Quark-Gluon Plasma Physics

3. Thermodynamics of the QGP

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3.1 QGP thermodynamics in bag model

Thermodynamics of
relativistic Bose gas
relativistic Fermi-gas
Bag model of hadrons
Constructing the phase diagram between pion gas and QGP
“realistic hadron gas and QGP
3.1.1 Thermodynamics of a relativistic Bose gas

probability density for occupation of state with relativistic energy $E$ and degeneracy $g$

$$N(E) = \frac{g}{(2\pi)^3} \left( \exp\left( \frac{E - \mu}{T} \right) - 1 \right)^{-1}$$

with energy $E^2 = p^2 + m^2$

(note: here $\hbar = c = 1$)

and chemical potential $\mu$ controlling the average number of particles vs antiparticles

neglecting the particle mass (okay since in interesting region $E = 3T \gg m$)

and chemical potential (good as long as no additive quantum number)

Boson number density

$$n = \int N(E) d^3p = \frac{4\pi g}{(2\pi)^3} \int \frac{p^2 dp}{\exp\left( \frac{p}{T} \right) - 1}$$

$$n = \frac{g}{\pi^2} T^3 \zeta(3)$$

with Riemann $\zeta$-function $\zeta(3) \approx 1.2$
Thermodynamics of a relativistic Bose gas

**Boson energy density**

\[ \epsilon = \int N(E) p^3 p = \frac{4\pi g}{(2\pi)^3} \int \frac{p^3 dp}{\exp\left(\frac{p}{T}\right) - 1} \]

\[ \epsilon = \frac{3g}{\pi^2} T^4 \zeta(4) \]

with Riemann \( \zeta \)-function \( \zeta(4) = \frac{\pi^4}{90} \approx 1.08 \)

\[
\epsilon = \frac{\pi^2}{30} g T^4
\]

and we get the **Energy per particle**

\[ \epsilon/n = 3T \frac{\zeta(4)}{\zeta(3)} \approx 2.7 T \]

**Boson pressure**

\[ P = n^2 \frac{\partial \epsilon}{\partial n} \rightarrow P = \frac{1}{3} \epsilon \]

**Entropy density**

\[ d\sigma = \frac{d\epsilon}{T} \quad \text{and} \quad d\epsilon = \text{const.} T^3 dT \]

\[ \sigma = \int d\sigma = \text{const.} \int T^2 dT = \frac{1}{3} \text{const.} T^3 \]

\[ \sigma = \frac{4\pi^2}{90} g T^3 \]

\[ \rightarrow \sigma = \frac{1}{3} \frac{d\epsilon}{dT} \]
Thermodynamics of a relativistic Bose gas

and the entropy per particle (boson) \( \frac{\sigma}{n} = 4\zeta(4)/\zeta(3) \approx 3.6 \)

cf. old Landau formula for pions: \( S = 3.6 \, dN/dy \)
3.1.2 Thermodynamics of a relativistic Fermi gas

probability density for occupation

\[ N(E) = \frac{g}{(2\pi)^3} \left( \exp\left( \frac{E - \mu}{T} \right) + 1 \right)^{-1} \]

but now \( \mu \) not generally 0

Fermion number density

\[ n = \frac{4\pi g}{(2\pi)^3} \int \frac{p^2 dp}{\exp\left( \frac{p - \mu}{T} \right) + 1} \]

Energy density

\[ \epsilon = \frac{4\pi g}{(2\pi)^3} \int \frac{p^3 dp}{\exp\left( \frac{p - \mu}{T} \right) + 1} \]

cannot be solved analytically, only numerically

but there is analytic solution for sum of particle and antiparticle (e.g. quark and antiquark) (Chin, PLB 78 (1978) 552)

\[ \epsilon_q + \epsilon_{\bar{q}} = g \left( \frac{7\pi^2}{120} T^4 + \frac{\mu^2}{4} T^2 + \frac{\mu^4}{8\pi^2} \right) \]

and

\[ n_q - n_{\bar{q}} = g \left( \frac{\mu}{6} T^2 + \frac{\mu^3}{6\pi^2} \right) \]
specific example for fermions: quarks in QGP with no net baryon density (LHC)

\[ \langle q \rangle = \langle \bar{q} \rangle \leftrightarrow \mu = 0 \]

in that case quark number density \( n_q = \frac{g}{\pi^2} T^3 d(3) \)

Note: \( d(\alpha + 2) = \int \frac{x^\alpha dx}{e^x + 1} \) and \( d(3) \approx 0.9 \)

and quark and antiquark energy density \( \epsilon_q = \epsilon_{\bar{q}} = \frac{3g}{\pi^2} T^4 d(4) = \frac{7\pi^2}{240} g T^4 \)

with \( d(4) = \frac{7\pi^4}{720} \)

the energy per quark is then \( \epsilon/n = 3T \frac{d(4)}{d(3)} \approx 3.2T \)

Entropy density (computed as above for bosons)

\[ \sigma = \frac{7\pi^2}{180} g T^3 \]

and the entropy per fermion (quark) \( \sigma/n = 4 \frac{d(4)}{d(3)} \approx 4.2 \)
Summary relativistic bosons and fermions (no chem.pot.)

- **Energy density** \( \epsilon \propto T^4 \)

- **Pressure** \( P = \frac{1}{3} \epsilon \propto T^4 \)

- **Entropy density** \( \sigma \propto T^3 \)

- **Particle number density** \( n \propto T^3 \)

- to obtain physical units of GeV/fm\(^3\) or fm\(^{-3}\), multiply with appropriate powers of \( \hbar c \)

- all are proportional to the number of degrees of freedom

- between bosons and fermions there is a factor 7/8

  \[ \epsilon_f = \frac{7}{8} \epsilon_b \]

  etc.
3.1.3 Short excursion: the bag model

to deal with QCD in the nonperturbative regime (i.e. where $\alpha_s$ is not negligible) one needs to make models (alternative: lattice QCD see below) for instance to treat the nucleon and its excitations.

**MIT bag model**: build confinement and asymptotic freedom into simple phenomenological model


hadrons considered as bags embedded into a non-perturbative QCD vacuum also called “physical vacuum” or “normal QCD vacuum”

space divided into 2 regions

**Interior of bag**: quarks have very small (current) masses, interaction weak

**Exterior of bag**: quarks are not allowed to propagate there, lower vacuum energy, no colored quarks or gluons but quark and gluon condensates
Hadrons in MIT bag model

Hadrons are considered drops of another, perturbative phase of QCD immersed into normal QCD vacuum

all non-perturbative physics included in one universal quantity, the bag constant $B$ defined as the difference in energy density between perturbative and physical vacua:

$$\epsilon_{\text{bag}} - \epsilon_{\text{vac}} \equiv B > 0$$

solve Dirac equation for massless quarks inside bag with volume $V$ and surface $S$ with special boundary conditions at the surface that

i) enforce confinement: quark current normal to bag surface = 0

ii) define a stability condition for bag: pressure of Dirac particles inside is balanced by difference in energy density inside and outside

$$H = H_{\text{kin}} + H_{\text{spin-spin}} + BV$$

kin. Energy of quarks confined in bag
spin-spin interaction
energy to make hole of volume $V$ in phys. vacuum
for (nearly) massless quarks (spherical bag with radius $R$)

\[ E_{\text{kin}} \propto \frac{1}{R} \quad \rightarrow \quad \text{tries to extend bag} \]

\[ B \frac{4\pi}{3} R^3 \quad \rightarrow \quad \text{tries to contract bag} \]

 equilibrium is reached

obtain e.g. for nucleon mass (spherical bag with 3 quarks in s-state)

\[ E = 3 \frac{\omega_n}{R} - \frac{4\pi}{3} BR^3 \quad \text{with} \quad \omega_{1,-1} = 2.04 \quad \omega_{2,-1} = 5.40 \quad \ldots \]

\[ \frac{\partial E}{\partial R} = 0 \]

internal energy determines the radius of the bag, if $B$ is a universal constant

- determines masses and sizes of all hadrons
  rather successful with $B_{\text{MIT}} = 56 \text{ MeV/fm}^3$
  baryon octet and decuplet as well as vector mesons well reproduced

note: often instead of $B$, $B^{1/4}$ in MeV is quoted $B_{\text{MIT}}^{1/4} = 146 \text{ MeV}$
3.1.4 Thermodynamics of pion gas and QGP

**pion gas**: massless bosons with degeneracy \( g_\pi = 3 \) for \( \pi^+, \pi^0, \pi^- \)

Energy density of pion gas \( \epsilon_\pi = \frac{\pi^2}{30} g_\pi T^4 = 129 T^4 \) and pressure \( P = \frac{1}{3} \epsilon = 43 T^4 \)

After properly inserting missing powers of \( \hbar c \) and using \( T \) in GeV

**quark-gluon plasma**:  

Gluons as massless bosons with degeneracy \( g_g = 2(\text{spin}) \times 8(\text{color}) = 16 \)

Quarks massless fermions with degeneracy \( g_q = N_f \times 2(\text{spin}) \times 3(\text{color}) = 6 N_f \)

And same for antiquarks (here \( N_f \) is number of massless/light flavors)

**Additional contribution to energy density**: to make quark-gluon gas, need to create cavity in vacuum

Energy needed is given by the **bag constant** \( B \) “pressure of vacuum on color field”

Analogy to Meissner effect: superconductor expells magnetic field

\[ \leftrightarrow \text{QCD vacuum expels color field into bags} \]

\[ \epsilon = \epsilon_{\text{thermal}} + B \]

And deriving pressure as above

\[ P = \frac{1}{3}(\epsilon - 4B) \]
3.1.4 Thermodynamics of pion gas and QGP

What value to use for the bag constant?
from hadron phenomenology at T=0 and normal nuclear matter density

\[ B \approx 50 - 100 \text{ MeV/fm}^3 \]

but there are a number of problems with MIT bag model

and there is good indication that B derived there is not the energy density of the QCD vacuum; conclusion: hadrons are not small drops of the new QCD phase but only a relatively small perturbation of the QCD vacuum

also \( B = B(T,n) \) (see e.g. Shuryak, the QCD vacuum...)

basic argument: at large \( T,n \) all non-perturbative phenomena suppressed

\[ B_{\text{eff}} \approx 500 - 1000 \text{ MeV/fm}^3 \]

vacuum energy density

energy density of quark-gluon gas

\[ \epsilon_{\text{qg}} = \frac{\pi^2}{30} (g_g + \frac{7}{8} g_q) T^4 + B = \frac{\pi^2}{30} (16 + \frac{21}{2} N_f) T^4 + B \]

for \( N_f = 2 \) (u,d)

\[ \epsilon_{\text{qg}} = 1592 T^4 + 0.5 \ (\text{GeV/fm}^3) \]
Constructing the phase diagram

System always in phase with highest pressure

Gibbs conditions for critical point

\[ P_{\text{QGP}} = P_{\text{piongas}} \]

and

\[ \mu_{\text{QGP}} = \mu_{\text{piongas}}(=0) \]

For \( N_f = 2 \)

\[ \frac{3\pi^2}{90} T_c^4 = \frac{\pi^2}{90} (16 + \frac{21}{2} N_f) T^4 - B \]

\[ \frac{34\pi^2}{90} T_c^4 = B \quad T_c = \left( \frac{90 \cdot 0.5 \text{GeV} \cdot 0.197^3 \text{GeV}^3 \text{fm}^3}{34\pi^2 \text{fm}^3} \right)^{1/4} = 0.18 \text{ GeV} \]

Latent heat:

\[ \epsilon_{\text{qg}} - \epsilon_{\text{pion}}(\text{at } T_c) = \frac{34\pi^2}{30} T_c^4 + B \]

\[ = 1.54 + 0.5 = 2 \frac{\text{GeV}}{\text{fm}^3} \]

Change in entropy density:

\[ \sigma_{\text{qg}} - \sigma_{\text{pion}} = \frac{34 \cdot 4\pi^2}{90} T_c^3 = 11.4/\text{fm}^3 \]
Now check the high baryon density limit

compute a \( T = 0 \quad \mu \neq 0 \) point

cannot do this with pions alone, need nucleons

\[
P_{\text{pion}} = 0 \quad P_{\text{nucleon}} = \frac{g\mu^4}{3 \cdot 8\pi^2}
\]

with \( g = 4 \) (2(spin) \times 2(\text{isospin}))

for the quark-gluon side at \( T = 0 \)

\[
P_{\bar{q}q} = \frac{g\mu^4}{3 \cdot 8\pi^2} - B \quad \text{with } g = 12 \text{ (quarks and antiquarks)}
\]

\[
P_{\text{nucleon}} = P_{\bar{q}q} \quad \mu = \left(\frac{3\pi^2 \cdot 0.5 \text{GeV} \cdot 0.197^3 \text{GeV}^3 \text{fm}^3}{\text{fm}^3}\right)^{1/4}
\]

\[= 0.58 \text{ GeV}\]

simple thermodynamik model gives first order phase transition,

**Caution:** this sets the scale, but there are a number of approximations

- pion gas is oversimplification for hadronic matter
- \( B = 0.5 \text{ GeV/fm}^3 \) (should use \( B(T,n) \))
3.1.5 more realistic: replace pion gas by hadron gas

implement all know hadrons up to 2 GeV in mass
ideal gas of quarks and gluons, u,d massless, s 150 MeV
fix bag constant to match lattice QCD result (see below) at $\mu_b=0 \rightarrow B=262$ MeV/fm$^3$

compute $P(\mu_b,T)$ with
with pressure and $\mu_b$ continuous
to obtain $T_c$

Phase diagram constructed with hadron gas and QGP

Note: chemical potential is continuous at phase transition but not the baryon density!
Speed of sound

in relativistic gas without interactions, speed of sound squared

\[ c_s^2 = \frac{dP}{d\epsilon} = \frac{1}{3} \]

but in vicinity of phase transition strong deviation of P from 1/3 \( \epsilon \)

there is always a minimum in speed of sound

leading to a so-called 'softest point'

3.2 Lattice QCD

QCD asymptotically free at large T and/or small distances
at low T and for finite size systems \( \alpha_s = O(1) \)

\[ \rightarrow \] cannot use perturbation theory

instead define QCD at zero and finite temperature by putting gauge field on a space-time lattice and find solutions by solving path integrals numerically

“lattice QCD”


recent reviews:
arXiv: 1504.05274 [hep-lat]
Lattice QCD - schematic outline of basic (3) steps

i) use evolution in Euclidean time $\tau = it$ instead of Minkowski time to eliminate oscillations due to complex action

ii) replace Euclidean $x, \tau$ continuum by finite lattice

field theory with infinite number of degrees of freedom $\rightarrow$ finite many body problem
quantum field theory equivalent to classical statistical mechanics
with $\exp(-iHt) \rightarrow \exp(-H\tau) = \exp(-S)$

Powerful connection between quantum field theory and statistical mechanics (already realized in 1960ies by K. Symanzik)
Lattice QCD basic steps

iii) evaluate partition function $Z$ by using Feynman path integrals

$$Z = \text{Tr} \exp(-H_{\text{QCD}} \tau)$$

$$\Rightarrow Z = \int \Pi_{\text{links}} dU_{ij} \exp(-S(U))$$


iv) lattices need to be big! e.g. $16^3 \times 32$ sites

have to sum over all color indices at each link \( \Rightarrow \) integral \( 10^7 \) dimensional

start with some values $U_{ij}$ for all links, successively reassign new elements to reduce computing time: use stochastic technique with clever weighting (\( \exp(-S(U) \) favors small action)

have to sweep through entire lattice a few hundred times to evaluate thermodynamic quantities, baryon masses, wave functions
Huge increase in computing power since Wilson 1974

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**Graph:**
- **Graph Title:** Evolution of peak speed of supercomputers and improved algorithms.
- **Graph Details:**
  - **Y-axis:** PFLOPS, TFLOPS, GFLOPS
  - **X-axis:** Year
  - **Legend:**
    - JPN project machines: NWT, CP-PACS, Earth Simulator, K
    - Vector machines: CRAY/CDC, Hitachi/Fujitsu/NEC
    - Vector parallel machines: Fujitsu, NEC, CRAY
    - Parallel machines: CRAY, IBM, Intel, TMC, China
    - QCD parallel machines: Tsukuba, Columbia, APE, GF11 (IBM)

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**Text:**
- **Graph Description:**
  - CRAY-1
  - arXiv: 1501.04215

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**Evolution of Peak Speed:**
- Now lattice QCD is a mature technique with increasingly reliable results.
State-of-the-art light hadron spectrum from lattice QCD

S. Dürr, Z. Fodor et al. (Budapest-Marseille–Wuppertal Coll.) Science 322 (2008) 1225
Finite temperature lattice QCD

Variation of temperature:

temperature is changed e.g. by keeping $N_t$ constant

and changing the lattice spacing

(and thereby the coupling $g^2$)

$$a \to 0, \quad g^2 \to 0, \quad T \to \infty$$

approach asymptotic freedom

absolute scale is set by calculating a baryon or meson mass in units of $a$ (or $g^2$)
Indicators of the phase transition

Order parameter of deconfinement:

\[ \langle \Omega \rangle \]

\[ T_c \rightarrow T \]

free energy of static quark

\[ F_q = \infty \] confined

\[ F_q < \infty \] deconfined

Wilson-Polyakov loop

\[ \langle \Omega \rangle = \exp(-F_q/T) = 0 \] confined

\( \neq 0 \) deconfined

Order parameter of chiral symmetry restoration:

\[ \langle \bar{\Psi} \Psi \rangle \]

\[ T_c \rightarrow T \]

quark condensate

\[ \langle \bar{\Psi} \Psi \rangle \rightarrow 0 \] broken chiral symmetry

\( = 0 \) chirally symmetric
Indicators of the phase transition

susceptibilities $\chi$

e.g. chiral susc.

$$\chi_m = T \frac{\partial \langle \bar{\psi} \psi \rangle}{\partial m}$$

response of quark condens.
to mass

similarly Polyakov susceptibility

response to addition of
a quark
Deconfinement and chiral phase transition in lattice QCD

Polyakov loop - deconfinement

\[ \chi_L = N_c^3 \langle L^2 \rangle - \langle L \rangle^2 \]

F. Karsch, E. Laermann, hep-lat/0305025

Suceptibilities \( \chi \): measure of fluctuations

\[ \chi_{\psi\psi} = \partial \langle \psi\psi \rangle / \partial m = \partial^2 p / \partial m^2 \]
How to display equation of state?

divide by $T^4$ dependence for relativistic Bose/Fermi-gas
Equation of state in lattice QCD

consolidated results from different groups, extrapolated to continuum and chiral limit

- rapid rise of energy density (normalized to $T^4$ rise for relativistic gas)
- signals rapid increase in degrees of freedom due to transition from hadrons to quarks and gluons

H.T. Ding. arXiv:1504.05274
What is most realistic value of the critical temperature?

order parameter: chiral condensate, smooth behavior across phase conversion its susceptibility peaks at $T_c$


$T_c$ from peak in chiral susceptibility

$= 154 \pm 9$ MeV for chiral restoration
Measure of deconfinement in IQCD

\[ \chi_n^B = \frac{d^n P / T^4}{d \mu_B^n} \]

\[ \chi_4^B / \chi_2^B \propto \text{baryon number}^2 \]

H.T. Ding, NPA931 (2014) 52

rapid drop suggests:
chiral cross over and deconfinement appear in the same narrow temperature range

confined: 1

measure suggested by Ejiri, Karsch, Redlich (2006)

deconfined: 6/9\pi^2
Order of phase transition

Present state-of-the-art lattice QCD simulations give smooth cross over for realistic quark masses.

Critical role of strange quark mass.

Phase diagram in 2+1 flavor QCD (arXiv: 1504.05273)
Lattice QCD at finite baryon density

Lattice QCD at non-equal numbers of fermions and antifermions (non-zero baryon chemical potential) has a problem:
- for Fermions the partition function contains a Slater determinant
- for non-zero chemical potential this Slater determinant is complex
- straight forward importance sampling not possible
- oscillations i.e. the lattice QCD sign problem

\[ \det M(\mu) = |\det M(\mu)|e^{i\theta} \]

Use Taylor expansion to extrapolate into region of finite chemical potential
Lattice QCD at finite baryon density

for experimental points see chapter 5
Speed of sound from lattice QCD

similar to what was visible already for hadron gas – QGP using bag model equation of state:

in region of phase transition $P/\varepsilon$ not constant

softest point and minimum in speed of sound