1. Quantisation and Scattering

1) The action of a free real scalar field is given by

$$S[\phi] = \frac{1}{2}\int d^4x \left(\partial_\mu\phi(x)\,\partial^\mu\phi(x) - m^2\phi^2(x)\right) . \tag{1}$$

The field operator of a real scalar field is given by

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}\left\{e^{ik\,x}a^\dagger(\vec{k}) + e^{-ik\,x}a(\vec{k})\right\} , \tag{2}$$

with $\omega = \sqrt{\vec{k}^2 + m^2}$, and annihilation/creation operators $a, a^\dagger$ with

$$[a(\vec{k}), a^\dagger(\vec{k}')] = 2\omega(2\pi)^3\delta(\vec{k} - \vec{k}') . \tag{3}$$

a) Show that the Klein-Gordon equation $(\partial_\mu\partial^\mu + m^2)\phi(x) = 0$ follows from

$$\frac{\delta S[\phi]}{\delta\phi(x)} = 0 , \qquad \text{with} \qquad \frac{\delta\phi(y)}{\delta\phi(x)} = \delta(x-y) . \tag{4}$$

Show that $\phi(x)$ in (2) satisfies the Klein-Gordon equation, and show, that the canonical momentum operator $\Pi(x)$ is given by $\Pi(x) = \dot{\phi}(x)$.

b) Show with (3) that $\phi(x)$ and $\dot{\phi}(x)$ satisfy the canonical commutation relations

$$[\phi(\vec{x},t)\,,\,\dot{\phi}(\vec{y},t)] = i\delta(\vec{x}-\vec{y}) , \qquad [\phi(\vec{x},t)\,,\,\phi(\vec{y},t)] = [\dot{\phi}(\vec{x},t)\,,\,\dot{\phi}(\vec{y},t)] = 0 . \tag{5}$$

2) Consider the scattering of one particle in the presence of an interaction Lagrangian $\mathcal{L}'(x) = -1/2V(\vec{x}) : \phi(x)\phi(x) :$, where : . : stands for normal ordering, e.g.,

$$: a_1 \cdots a_n a_1^\dagger \cdots a_m^\dagger := a_1^\dagger \cdots a_m^\dagger a_1 \cdots a_n . \tag{6}$$

With $H'(t) = -\int d^3x\,\mathcal{L}'(x)$, the time evolution in the interaction picture is given by

$$i\partial_t|t\rangle = H'(t)|t\rangle . \tag{7}$$

a) Show by iterating (7) in its infinitesimal form, $|t + \Delta t\rangle = (1\!\!1 - iH'(t))|t\rangle$, that

$$|t\rangle = U(t,t_0)|t_0\rangle , \qquad \text{where} \qquad U(t,t_0) = Te^{-i\int_{t_0}^{t} dt' H'(t')} , \tag{8}$$

with time ordering T: $T\,H'(t_1)H'(t_2) = H'(t_1)H'(t_2)\theta(t_1-t_2) + H'(t_2)H'(t_1)\theta(t_2-t_1)$. Expand U up to the second order.

b) Show that

$$\langle\vec{k}'| : \phi(x)\phi(x) : |\vec{k}\rangle = 2\langle\vec{k}'| : \phi(x)|0\rangle\langle 0|\phi(x) : |\vec{k}\rangle , \tag{9}$$

and compute it. This matrix element is relevant for the first order perturbation theory.

3) The action of a free photon field is given by

$$S[A] = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\int d^4x\, A_\mu(x)\,(g^{\mu\nu}\partial_\rho\partial^\rho - \partial^\mu\partial^\nu)\,A_\nu(x)\,. \tag{10}$$

The field operator of the gauge field A_μ is given by

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k_0}\left\{e^{ik\,x}a_\mu^\dagger(\vec{k}) + e^{-ik\,x}a_\mu(\vec{k})\right\}\,, \tag{11}$$

with $k_0 = \sqrt{\vec{k}^2}$, and annihilation/creation operators $a_\mu, a_\mu^\dagger$ with

$$[a_\mu(\vec{k}), a_\nu^\dagger(\vec{k}')] = -g_{\mu\nu}2k_0(2\pi)^3\delta(\vec{k}-\vec{k}')\,. \tag{12}$$

a) Show that the Klein-Gordon equation (wave equation) $\partial_\rho\partial^\rho A_\mu(x) = 0$ follows from

$$\frac{\delta S[A]}{\delta A_\mu(x)} = 0\,, \qquad \text{with} \qquad \frac{\delta A_\nu(y)}{\delta A_\mu(x)} = g_\nu{}^\mu\delta(x-y) \qquad \text{for} \qquad \partial^\mu A_\mu = 0\,. \tag{13}$$

Show that $A_\mu(x)$ in (11) satisfies the wave equation, and show, that the canonical momentum operator $\Pi_i(x)$ is given by $\Pi_i(x) = E_i(x) = -F_{0i}(x) = -[\partial_0 A_i(x) - \partial_i A_0(x)]$.

b) Show with (12) that $A_i(x)$ and $E_j(x)$ satisfy the canonical commutation relation

$$[A_i(\vec{x},t)\,,\,E_j(\vec{y},t)] = ig_{ij}\delta(\vec{x}-\vec{y})\,. \tag{14}$$

4) Physical states are defined by

$$\alpha_0|\text{phys. states}\rangle = 0\,, \tag{15}$$

with

$$\begin{aligned}
\alpha_0^\dagger(\vec{k}) &= \frac{1}{\sqrt{2}}\left(a_0^\dagger(\vec{k}) - \hat{k}\vec{a}^\dagger(\vec{k})\right)\,, &&\text{with} \quad \hat{k} = \frac{\vec{k}}{|\vec{k}|}\,,\\
\alpha_i^\dagger(\vec{k}) &= \hat{e}_i\vec{a}^\dagger(\vec{k})\,, &&\text{with} \quad \hat{e}_i\hat{k} = 0\,, \quad i = 1,2\,,\\
\alpha_3^\dagger(\vec{k}) &= \frac{1}{\sqrt{2}}\left(a_0^\dagger(\vec{k}) + \hat{k}\vec{a}^\dagger(\vec{k})\right)\,.
\end{aligned}$$

a) Show that the operator $\alpha^\dagger$ satisfy the commutation relations

$$\begin{aligned}
[\alpha_0(\vec{k})\,,\,\alpha_0^\dagger(\vec{k}')] &= 0 = [\alpha_3^\dagger(\vec{k})\,,\,\alpha_3(\vec{k}')]\\
[\alpha_0(\vec{k})\,,\,\alpha_i^\dagger(\vec{k}')] &= 0 = [\alpha_3^\dagger(\vec{k})\,,\,\alpha_i(\vec{k}')]\\
[\alpha_0(\vec{k})\,,\,\alpha_3^\dagger(\vec{k}')] &= -2k_0(2\pi)^3\delta(\vec{k}-\vec{k}')\\
[\alpha_i(\vec{k})\,,\,\alpha_i^\dagger(\vec{k}')] &= 2k_0(2\pi)^3\delta(\vec{k}-\vec{k}')\,, \qquad i = 1,2\,.
\end{aligned}$$

b) Show that the one particle states $\alpha_0^\dagger(\vec{k})|0\rangle\,,\,\alpha_i^\dagger(\vec{k})|0\rangle$ with $i = 1,2$ are physical states with (15). Show that $\alpha_3^\dagger(\vec{k})|0\rangle$ is not a physical state.