Scattering Matrix

- scattering process
- Mandelstam variables
- scattering operator
- cross section and transition amplitude (Fermi's golden rule)
- cross section for 2 → 2 process
- (optical theorem)

Lorentz Invariance of Phase Space Element dL

transformation of energy E and momentum p_i

$$dp_x' dp_y' dp_z' = D_J dp_x dp_y dp_z$$

note: spatial rotations leave E and |p| unchanged

$$D_J = \begin{vmatrix} \frac{\partial p_x'}{\partial p_x} & \frac{\partial p_y'}{\partial p_x} & \frac{\partial p_z'}{\partial p_x} \\ \frac{\partial p_x'}{\partial p_y} & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix}$$

Jacobi determinant

Possible transformations:

- rotation $|D_J|=1$
- Lorentz boost
- discrete transformations (parity) $|D_J|=1$
- combination of above trafos (Poincare group)

Boosts:

$$E' = +\gamma E - \gamma\beta p_z \qquad p_x' = p_x$$

$$p_z' = -\gamma\beta E + \gamma p_z \qquad p_y' = p_y$$

$$\frac{\partial p_z'}{\partial p_z} = \gamma - \gamma\beta \frac{\partial E}{\partial p_z}$$

$$\frac{\partial p_x'}{\partial p_x} = \frac{\partial p_y'}{\partial p_y} = 1$$

using $E\,dE = p_z dp_z$

$$D_J = \gamma\left(1 - \beta\frac{p_z}{E}\right) = \frac{E'}{E}$$

$$\rightarrow \frac{d^3 p}{E} = \frac{d^3 p'}{E'}$$

qed

Optical Theorem I

Process: A+B $\rightarrow$ X, X represents an infinite number of final states

Total cross section: $\sigma = \int_{X_i} d\sigma_i$ i =specific final state

From unitarity of scattering operator: $S^+ = S^{-1}$

$$S^+ S = S S^+ = 1$$

Inserting reaction operator $S = 1 + R$ yields:

$$(1+R)(1+R^+) = 1 + R R^+ + R + R^+ = 1$$

=> $$R R^+ = -(R + R^+) = -\Re(R + R^+)$$

Optical Theorem II

$2 \rightarrow 2$ reaction: $$R_{fi} = -i(2\pi)^4 \cdot \prod_{i=1}^{4} N_i \, \delta(p_1 + p_2 - p_3 - p_4) T_{fi}$$

transition amplitude:

$$\sum_f \langle i|R|f\rangle\langle f|R|i\rangle = \sum_f |R_{fi}|^2 = -2(2\pi)^4 \left(\prod_{i=1}^{4} N_i\right)^2 \Im\left(T^{elastic}(0)\right)$$

using:

$\sum_f |f\rangle\langle f| = 1$ and defining $\sum_f \langle i|i\rangle = T^{elastic}(0)$

$T^{elastic}(\Theta=0)$ is the elastic forward scattering amplitude

The imaginary part of the transition amplitude describes the amount of scattered (absorbed) particles like a refrection index

The total cross section is given by the imaginary part of the forward scattering amplitude → optical theorem

Optical Theorem III

$$\sum_f \int |T_{fi}|^2 \, dL = -2\,\Im\left(T^{elastic}(0)\right)$$

=> $$\sigma_{total} = -\frac{1}{s_{12}}\,\Im\left(T_{fi}^{elastic}(0)\right)$$

Example: luminosity measurement at LHC is done using the total cross section of:

p + p $\rightarrow$ p + p (reference process)

The total cross section is determined by measuring the forward scattering amplitude!

LHC proton-proton forward scattering

limit t → 0

$$\frac{d\sigma}{dt} = \sigma_{tot}^2 \frac{1+\rho^2}{16\pi(\hbar c)^2} \times e^{-B|t|}$$

B=slope parameter

$$\rho = \frac{\mathrm{Re}(f_{el})}{\mathrm{Im}(f_{el})}$$

ratio of scattering amplitudes from theory