Dirac Equation I

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Gamma Matrices I

Gamma matrices γ^μ are chosen such that γ^0 is hermitian while γ^k (k=1,2,3) are anti-hermitian

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma^0)^2 = 1,$$

$$(\gamma^k)^\dagger = -(\gamma^k), \quad (\gamma^k)^2 = -1 \quad (k=1,2,3)$$

In addition, we define γ^5 as the hermitian matrix: $\gamma^5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3$, $\quad (\gamma^5)^2 = 1$,

The 4x4 gamma matrices can be represented by (diagonal γ^0 representation):

$$\gamma^k = \begin{pmatrix} 0 & \underline{\sigma}^k \\ -\underline{\sigma}^k & 0 \end{pmatrix}, \quad \gamma^0 \stackrel{\text{def}}{=} \beta = \begin{pmatrix} \underline{1} & 0 \\ 0 & -\underline{1} \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \underline{1} \\ \underline{1} & 0 \end{pmatrix}$$

(note several representations exist)

With the 2x2 Pauli matrices:

$$\underline{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \underline{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \underline{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

All gamma matrices anti-commute according to: $\gamma^i \gamma^k + \gamma^k \gamma^i = 0$ for $i \neq k$

Gamma Matrices II

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(in other representations γ^5 is diagonal)

Spinors

positive energy

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

negative energy

$$u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \qquad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

Spinors fermions / anti-fermions

(spin polarisation along z-axis)

fermions:

$$u_{1(R)} = \sqrt{E+m} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad u_{2(L)} = \sqrt{E+m} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

anti-fermions:

$$v_{2,(L)} = \sqrt{E+m} \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \qquad v_{1(R)} = \sqrt{E+m} \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}$$

Spinors fermions / anti-fermions

$\vec{p} = p\,(s\,\cos\varphi, s\,\sin\varphi,\, c)$ with $c = \cos\Theta$ $s = \sin\Theta$

fermions:

$$u_{1(R)} = \sqrt{E+m} \begin{pmatrix} c \\ s e^{i\varphi} \\ \frac{p}{E+m} c \\ \frac{p}{E+m} s e^{i\varphi} \end{pmatrix}$$

$$u_{2(L)} = \sqrt{E+m} \begin{pmatrix} -s \\ c\, e^{i\varphi} \\ \frac{p}{E+m} s \\ \frac{-p}{E+m} c\, e^{i\varphi} \end{pmatrix}$$

anti-fermions:

$$v_{2,(L)} = \sqrt{E+m} \begin{pmatrix} \frac{p}{E+m} c \\ \frac{p}{E+m} s\, e^{i\varphi} \\ c \\ s\, e^{i\varphi} \end{pmatrix}$$

$$v_{1(R)} = \sqrt{E+m} \begin{pmatrix} \frac{p}{E+m} s \\ \frac{-p}{E+m} c\, e^{i\varphi} \\ -s \\ c e^{i\varphi} \end{pmatrix}$$

Spinors of Helicity States

$u_R = |\vec{p},\ \lambda = +1/2\rangle$ (u$_1$)

$u_L = |\vec{p},\ \lambda = -1/2\rangle$ (u$_2$)

$v_L = |\vec{p},\ \lambda = -1/2\rangle$ (v$_1$)

$v_R = |\vec{p},\ \lambda = +1/2\rangle$ (v$_2$)

fermions:

$$u_R = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{|\vec{p}|}{E+m} \\ 0 \end{pmatrix} \qquad u_L = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-|\vec{p}|}{E+m} \end{pmatrix}$$

anti-fermions:

$$v_L = \sqrt{E+m}\begin{pmatrix} \frac{|\vec{p}|}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad v_R = \sqrt{E+m}\begin{pmatrix} 0 \\ \frac{-|\vec{p}|}{E+m} \\ 0 \\ 1 \end{pmatrix}$$

limit $p \rightarrow \infty$

$u_R \rightarrow v_L$

$u_L \rightarrow v_R$