Particle physics lecture: Dirac equation

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additional material

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Flash-back

Klein-Gordon:

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

- We've derived and identified the corresponding continuity equation
- Has problems: negative solutions in relativistic case

Dirac attacked this by looking out for

$$\hat{H} = \widehat{\sqrt{p^2 + m}}$$

and used

$$\hat{H}\psi = (\boldsymbol{\alpha}\hat{\mathbf{p}} + \beta m)\psi$$

where $\boldsymbol{\alpha}$ and β were some yet to be found mathematical objects.

Flash-back

Obeying the following rules would do the trick:

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij}$$
$$\alpha_i \beta + \beta \alpha_i = 0$$
$$\beta^2 = 1$$

Matrices of rank 4 are the minimal objects doing that.

The matrices can be constructed from Pauli matrices (which hints to spin-$\frac{1}{2}$ already...)

Flash-back

Use of γ-matrices allows to write the Dirac equation in covariant form:

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0$$

Or even more compact using *Feynman slash notation*:

$$(i\not{\partial}_\mu - m)\,\psi = 0$$

Both ways are *very compact* notations. Keep in mind that due to *Einstein convention*, this is...

$$(i\gamma^0\partial_0 + i\gamma^1\partial_1 + i\gamma^2\partial_2 + i\gamma^3\partial_3 - m)\,\psi$$

...and which is a system of **four** differential equations because ψ is a spinor with four components and the γ-matrices are 4×4 in size.

Flash-back

Covariance is built into the γ-matrices:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

Four components hints somewhat naturally to two families, particles and antiparticles, in two spin states $\pm\frac{1}{2}$.

Overview of matrices used

Pauli matrices:

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

α, β **matrices:**

$$\alpha_i = \begin{pmatrix} \mathbf{0} & \sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix} \qquad \beta = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

γ **matrices:**

$$\gamma^0 = \beta \qquad \gamma^i = \beta\alpha_i = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \sigma_i \\ -\sigma_i & \mathbf{0} \end{pmatrix}$$

$i \in \{1, 2, 3\}$

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- **and some more properties**

  . . . to see what else the Dirac equation has on offer.

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Spin and Dirac equation

Recall quantum mechanics:

- Time dependence of some observable $\mathcal{O}$

$$\frac{d\mathcal{O}}{dt} = \frac{d}{dt}\langle\hat{\mathcal{O}}\rangle = i\langle\psi|[\hat{H}, \hat{\mathcal{O}}]|\psi\rangle$$

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- Non-relativistic QM for a free particle:

$$\hat{H}_{\text{Schrödinger}} = \frac{\hat{p}^2}{2m}$$

One can show: $[\hat{H}_{\text{Schrödinger}}, \hat{L}] = 0$ hence *angular momentum* is conserved.

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- How about conservation in relativistic QM using the Dirac equation? $\Rightarrow$ blackboard

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Quantisation of spin

- From QM we know
$$[\hat{L}_x, \hat{L}_y] = i\hat{L}_z$$
$$\hat{L}^2\,\psi = \ell(\ell+1)\,\psi$$
Angular momentum is **quantised**.
- **Spin operator** $\hat{\boldsymbol{s}}$ depends linearly on $\boldsymbol{\sigma}$, which have well-known commutator rules.
- In relativistic QM, one can show:
$$[\hat{S}_x, \hat{S}_y] = i\hat{S}_z$$
$$\hat{\boldsymbol{S}}^2\,\psi = S(S+1)\,\psi$$
Hence, particles have quantised spin as well.

Quantisation of spin

- Furthermore, Pauli matrices cause

$$\hat{\boldsymbol{S}}^2 = \frac{3}{4}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad \Rightarrow \quad s = \frac{1}{2}$$

$\Rightarrow$ The Dirac equation describes **fermions**.