\psi(\vec x, t)\,,\, \psi(\vec y, t)\} = \{\bar\psi(\vec x, t)\,,\, \bar\psi(\vec y, t)\} = 0\,. \tag{5}$$

**c)** Show that the Feynman propagator $G_{F,A}(x-y) = \langle 0|TA_\mu(x)A_\nu(y)|0\rangle$ is the propagator of the wave equation,

$$-\partial_\mu\partial^\mu G_{F,A}(x-y) = ig_{\mu\nu}\delta(x-y)\,. \tag{6}$$

Show that $G_{F,A}(x-y)$ is given by

$$G_{F,A}(x-y) = ig_{\mu\nu}\lim_{\epsilon\to 0_+}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2 + i\epsilon}\,. \tag{7}$$

**c)** Show that the Feynman propagator $G_{F,\psi}(x-y) = \langle 0|T\psi(x)\bar\psi(y)|0\rangle$ is the propagator of the Dirac equation,

$$(i\partial\!\!\!/^x - m)\, G_{F,\psi}(x-y) = i\delta(x-y)\,. \tag{8}$$

Show that $G_{F,\psi}(x-y)$ is given by

$$G_{F,\psi}(x-y) = i\lim_{\epsilon\to 0_+}\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}(p\!\!\!/ + m)\,. \tag{9}$$

**2)** The $S$-Matrix of QED is given by

$$S = T\ \exp\left(-i\int d^4x\, A_\nu(x) j^\nu(x)\right)\,, \tag{10}$$

where $T$ stands for time ordering and the fermionic current

$$j^\nu(x) = -e : \bar\psi\gamma^\nu\psi(x) :=: \psi_e\gamma^\nu\psi_e(x) : + : \psi_\mu\gamma^\nu\psi_\mu(x) :, \tag{11}$$

with standard four-component Dirac spinors $\psi_e, \psi_\mu$ and $\psi = (\psi_e, \psi_\mu)$. For the $e^-e^+ \to \mu^-\mu^+$-scattering we are interested in the $S$-Matrix element

$$\langle\mu^+(p_4)\mu^-(p_3)|\, S\, |e^+(p_2)e^-(p_1)\rangle\,, \tag{12}$$

with the initial and final states

$$|e^+(p_2)e^-(p_1)\rangle = b_e^\dagger(\vec p_2)a_e^\dagger(\vec p_1)|0\rangle\,, \qquad \langle\mu^+(p_4)\mu^-(p_3)| = \langle 0|b_\mu(\vec p_4)a_\mu(\vec p_3)\,, \tag{13}$$

respectively.

**a)** Expand the $S$-matrix up to the second order, and perform the time-ordering.

**b)** Show, that the following relations are valid,

$$\langle 0|\psi_e(x)a_e^\dagger(\vec p_1)|0\rangle = e^{-ip_1x}u_e(p_1)\,, \tag{14}$$

$$\langle 0|\bar\psi_e(x)b_e^\dagger(\vec p_2)|0\rangle = e^{-ip_2x}\bar v_e(p_2)\,, \tag{15}$$

and derive the corresponding Feynman rules for the initial state. Apply the same reasoning to the final state.

**c)** Reduce the matrix element

$$\langle\mu^+(p_4)\mu^-(p_3)|\, T : \bar\psi\gamma^\nu\psi(x) :: \bar\psi\gamma^\mu\psi(x') : \, |e^+(p_2)e^-(p_1)\rangle$$

to a product of the simple matrix elements such as $\langle 0|\psi_e(x)a_e^\dagger(\vec p_1)|0\rangle$ in (14).

**d)** Show that

$$\sum_{s,s',r,r'} |\bar u_s(p_3)\gamma^\mu v_{s'}(p_4)\, \bar v_r(p_2)\gamma_\mu u'_r(p_1)|^2$$
$$= \mathrm{tr}\,[\gamma^\mu(\not p_4 - m)\gamma_\rho(\not p_3 + m)]\,\mathrm{tr}\,[\gamma^\mu(\not p_1 + m)\gamma_\rho(\not p_2 - m)]\,. \tag{16}$$

The spin sums in (16) are relevant for $e^+e^- \to e^+e^-$ and $e^+e^- \to \mu + \mu^-$-scattering matrix elements.

**3)** Consider the scattering process $a(p_a) + b(p_b) \to c(p_c) + d(p_d)$, where the 4-momenta $p_i$ are that in the lab-system. The Mandelstam-variables $s,\ t,\ u$ are Lorentz-invariant variables,

$$s = (p_a + p_b)^2\,, \quad t = (p_a - p_c)^2\,, \quad u = (p_a - p_d)^2\,, \tag{17}$$

with

$$s = (p_a + p_b)^2 = (p_a^* + p_b^*)^2\,, \tag{18}$$

where the $p_i^*$ are the momenta of the particles in the centre of mass (CM) system: $\vec{p}_a^* = -\vec{p}_b^*$.

**a)** The particle b is at rest in the lab system. Show that

$$|\vec{p}_a| = \frac{1}{2m_b} w(s, m_a^2, m_b^2)\,, \tag{19}$$

with

$$w(x,y,z) = (x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)^{\frac{1}{2}} \tag{20}$$

Show also that the energy $E_a^*$ ($E_b^*$) of the particle a (b) in the CMS is given by

$$E_a^* = \frac{s + m_b^2 - m_a^2}{2\sqrt{2}} \quad \text{and} \quad E_b^* = \frac{s + m_a^2 - m_b^2}{2\sqrt{2}} \tag{21}$$

Equivalent relations hold for $E_c^*$ and $E_d^*$.

**b)** Show that

$$|\vec{p}_b^*| = |\vec{p}_a^*| = \frac{1}{2\sqrt{s}} w(s, m_a^2, m_b^2) \qquad |\vec{p}_c^*| = |\vec{p}_d^*| = \frac{1}{2\sqrt{s}} w(s, m_c^2, m_d^2)\,, \tag{22}$$

and write $t$ as a function of $s$ and the scattering angle $\theta^*$ in the CMS.

**c)** The flow F for the above configuration is (b is at rest, volume $V = 1$):

$$F = |\vec{v}| 2E_a 2m_b\,. \tag{23}$$

Express the flow as a function of the momentum $|\vec{p}_a^*|$ in CMS.

**d)** Perform the integration of the 2-particle phase space for the above scattering process in the CMS. Show that

$$\int d\Phi_2 = \frac{1}{16\pi^2} \frac{|\vec{p}_c^*|}{\sqrt{s}} \int d\Omega_C\,, \tag{24}$$

by using the relation

$$\int \delta[f(\omega)] g(\omega) d\omega = \left( g \left| \frac{df}{d\omega} \right|^{-1} \right)_{f=0}\,. \tag{25}$$