Quark-Gluon Plasma Physics

4. Thermodynamics of the QGP

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Quantum statistics: Occupation number

For n identical particles, the wave function

$$\psi(\vec{r}_1, \ldots, \vec{r}_n)$$

must be an even (odd) function under interchange of any pair of coordinates for bosons (ferminon)

n-body wave function:
symmetrize (→ bosons) or antisymmetrize (→ fermion) product of n single-particle wave functions

Occupation number

$$n_\alpha = \begin{cases} 0, 1, 2, \ldots, \infty & \text{for bosons,} \\ 0, 1 & \text{for fermions.} \end{cases}$$

label for single-particle state

We have

$$\sum_\alpha n_\alpha = n$$

Average occupation number in thermal equilibrium

Average occupation number of single-particle state α:

$$n_\alpha = \frac{1}{z^{-1} e^{\beta E_\alpha} \pm 1} = \frac{1}{e^{\frac{E_\alpha - \mu}{kT}} \pm 1}$$

$+$: Fermi, $-$: Bose

$$\beta = \frac{1}{kT}$$

where the fugacity *z* defines the chemical potential μ:

$$z = e^{\beta \mu}$$

Chemical potential μ controls average particle number of the system

Number of states

Number of states between momentum p and p+dp
(each state occupies a volume h^3 in phase space):

number of states

physical volume

$$dN = \frac{V}{h^3} \boxed{4\pi p^2 dp}$$

volume of a spherical shell with radius p
and thickness dp in momentum space

Degeneracy for gluons and quarks

Gluons (spin-1 bosons):

$$g_\mathrm{g} = 8_\mathrm{color} \times 2_\mathrm{spin} = 16$$

Quarks (spin 1/2 fermions):

$$\begin{aligned} g_\mathrm{q} &= g_\mathrm{quark} + g_\mathrm{anti\text{-}quark} \\ &= 2 \times g_\mathrm{quark} \\ &= 2 \times 2_\mathrm{spin} \times 2_\mathrm{flavor} \times 3_\mathrm{color} = 24 \end{aligned}$$

Occupation number:

$$n_\alpha = \frac{g}{e^{\frac{E_\alpha - \mu}{kT}} \pm 1}$$

Non-interacting gluon gas

Gluons ($\mu = 0$), Bose-Einstein distribution ($m_{\text{gluon}} = 0 \rightarrow E = p$):

$$n_g = g_g \frac{4\pi}{(2\pi)^3} \int_0^\infty dE \frac{E^2}{e^{E/T} - 1}, \qquad \varepsilon_g = g_g \frac{4\pi}{(2\pi)^3} \int_0^\infty dE \frac{E^3}{e^{E/T} - 1} \quad (\hbar = k = 1)$$

Solution:

energy density:	pressure:	gluon density:
$\varepsilon_g = g_g \frac{\pi^2}{30} T^4,$	$p_g = \frac{1}{3} \varepsilon_g,$	$n_g = \frac{g_g}{\pi^2} \zeta(3) T^3$

$\zeta(3) = 1.20205$

Example: $T = 200\,\text{MeV},\ g_q = 16 \quad \Rightarrow \quad n_g = 2.03\ \text{gluons}/\text{fm}^3$

Non-interaction gas of massless quarks and antiquarks

Quark density (Fermi-Dirac distribution, massless quarks, i.e., $E = p$):

$$n_q(\mu_q) = \frac{N_q}{V} = g_q \frac{4\pi}{(2\pi)^3} \int_0^\infty dE \, \frac{E^2}{e^{(E-\mu_q)/T} + 1} \qquad (\hbar = k = 1)$$

no analytic solution for $\mu_q \neq 0$

pair creation: $q + \bar{q} \rightleftarrows \text{radiation}$

→ only the difference of the quark and antiquark density is fixed

→ $\mu_q + \mu_{\bar{q}} = 0$

For antiquarks we thus obtain:

$$n_{\bar{q}}(\mu_q) = g_{\bar{q}} \frac{4\pi}{(2\pi)^3} \int_0^\infty dE \, \frac{E^2}{e^{(E+\mu_q)/T} + 1} \qquad (\hbar = k = 1)$$

Non-interacting quark gas with μ = 0

Quark density ($\mu_q = 0$):

$$\boxed{n_q = n_{\bar{q}} = \frac{g_q}{2\pi^2} \frac{3}{2} \zeta(3)\, T^3}$$

Total energy of the quarks ($E = p$ for massless quarks):

$$E = \int_0^\infty E \, dN_q$$

Energy density and pressure ($\mu_q = 0$):

$$\boxed{\varepsilon_q = \frac{E_q}{V} = \frac{7}{8} g_q \frac{\pi^2}{30} T^4, \qquad p_q = \frac{1}{3} \varepsilon_q}$$

(identical result for antiquarks)

Example: $T = 200\,\text{MeV}, \; g_q = 18 \quad \Rightarrow \quad n_q = n_{\bar{q}} = 1.71/\text{fm}^3$

Non-interacting QGP at μ = 0

Pressure and energy density in a quark-gluon plasma at μ = 0 without particle interactions:

$$p_{\mathrm{QGP}} = \left(g_g + \frac{7}{8}(g_q + g_{\bar{q}})\right)\frac{\pi^2}{90}T^4, \qquad \varepsilon_{\mathrm{QGP}} = 3p_{\mathrm{QGP}}$$

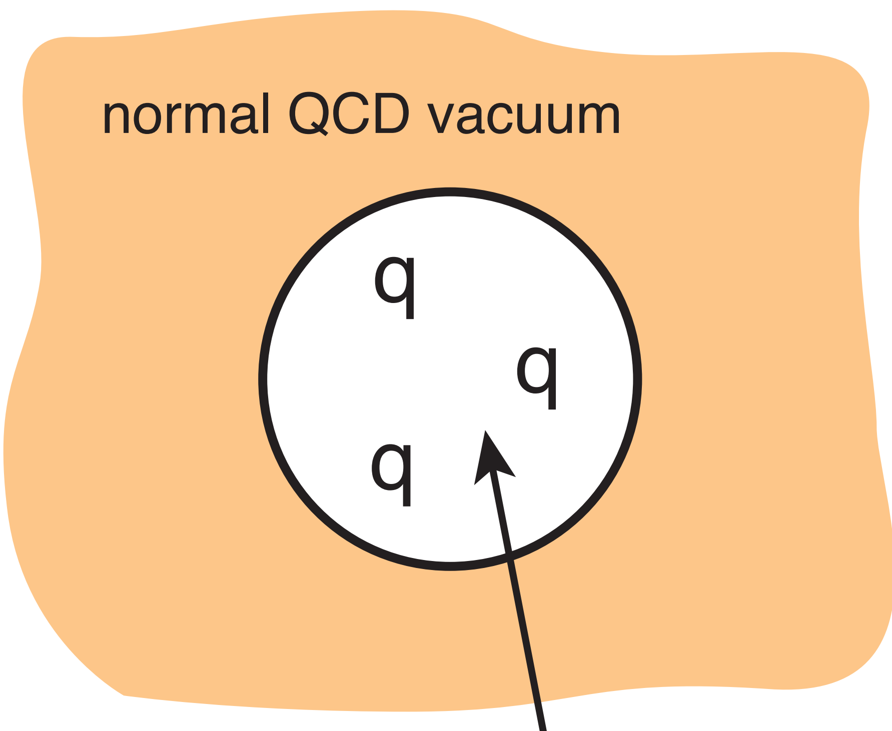
$$= \begin{cases} 37\dfrac{\pi^2}{90}T^4 & \text{for } u, d \\[2ex] 47.5\dfrac{\pi^2}{90}T^4 & \text{for } u, d, s \end{cases} \qquad = \begin{cases} 37\dfrac{\pi^2}{30}T^4 & \text{for } u, d \\[2ex] 47.5\dfrac{\pi^2}{30}T^4 & \text{for } u, d, s \end{cases}$$

Example:

$$T = 200\,\mathrm{MeV},\ \text{two active quark flavors} \quad \Rightarrow \quad \varepsilon_{\mathrm{QGP}}^{\mathrm{id.\,gas}} = 2.55\ \mathrm{GeV/fm^3}$$

Bag Model

A. Chodos et al., Phys. Rev. D10 (1974) 2599
T. DeGrand et al. Phys. Rev. D12 (1975) 2060



- Build confinement and asymptotic freedom into simple phenomenological model
- Hadron = „bag" filled with massless quarks
- Two kinds of vacuum
  - Normal QCD vacuum outside of the bag
  - Perturbative QCD vacuum within the bag

Energy density in the bag is higher than in the vacuum: $\varepsilon_{\text{pert}} - \varepsilon_{\text{vacuum}} =: B$

Energy of N quarks in a bag of radius R: $$E = \underbrace{\boxed{\frac{2.04 N}{R}}}_{\text{kinetic energy of } N \text{ particles in a sphere of radius } R} + \frac{4}{3}\pi R^3 B$$

Condition for stability: dE/dR = 0 (minimum):

$$B^{1/4} = \left(\frac{2.04 N}{4\pi}\right)^{1/4} \frac{1}{R} \quad \overset{N=3,\, R=0.8\,\text{fm}}{\Longrightarrow} \quad B^{1/4} = 206\,\text{MeV} \quad (\hbar = c = 1)$$

Critical temperature for an ideal QGP with μ = 0

Modeling the complicated aspects of the QCD vacuum with one number:

$$\varepsilon_{\text{QGP}}^{\text{QCD vac.}} = \varepsilon_{\text{QGP}} + B \qquad E = TS - pV \quad (\mu = 0)$$

$$p_{\text{QGP}}^{\text{QCD vac.}} = p_{\text{QGP}} - B \qquad \Rightarrow p + \varepsilon = Ts$$

So we have (here: HG = massless pion gas → $g = 3$ [3 species, π^+,π^-,π^0]):

$$p_{\text{HG}} = 3aT^4 \qquad \varepsilon_{\text{HG}} = 9aT^4$$

$$p_{\text{QGP}}^{\text{QCD vac.}} = 37aT^4 - B \qquad \varepsilon_{\text{QGP}}^{\text{QCD vac.}} = 111aT^4 + B$$

$$a = \frac{\pi^2}{90}$$

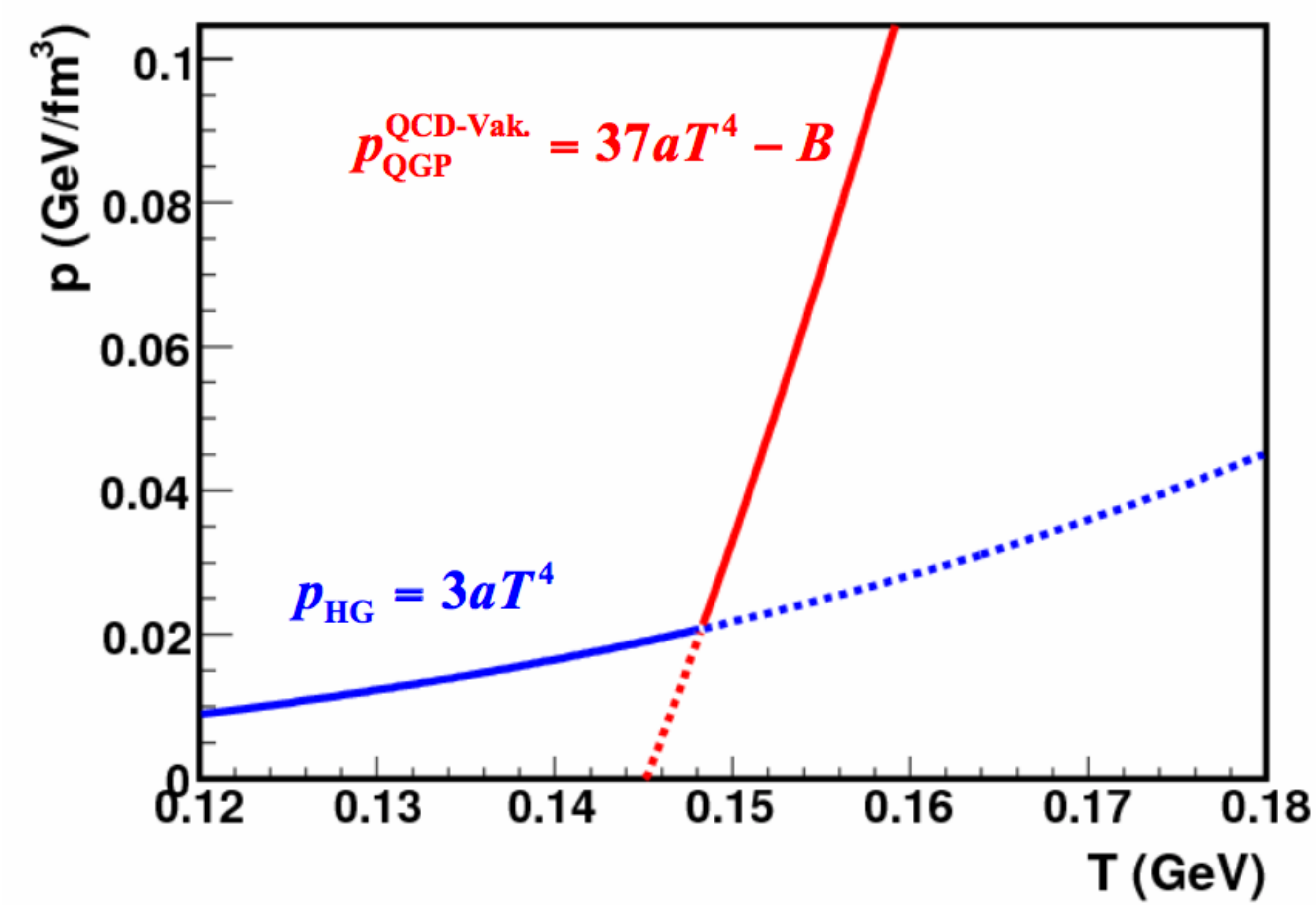
Gibbs criterion for the phase transition:

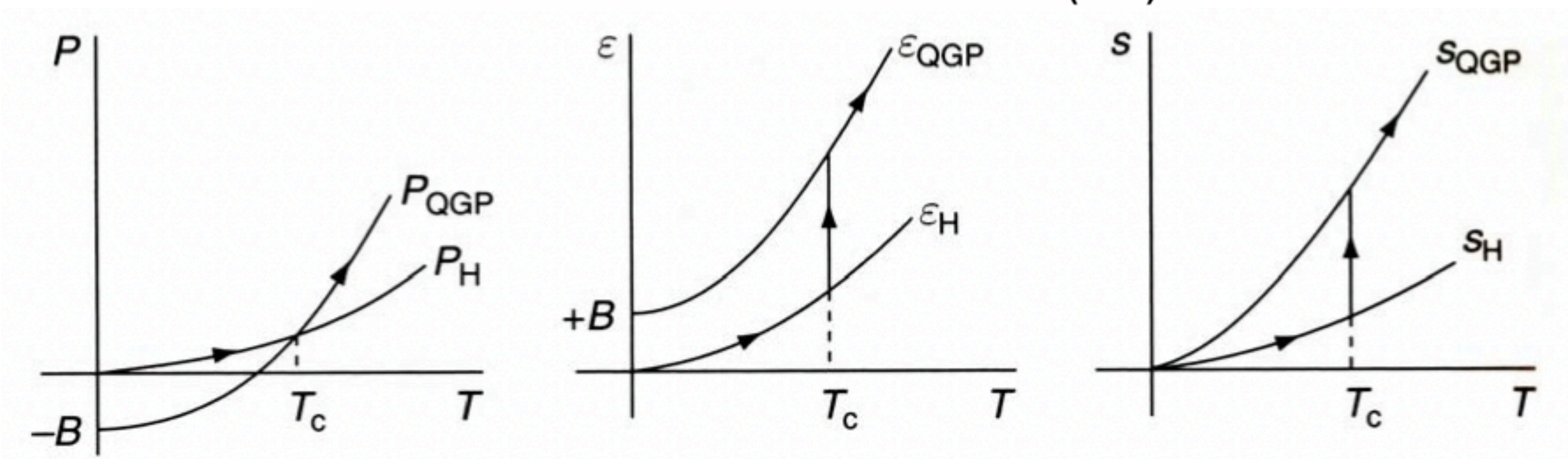
$$p_{\text{HG}}(T_c) = p_{\text{QGP}}^{\text{QCD vac.}}(T_c) \qquad \Rightarrow \qquad T_c = \left(\frac{B}{34a}\right)^{1/4} \approx 150\ \text{MeV}$$

Phase transition in the bag model is of first order. Latent heat:

$$\varepsilon_{\text{QGP}}^{\text{QCD vac.}}(T_c) - \varepsilon_{\text{HG}}(T_c) = 102aT_c^4 + B = 4B$$

Ideal QGP with with μ = 0: Pressure, energy density, and entropy vs. *T*





QGP with μ ≠ 0: Entropy

Entropy density:

$$dE = T dS - p dV \quad (\mu = 0) \quad \Rightarrow \quad \frac{dS}{dV} = s = \frac{\varepsilon + p}{T} = \frac{4p}{T}$$

Ratio entropy density QGP / massless pion gas:

$$s_{\mathrm{QGP}} = 148 a T^3, \quad s_{\mathrm{HG}} = 12 a T^3 \quad \Rightarrow \quad \frac{s_{\mathrm{QGP}}}{s_{\mathrm{HG}}} \approx 12.3$$

Large increase in volume at QGP → pion gas transition! (entropy conservation)

Entropy per particle:

Massless pion gas: $$\frac{s_\pi}{n_\pi} = \frac{12\pi^2/90 \cdot T^3}{g_\pi \cdot 1.202/\pi^2 \cdot T^3} = 3.6$$

Idel QGP: $$\frac{s_{\mathrm{q}}}{n_{\mathrm{q}}} = 1.4, \qquad \frac{s_{\mathrm{g}}}{n_{\mathrm{g}}} = 1.2$$

QGP with μ ≠ 0: Energy and particle number density of quarks

For $\mu_q \neq 0$ a solution in closed form can be found for $\varepsilon_q + \varepsilon_{\bar{q}}$ but not separately for ε_q and $\varepsilon_{\bar{q}}$: Chin, PL 78B (1978) 552

$$\varepsilon_q + \varepsilon_{\bar{q}} = g_q \times \left(\frac{7\pi^2}{120} T^4 + \frac{\mu_q^2}{4} T^2 + \frac{\mu_q^4}{8\pi^2} \right)$$

Accordingly one finds for the quark density:

$$n_q - n_{\bar{q}} = g_q \times \left(\frac{\mu_q}{6} T^2 + \frac{\mu_q^3}{6\pi^2} \right)$$

From this the net baryon density can be determined as (for $g_q = 12$):

$$n_B = \frac{n_q - n_{\bar{q}}}{3} = \frac{2\mu_q}{3} T^2 + \frac{2\mu_q^3}{3\pi^2} \quad = \quad \frac{2\mu_B}{9} T^2 + \frac{2\mu_B^3}{81\pi^2} \qquad (\mu_B = 3\mu_q)$$

QGP with μ ≠ 0:
Energy and particle number density of quarks

Energy density in a QGP with μ ≠ 0 (without particle interactions):

$$\varepsilon_{\mathrm{QGP}} = \varepsilon_q + \varepsilon_{\bar{q}} + \varepsilon_g = \frac{37\pi^2}{30} T^4 + 3\mu_q^2 T^2 + \frac{3\mu_q^4}{2\pi^2}$$

Condition for QGP stability:

$$p_{\mathrm{QGP}} = \frac{1}{3}\varepsilon_{\mathrm{QGP}} \stackrel{!}{=} B \quad \Rightarrow \quad T_c(\mu_q)$$

Condition for QGP: QGP-pressure ≥ pressure of the QCD-vacuum (similar, but not identical, to the previous condition $p_{\mathrm{HG}} = p_{\mathrm{QGP}}$)

Critical temperature / quark potential:

$$T_c(\mu_q = 0) = \left(\frac{90B}{37\pi^2}\right)^{1/4}$$

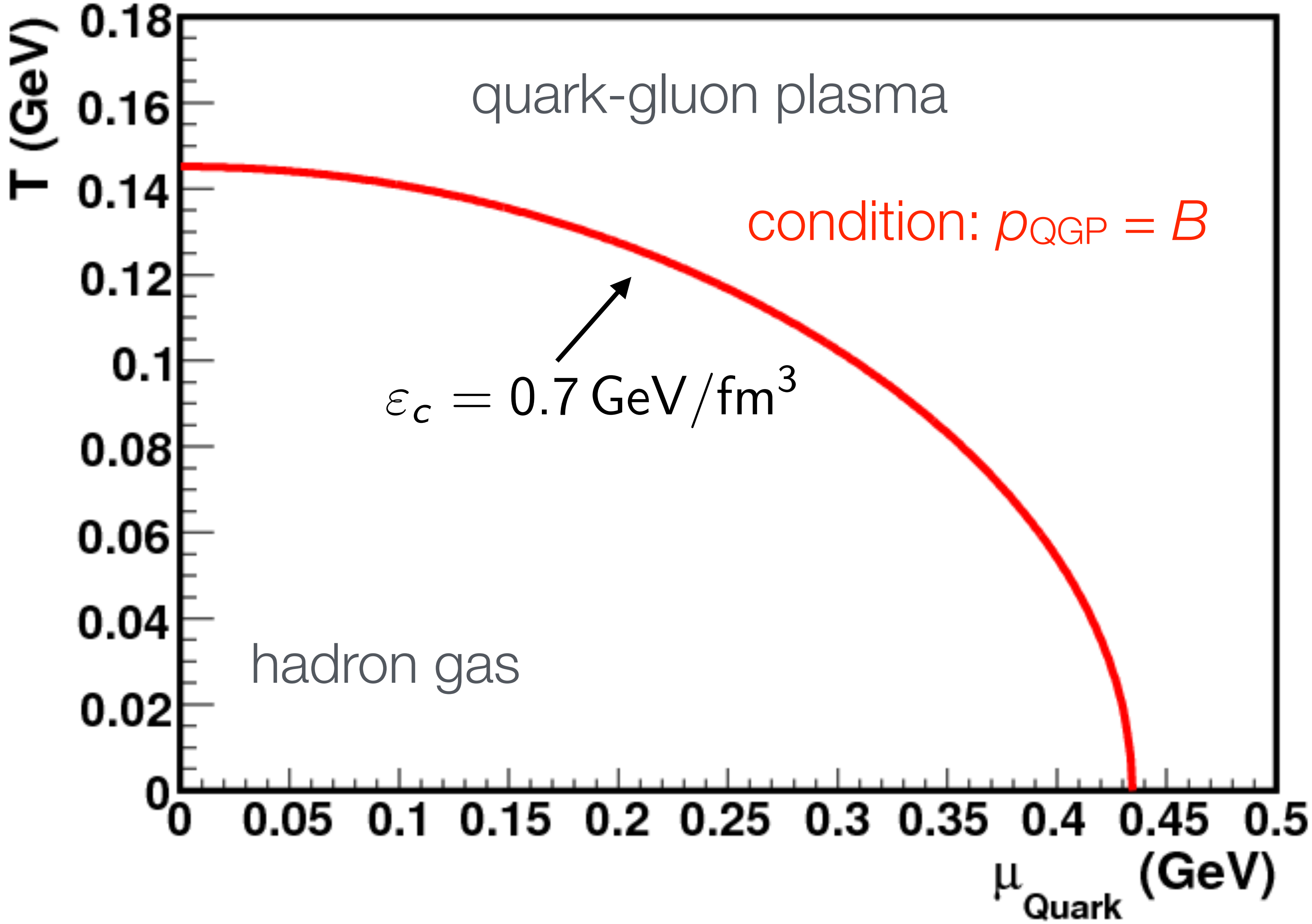
$$\mu_q^c(T = 0) = (2\pi^2 B)^{1/4} = 0.43\,\mathrm{GeV}$$

$$n_B^c(T = 0) = \frac{2}{3\pi^2}(2\pi^2 B)^{3/4}$$

$$= 0.72\,\mathrm{fm}^{-3} \approx 5 \times n_{\mathrm{nucleus}}$$

Possibly reached in neutron stars (?)

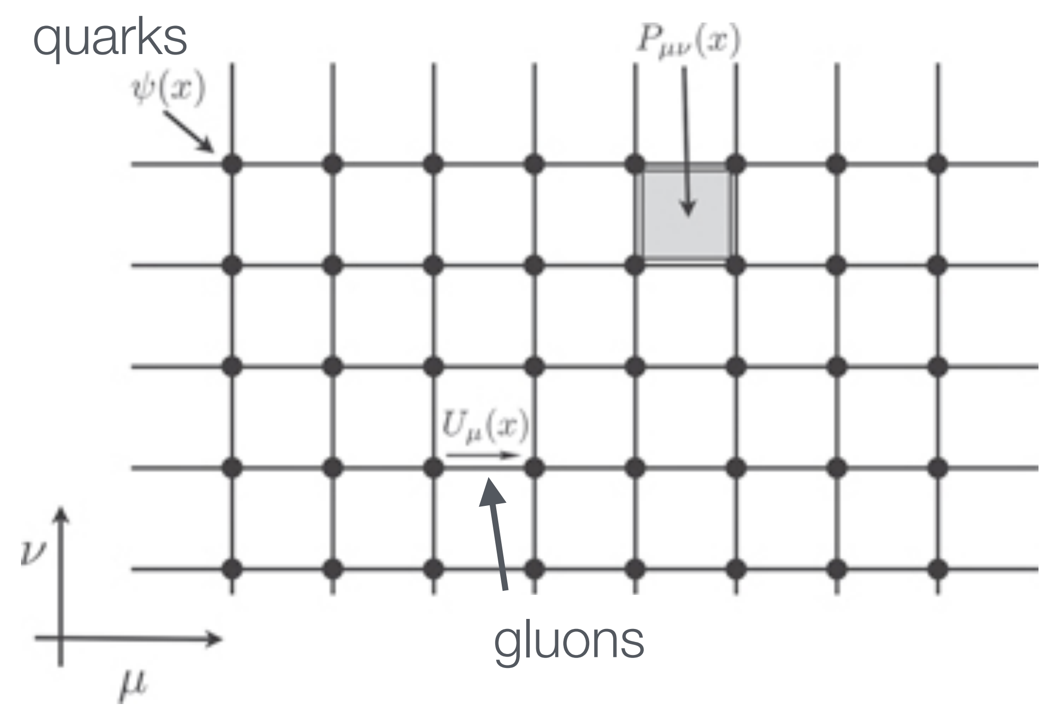
QGP with μ ≠ 0: Phase Diagram of the non-Interacting QGP



$\mu_b = 3 \cdot \mu_{\text{quark}}$

Lattice QCD

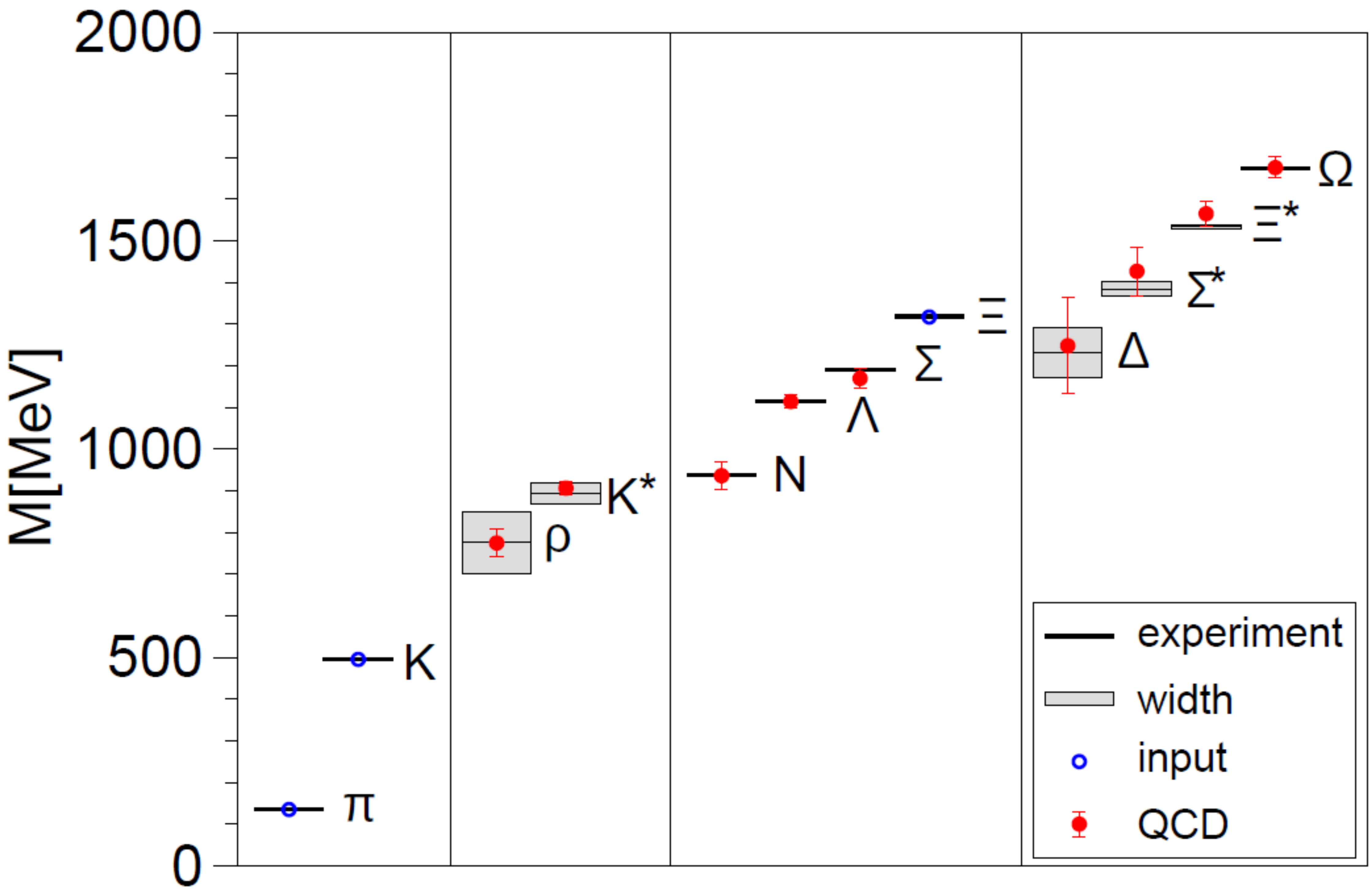
- Formulated in 1974 (K. Wilson), numerical Monte Carlo calculations started ca. 1980 (M. Creutz)
- First-principles non-perturbative calculation
- Benefitted from huge increase in computing power
- QCD thermodynamics on the lattice
  - So far restricted to $\mu_B \approx 0$
  - Two major groups (HotQCD coll., Wuppertal-Budapest coll.), results agree
- To be done:
  - Lattice QCD for finite baryon number
  - Transport properties of the QGP
  - Clarify existence and location of critical endpoint (CEP)



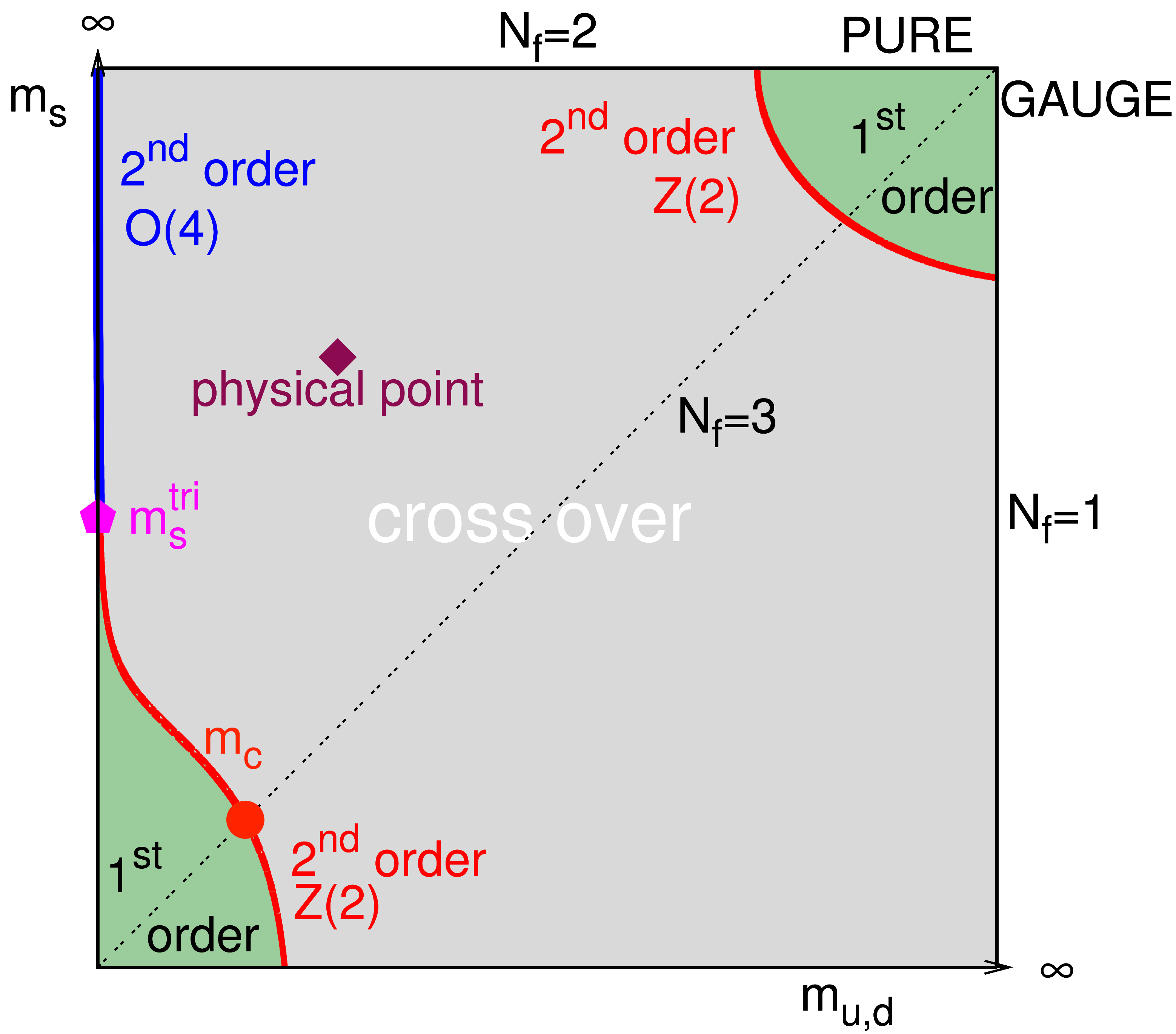
Example of a machine for lattice QCD: JUGENE in Jülich (294,912 processor cores, ~ 1 PetaFLOPS)

Lattice QCD correctly describes mass spectrum of hadrons

S. Dürr, Z.Fodor et al., Budapest-Marseille–Wuppertal Coll., Science 322 (2008) 1225



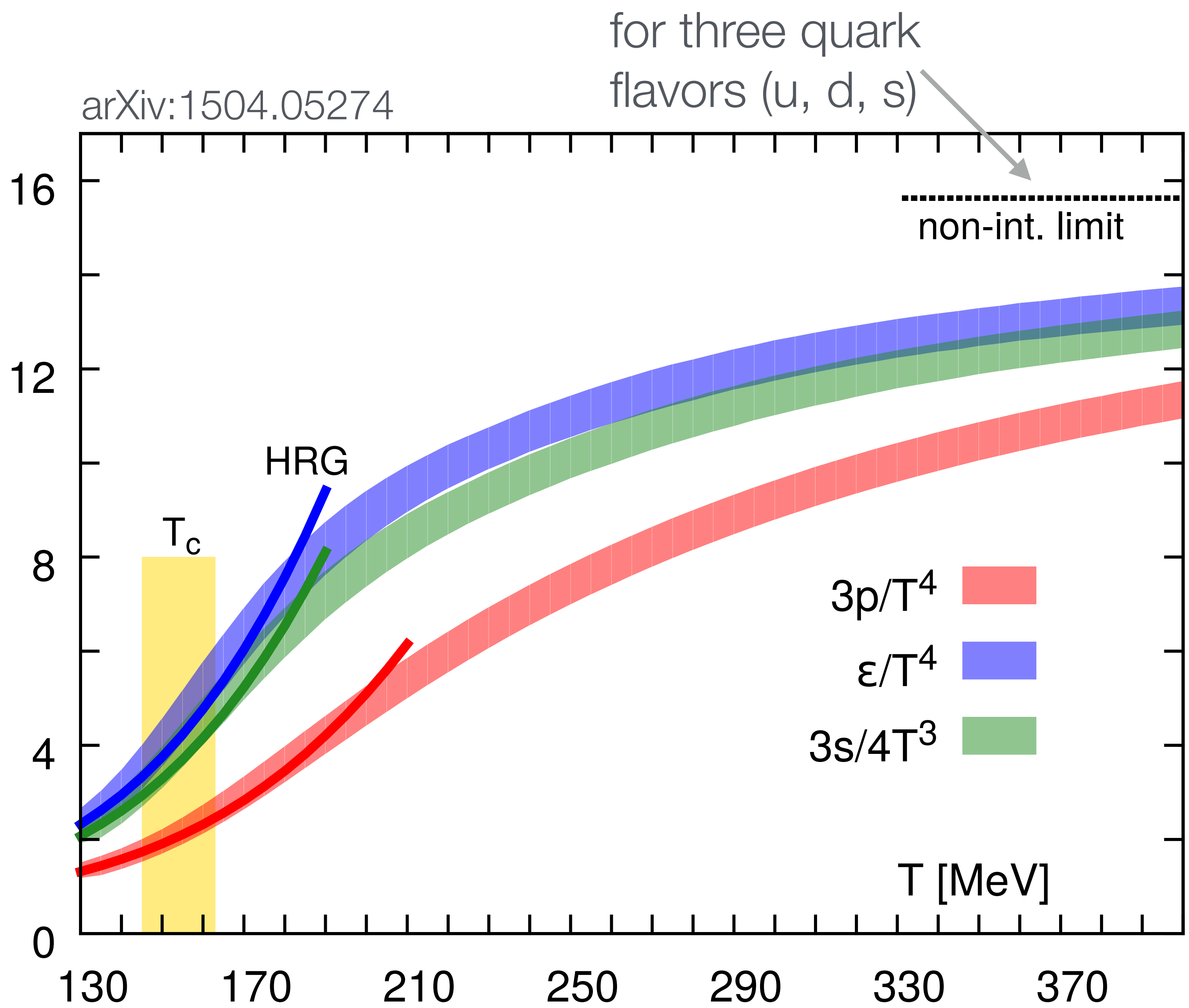
Lattice QCD:
Nature of the transition vs. quark mass



Ding, Karsch, Mukherjee, arXiv:1504.05274

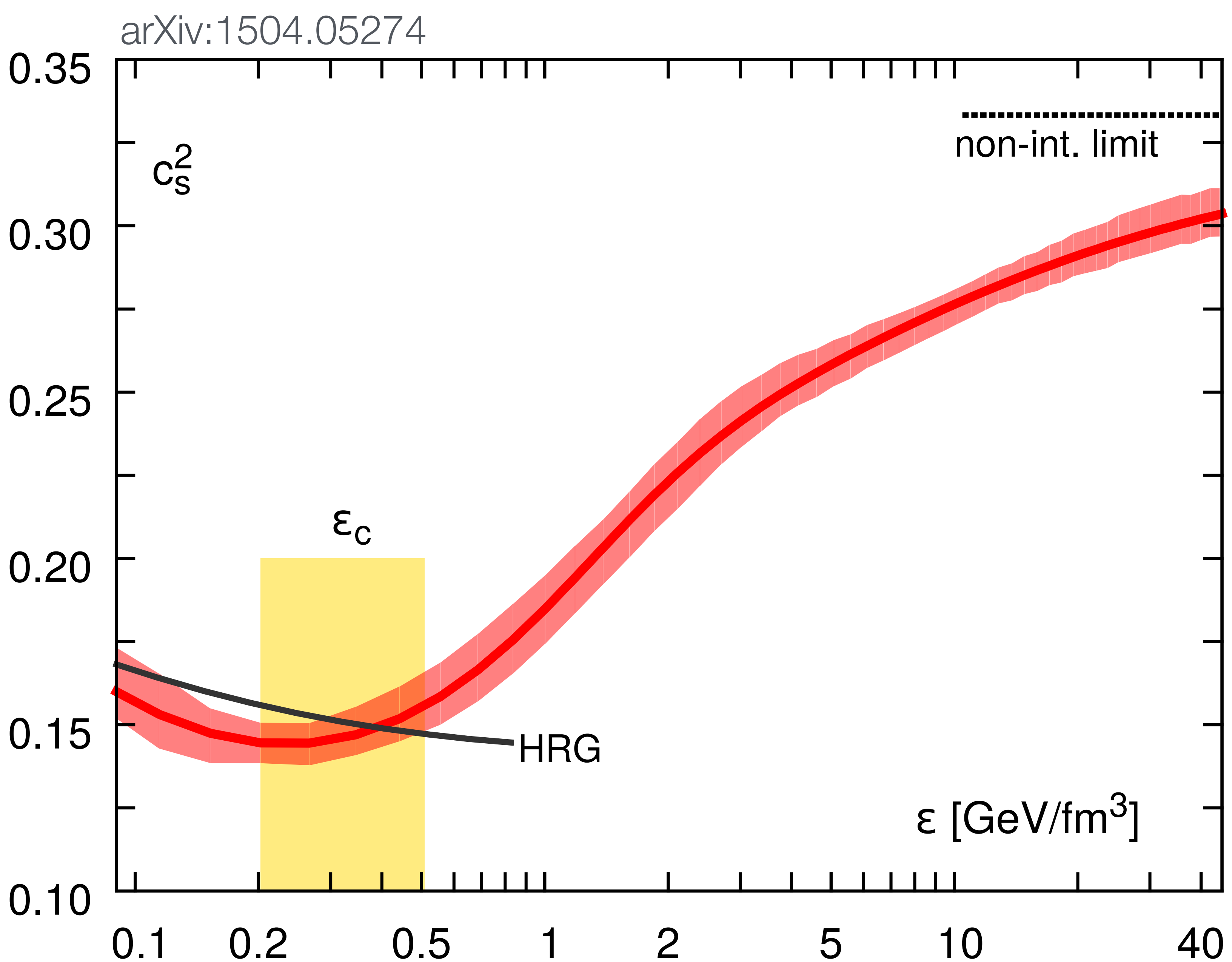
- Nature of the transition depends on quark masses
- Infinitely heavy quarks (pure gauge)
  - First order phase transition
  - $T_c \approx 270$ MeV
- Cross over transition for physical quark masses

Pressure, energy and entropy density from lattice QCD



- (2+1) flavor QCD
  - two light quarks (u,d) + 1 heavier quark (s)
- Results extrapolated to continuum limit
- Pseudo-critical temperature for chiral crossover transition
  - $T_c = (154 \pm 9)$ MeV
  - $\varepsilon_c \approx (0.34 \pm 0.16)$ GeV/fm^3
- Hadron resonance gas (HRG) agrees with lattice results for $T < T_c$
- State-of-the art hydro calc's use equation of from lattice QCD

Speed of sound

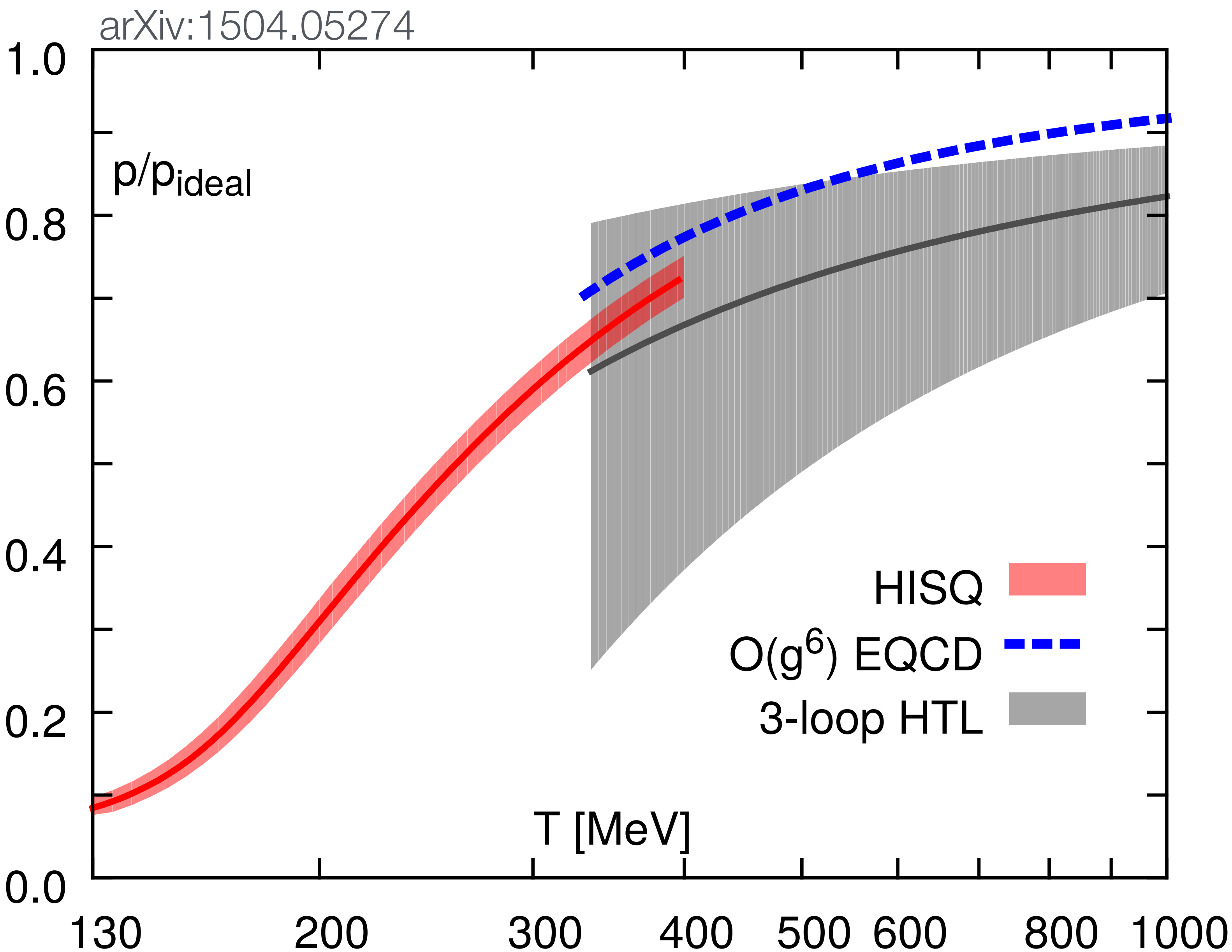


entropy density

$$c_s^2 = \frac{\mathrm{d}p}{\mathrm{d}\varepsilon} = \frac{\mathrm{d}p/\mathrm{d}T}{\mathrm{d}\varepsilon/\mathrm{d}T} = \frac{s}{c_V}$$

specific heat

Lattice QCD vs. perturbation theory



Lattice QCD agrees with perturbation theory (HLT) for T > 400 MeV

Summary QCD thermodynamics

- Toy model based on treating the QGP as a bag in the QCD vacuum filled with an ideal gas of quarks and gluons provides some intuitive insights into the phase diagram
- For T = 400 MeV the energy density of an ideal gas is only 20% above the lattice QCD results
- Lattice QCD results
  - For physical quark masses the transition at $\mu_B = 0$ is a crossover
  - Chiral symmetry transition coincides with deconfinement transition
  - Pseudo-critical temperature and energy density
    - $T_c = (154 \pm 9)$ MeV
    - $\varepsilon_c \approx (0.34 \pm 0.16)$ GeV/fm^3
- Not covered, but interesting: Thermodynamic fluctuations, especially fluctuations of conserved quantities (charge Q, baryon number B, …)
  - Measured via susceptibilities on the lattice, experimentally accessible