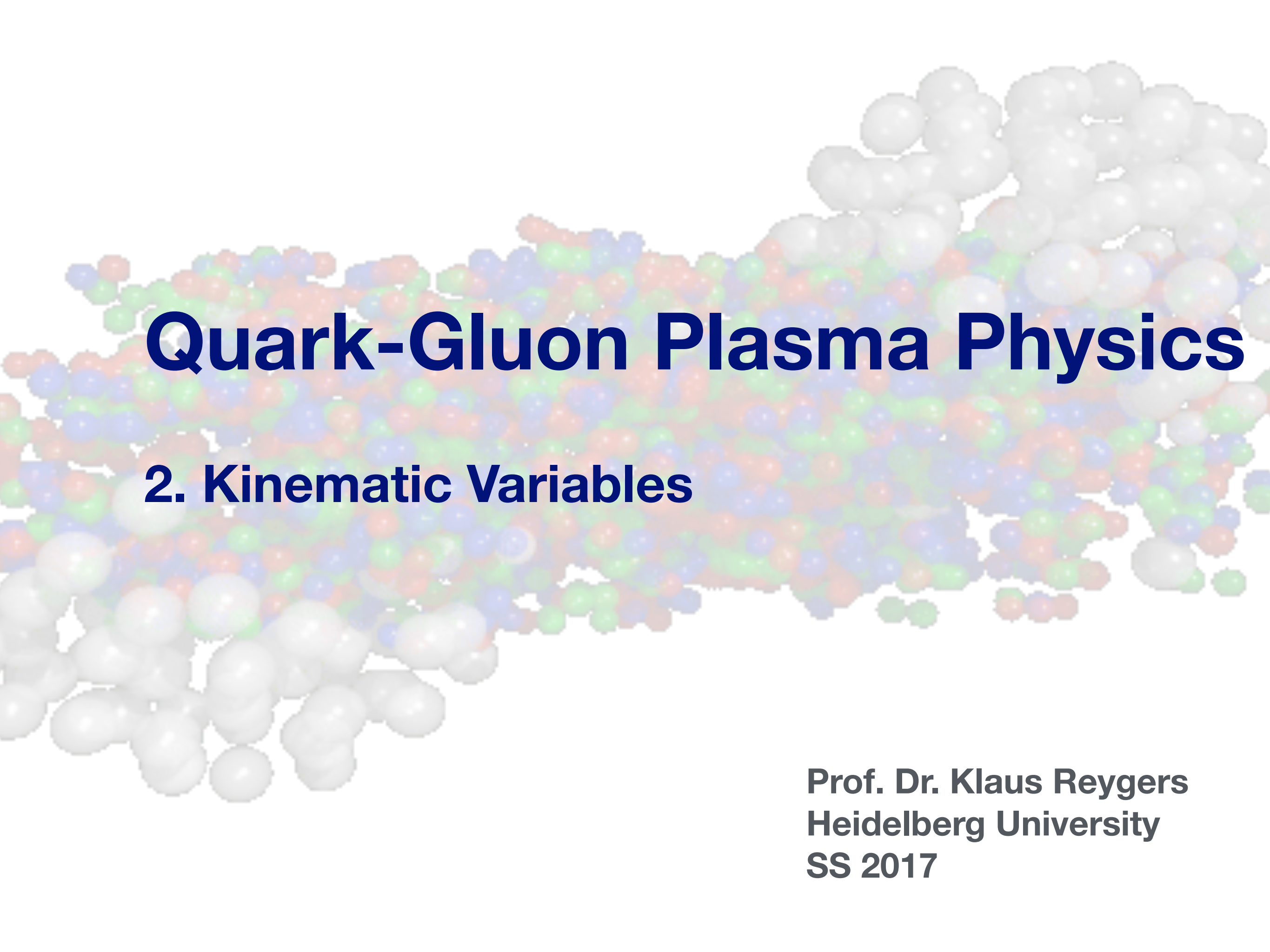




Quark-Gluon Plasma Physics

2. Kinematic Variables

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Lorentz transformation

- Postulates
1. There is no preferred inertial frame
 2. The speed of light in a vacuum has the same value c in all inertial frames of reference

Space-time four-vector in system S

$$x^\mu := (x^0, x^1, x^2, x^3) = (t, \vec{x}) = (t, x, y, z)$$

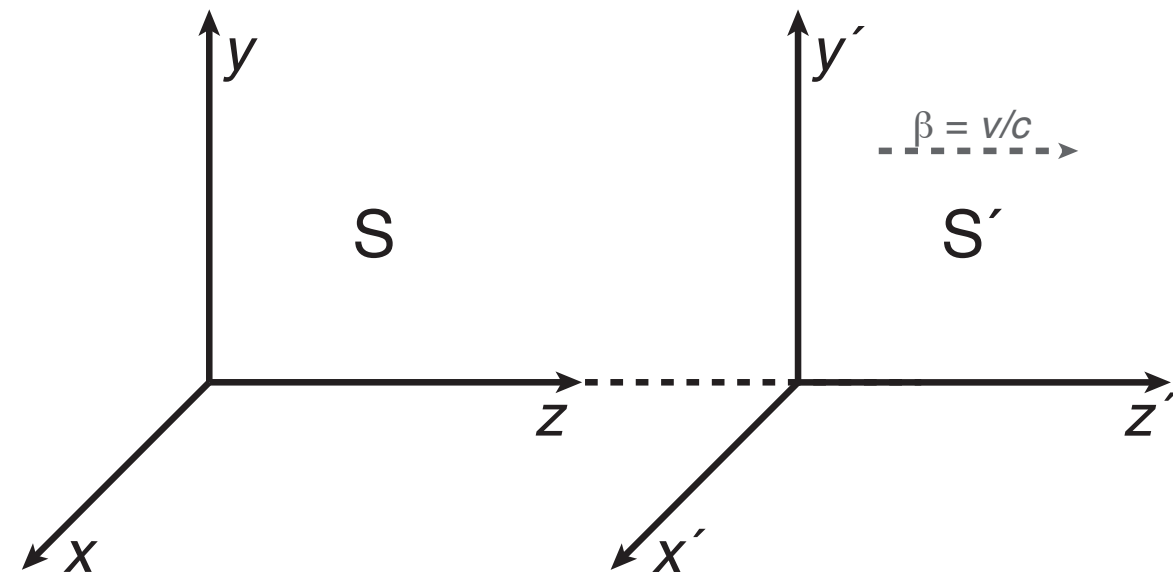
In system S'
(follows from the two postulates)

$$x^{0'} = \gamma(x^0 - \beta x^3)$$

$$x^{1'} = x^1$$

$$x^{2'} = x^2$$

$$x^{3'} = \gamma(x^3 - \beta x^0)$$



$$\beta = v/c \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

Energy-momentum 4-vector

General four-vector:

transforms under Lorentz transformation like the space-time four-vector

Relativistic energy and momentum:

$$E = \gamma m, \quad \vec{p} = \gamma \beta m, \quad m = \text{rest mass}$$

4-momentum vector:

$$p^\mu = (p^0, p^1, p^2, p^3) = (E, \vec{p}) = (E, \vec{p}_T, p_z) = (E, p_x, p_y, p_z)$$

Scalar product of two 4-vectors a and b :

$$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}$$

Relation between energy and momentum:

$$E^2 = p^2 + m^2$$

Center-of-Mass System (CMS)

Consider a collision of two particles. The CMS is defined by

$$\vec{p}_a = -\vec{p}_b$$

$$p_a = (E_a, \vec{p}_a) \qquad p_b = (E_b, \vec{p}_b)$$


The Mandelstam variable s is defined as

$$s := (p_a + p_b)^2 \stackrel{CMS}{=} (E_a + E_b)^2$$

$\sqrt{s}$ is the total energy in the center-of-mass frame ("center-of-mass energy")

Example: LHC. beam energy 6.5 TeV: $\sqrt{s} = 2 E = 13$ TeV (lab frame = CMS)

More on LHC energies

From 'centripetal force = Lorentz force' one obtains:

$$\begin{array}{c} \text{"/} \\ R \equiv \frac{p}{q} \\ \text{"rigidity"} \end{array} = \begin{array}{c} \\ r_{\text{LHC,bend}} \\ \text{1232 dipoles} \times 14.3 \text{ m} / (2 \pi) \end{array} \cdot B_{\text{LHC}}, \quad B_{\text{LHC,max}} \approx 8.3 \text{ T} \quad (\rightarrow \text{this limits } \sqrt{s})$$

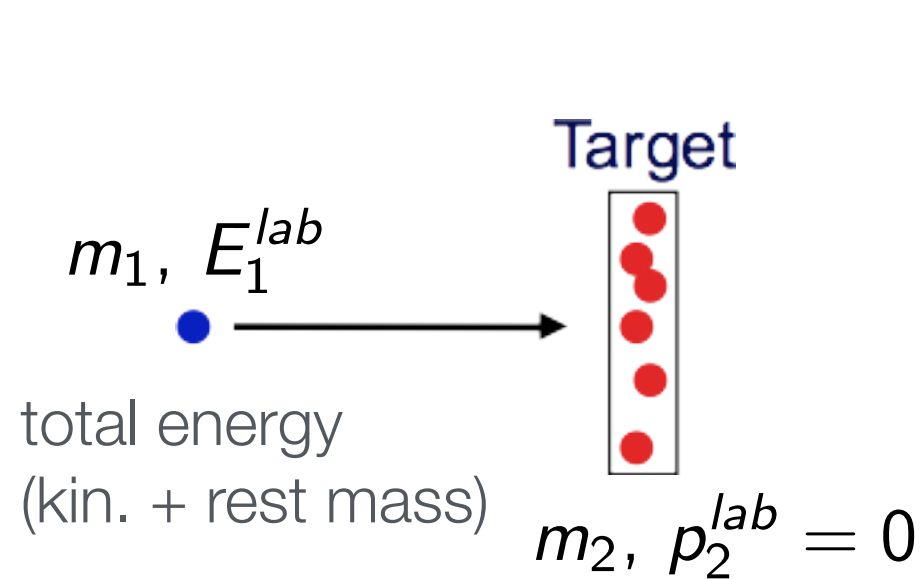
$$\text{protons: } R = p_{\text{proton}} \qquad \text{ions: } R = \frac{A \cdot p_{\text{nucleon}}}{Z}$$

$$\begin{array}{c} \text{2011/12: } p_{\text{proton}} = 3.5 \text{ TeV} \rightarrow p_{\text{nucleon}} \equiv p_{\text{Pb}}/A = \frac{Z}{A} \cdot p_{\text{proton}} = 1.38 \text{ TeV} \\ \text{corresponding energy of nucleons in} \\ \text{Pb ion for same } B \text{ field (same rigidity)} \end{array}$$

Center-of-momentum energy per nucleon-nucleon pair:

$$\text{Pb-Pb (2011/12): } \sqrt{s_{NN}} = 2.76 \text{ TeV} \qquad \text{Pb-Pb (2015): } \sqrt{s_{NN}} = 5.02 \text{ TeV}$$

$\sqrt{s}$ for Fixed-Target Experiments



$$s = \left[\begin{pmatrix} E_1^{\text{lab}} \\ \vec{p}_1 \end{pmatrix} + \begin{pmatrix} m_2 \\ \vec{0} \end{pmatrix} \right]^2$$

$$= m_1^2 + m_2^2 + 2E_1^{\text{lab}} m_2$$

$$\Rightarrow \sqrt{s} = \sqrt{m_1^2 + m_2^2 + 2E_1^{\text{lab}} m_2}$$

$$\stackrel{E_1^{\text{lab}} \gg m_1, m_2}{\approx} \sqrt{2E_1^{\text{lab}} m_2}$$

Example: antiproton production (fixed-target experiment):

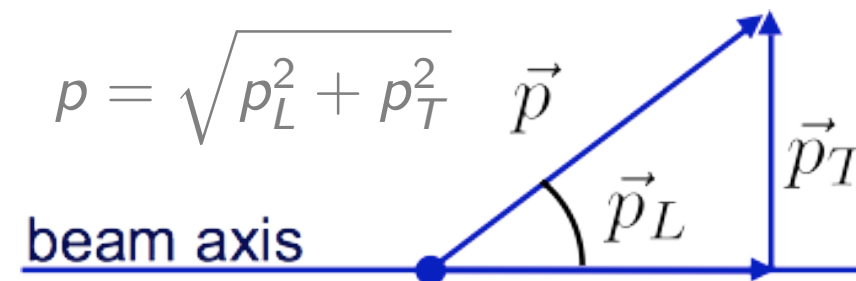
$$p + p \rightarrow p + p + p + \bar{p}$$

Minimum energy required to produce an antiproton: In CMS. all particles at rest after the reaction. i.e.. $\sqrt{s} = 4 m_p$. hence:

$$4m_p \stackrel{!}{=} \sqrt{2m_p^2 + 2E_1^{\text{lab},\text{min}} m_p} \Rightarrow E_1^{\text{lab},\text{min}} = \frac{(4m_p)^2 - 2m_p^2}{2m_p} = 7m_p$$

Rapidity

The rapidity y is a generalization of the (longitudinal) velocity $\beta_L = p_L / E$:



$$y := \operatorname{arctanh} \beta_L = \frac{1}{2} \ln \frac{1 + \beta_L}{1 - \beta_L} = \frac{1}{2} \ln \frac{E + p_L}{E - p_L}$$

$$y \approx \beta_L \text{ for } \beta_L \ll 1$$

With

$$e^y = \sqrt{\frac{E + p_L}{E - p_L}}, \quad e^{-y} = \sqrt{\frac{E - p_L}{E + p_L}}$$

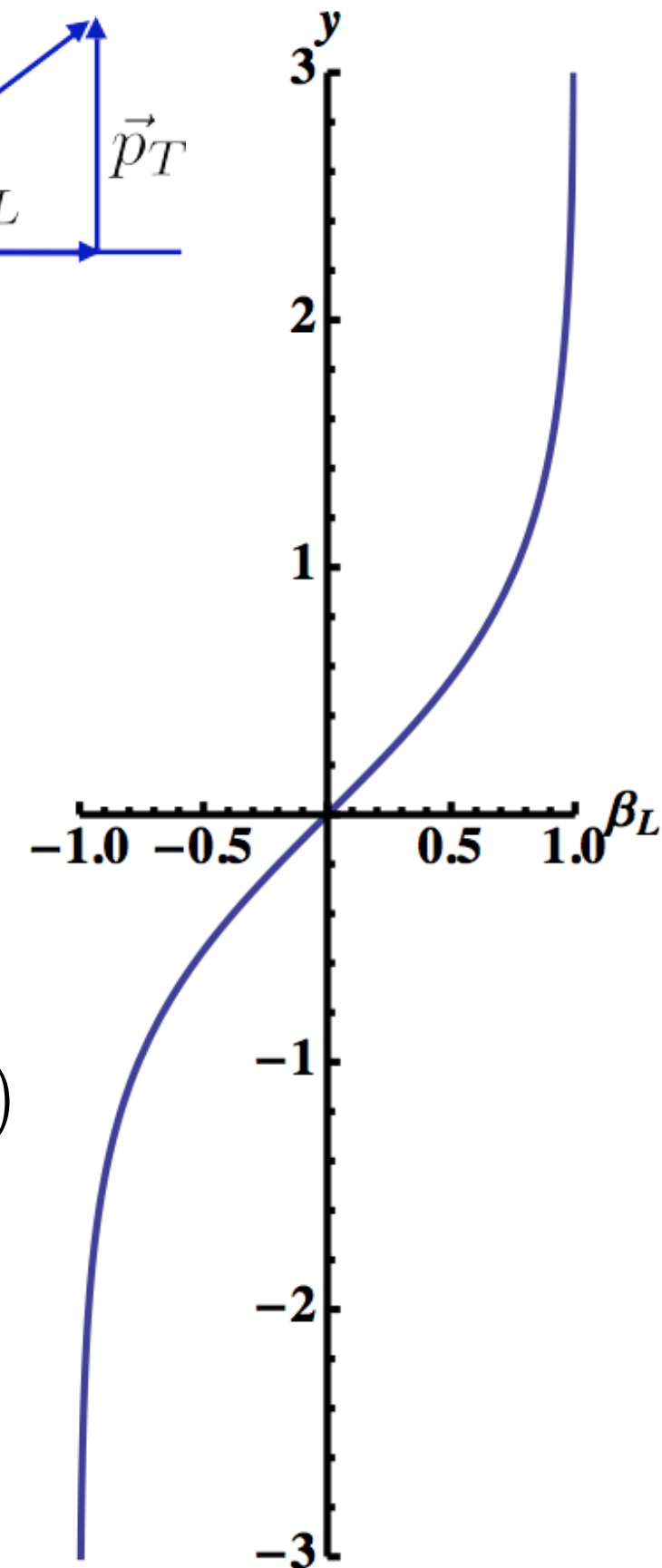
and

$$\sinh x = \frac{1}{2} (e^x - e^{-x}), \quad \cosh x = \frac{1}{2} (e^x + e^{-x})$$

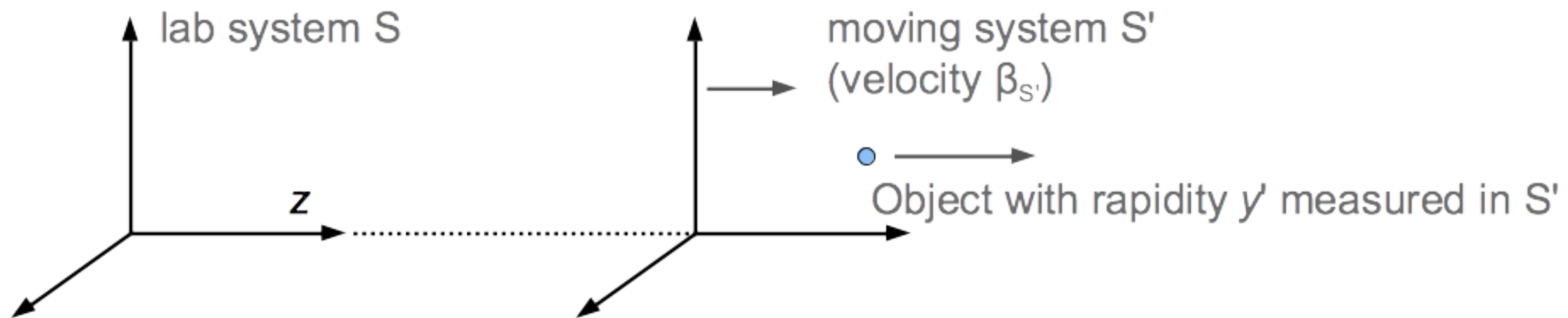
one obtains

$$E = m_T \cdot \cosh y, \quad p_L = m_T \cdot \sinh y$$

where $m_T := \sqrt{m^2 + p_T^2}$ is called *transverse mass*



Additivity of Rapidity under Lorentz Transformation



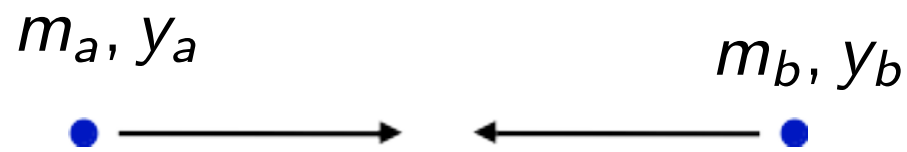
Lorentz transformation: $E = \gamma(E' + \beta p'_z)$, $p_z = \gamma(p'_z + \beta E')$ ($\beta \equiv \beta_{S'}$)

$$\begin{aligned}
 y &= \frac{1}{2} \ln \frac{E + p_z}{E - p_z} \\
 &= \frac{1}{2} \ln \frac{\gamma(E' + \beta p'_z) + \gamma(p'_z + \beta E')}{\gamma(E' + \beta p'_z) - \gamma(p'_z + \beta E')} \\
 &= \frac{1}{2} \ln \frac{(1 + \beta)(E' + p'_z)}{(1 - \beta)(E' - p'_z)} \\
 &= \underbrace{\frac{1}{2} \ln \frac{1 + \beta}{1 - \beta}}_{\text{rapidity of } S' \text{ as measured in } S} + \underbrace{\frac{1}{2} \ln \frac{E' + p'_z}{E' - p'_z}}_{y'}
 \end{aligned}$$

y is not Lorentz invariant.
however, it has a simple
transformation property:

$$y = y' + y_{S'}$$

Rapidity of the CMS (I)



$$a = (E_a, 0, 0, p_a)$$

$$b = (E_b, 0, 0, -p_b)$$

Velocity of the CMS:

$$a_z^* = \gamma_{\text{cm}}(a_z - \beta_{\text{CM}} a_0) \stackrel{!}{=} -b_z^* = -\gamma_{\text{cm}}(b_z - \beta_{\text{CM}} b_0) \quad \Rightarrow \quad \beta_{\text{cm}} = \frac{a_z + b_z}{a_0 + b_0}$$

Using the formula for the rapidity we obtain

$$y_{\text{cm}} = \frac{1}{2} \ln \left[\frac{a_0 + a_z + b_0 + b_z}{a_0 - a_z + b_0 - b_z} \right]$$

Writing energies and momenta in terms of rapidity:

$$\begin{aligned} y_{\text{cm}} &= \frac{1}{2} \ln \left[\frac{m_a e^{y_a} + m_b e^{y_b}}{m_a e^{-y_a} + m_b e^{-y_b}} \right] \\ &= \frac{1}{2} (y_a + y_b) + \frac{1}{2} \ln \left[\frac{m_a e^{y_a} + m_b e^{y_b}}{m_a e^{y_b} + m_b e^{y_a}} \right] \end{aligned}$$

Rapidity of the CMS (II)

For a collision of two particles with equal mass m and rapidities y_a and y_b . the rapidity of the CMS y_{cm} is then given by:

$$y_{\text{cm}} = (y_a + y_b)/2$$

In the center-of-mass frame. the rapidities of particles a and b are:

$$y_a^* = y_a - y_{\text{cm}} = -\frac{1}{2}(y_b - y_a) \qquad y_b^* = y_b - y_{\text{cm}} = \frac{1}{2}(y_b - y_a)$$

Examples (CMS rapidity of the nucleon-nucleon system)

a) fixed target experiment: $y_{\text{CM}} = (y_{\text{target}} + y_{\text{beam}})/2 = y_{\text{beam}}/2$

b) collider (same species and beam momentum): $y_{\text{CM}} = (y_{\text{target}} + y_{\text{beam}})/2 = 0$

c) collider (two different ions species. same B field):

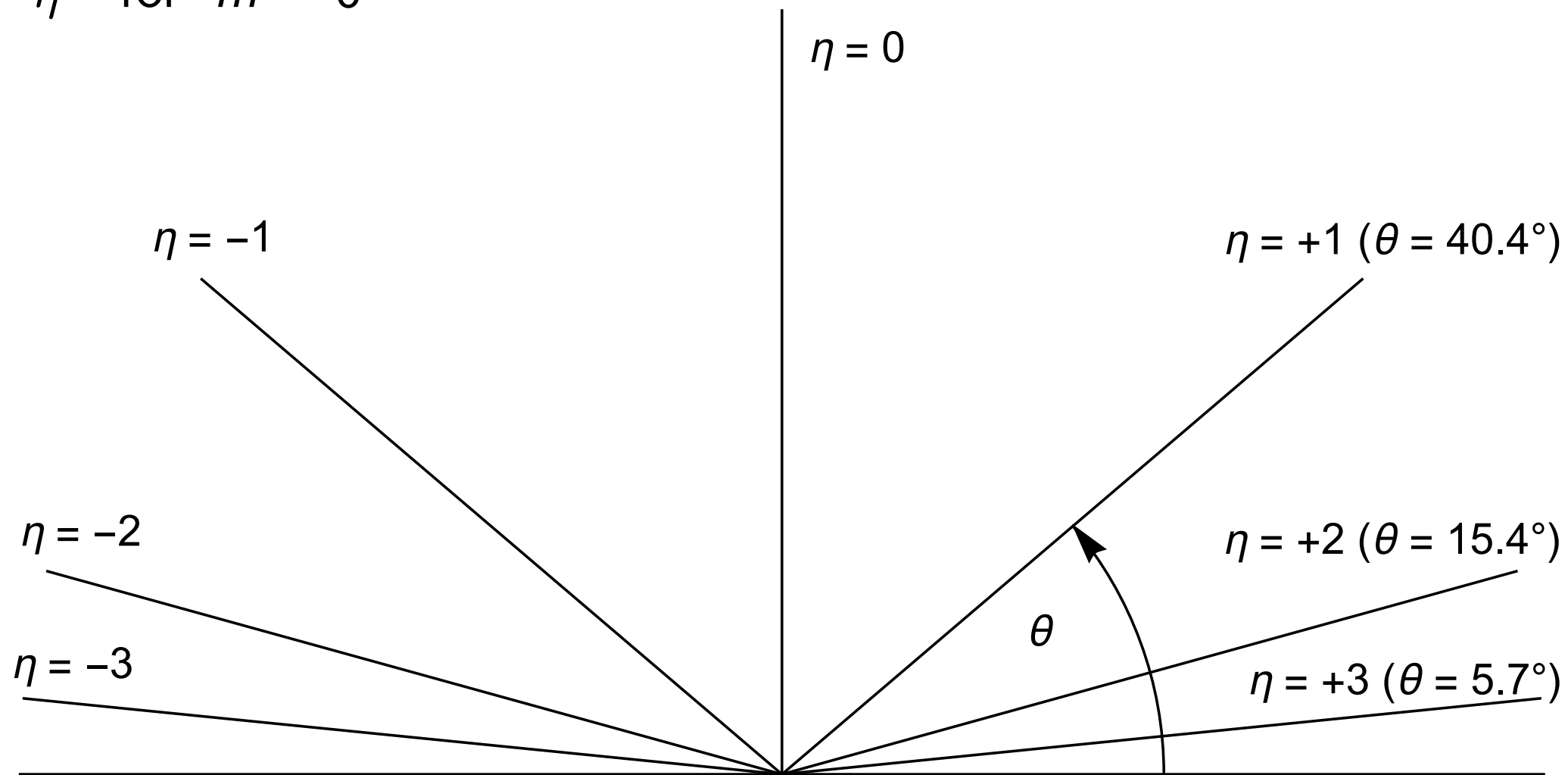
$$y_{\text{cm}} = \frac{1}{2} \ln \frac{Z_1 A_2}{A_1 Z_2} \quad [\text{exercise}] \qquad \text{p-Pb beam at LHC: } y_{\text{CM}} \approx 0.465$$

Pseudorapidity η

$$y = \frac{1}{2} \ln \frac{E + p \cos \vartheta}{E - p \cos \vartheta} \stackrel{p \gg m}{\approx} \frac{1}{2} \ln \frac{1 + \cos \vartheta}{1 - \cos \vartheta} = \frac{1}{2} \ln \frac{2 \cos^2 \frac{\vartheta}{2}}{2 \sin^2 \frac{\vartheta}{2}} = -\ln \left[\tan \frac{\vartheta}{2} \right] =: \eta$$

$\cos(2\alpha) = 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha$

$$y = \eta \quad \text{for } m = 0$$



Analogous to the relations for the rapidity we find:

$$p = p_T \cdot \cosh \eta, \quad p_L = p_T \cdot \sinh \eta$$

Example: Beam Rapidities

$$y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z} = \ln \frac{E + p_z}{\sqrt{E^2 - p_z^2}} = \ln \frac{E + p_z}{m} \approx \ln \frac{2E}{m}$$

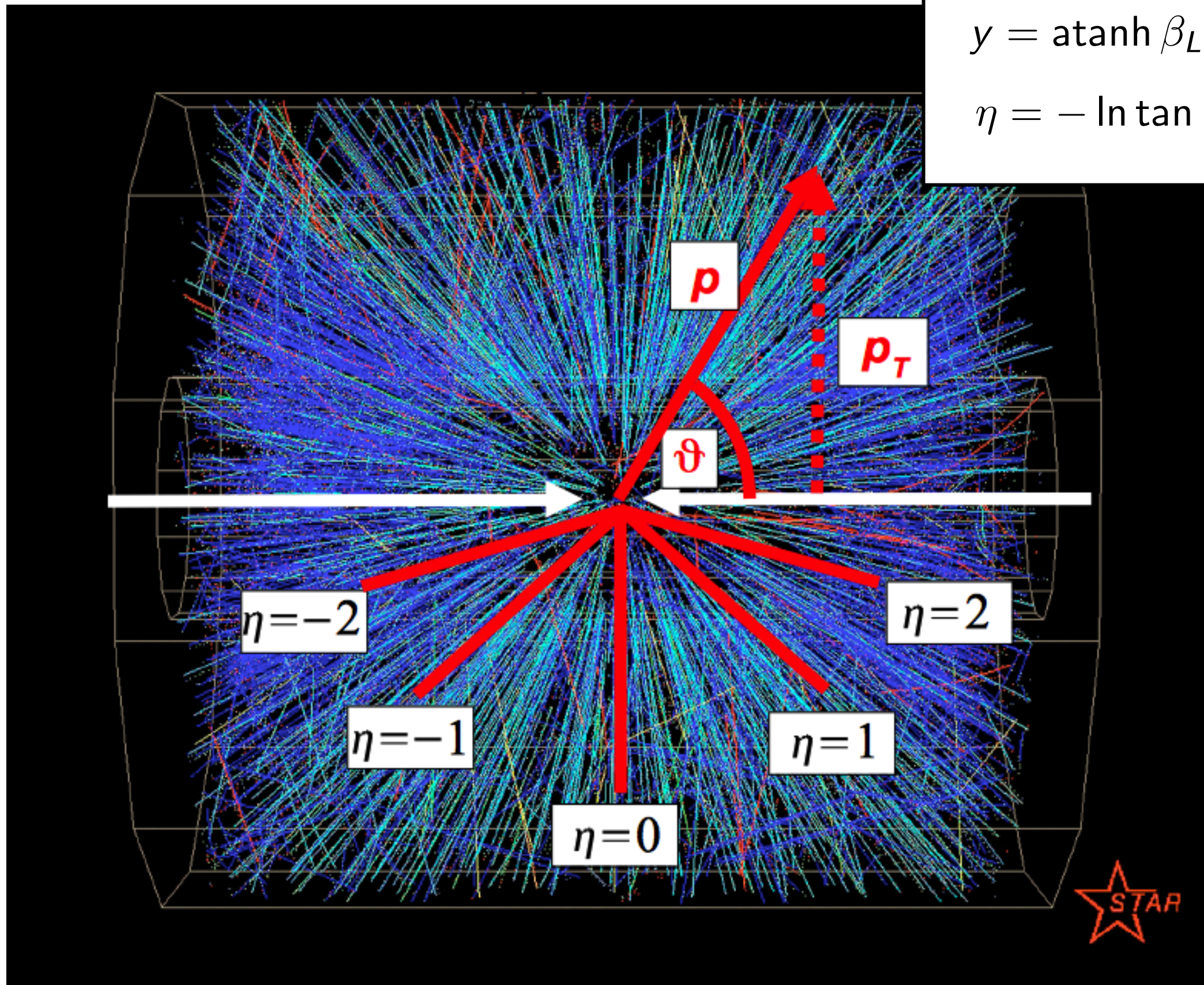
Beam momentum (GeV/c)	Beam rapidity
100	5.36
158	5.81
1380 (= 3500·82/208)	7.99
2760 (= 7000·82/208)	8.86
3500	8.92
6500	9.54
7000	9.61

Brief summary

$$p_T = p \sin \vartheta$$

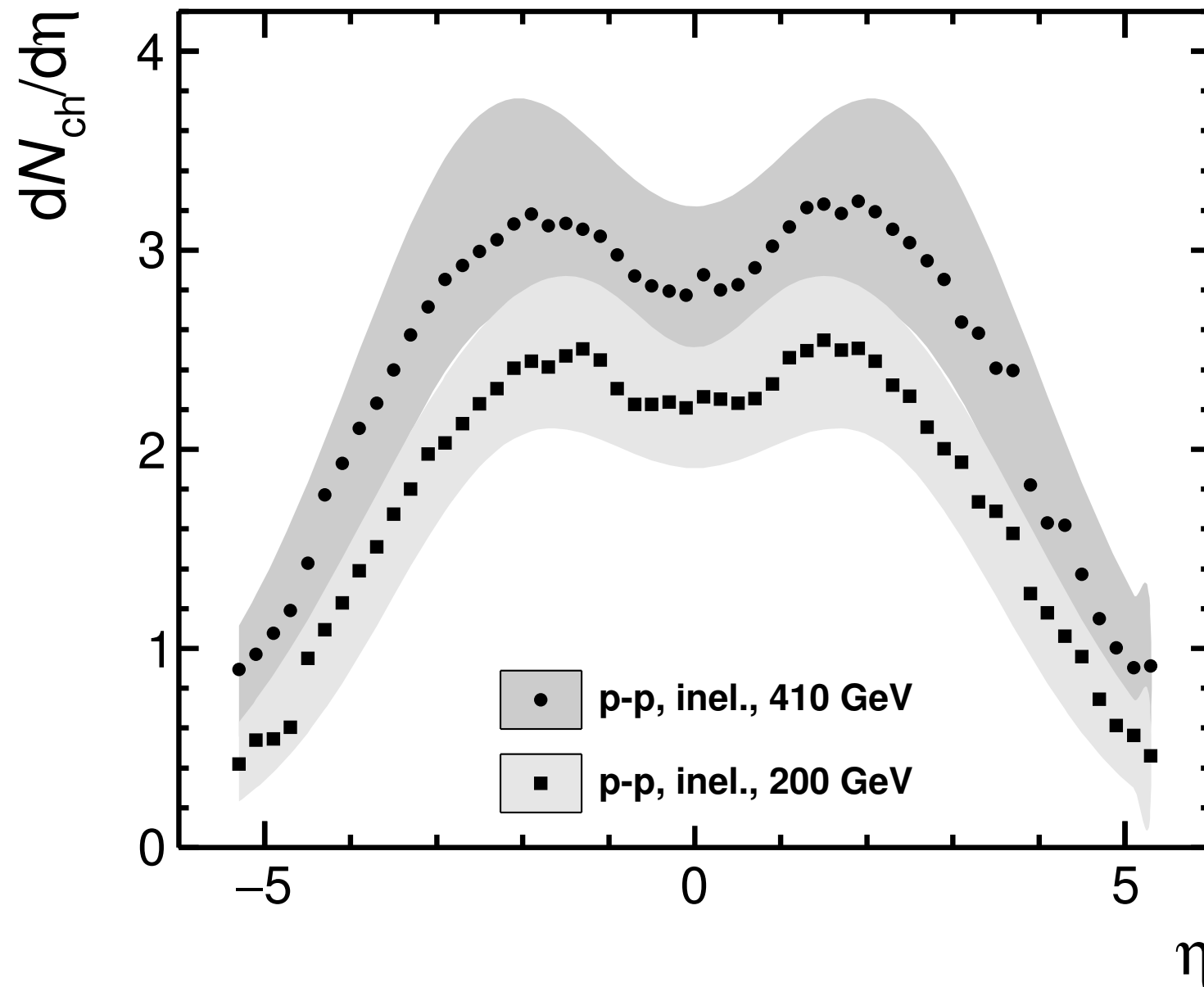
$$y = \operatorname{atanh} \beta_L$$

$$\eta = -\ln \tan \frac{\vartheta}{2}$$



Example of a Pseudorapidity Distribution of Charged Particles

PHOBOS. Phys.Rev. C83 (2011) 024913



Beam rapidity:

$$y_{\text{beam}} = \ln \frac{E + p}{m} = 5.36$$

Average number of charged particles per collision (pp at $\sqrt{s} = 200$ GeV):

$$\langle N_{ch} \rangle = \int \frac{dN_{ch}}{d\eta} d\eta \approx 20$$

Difference between dN/dy and $dN/d\eta$ in the CMS

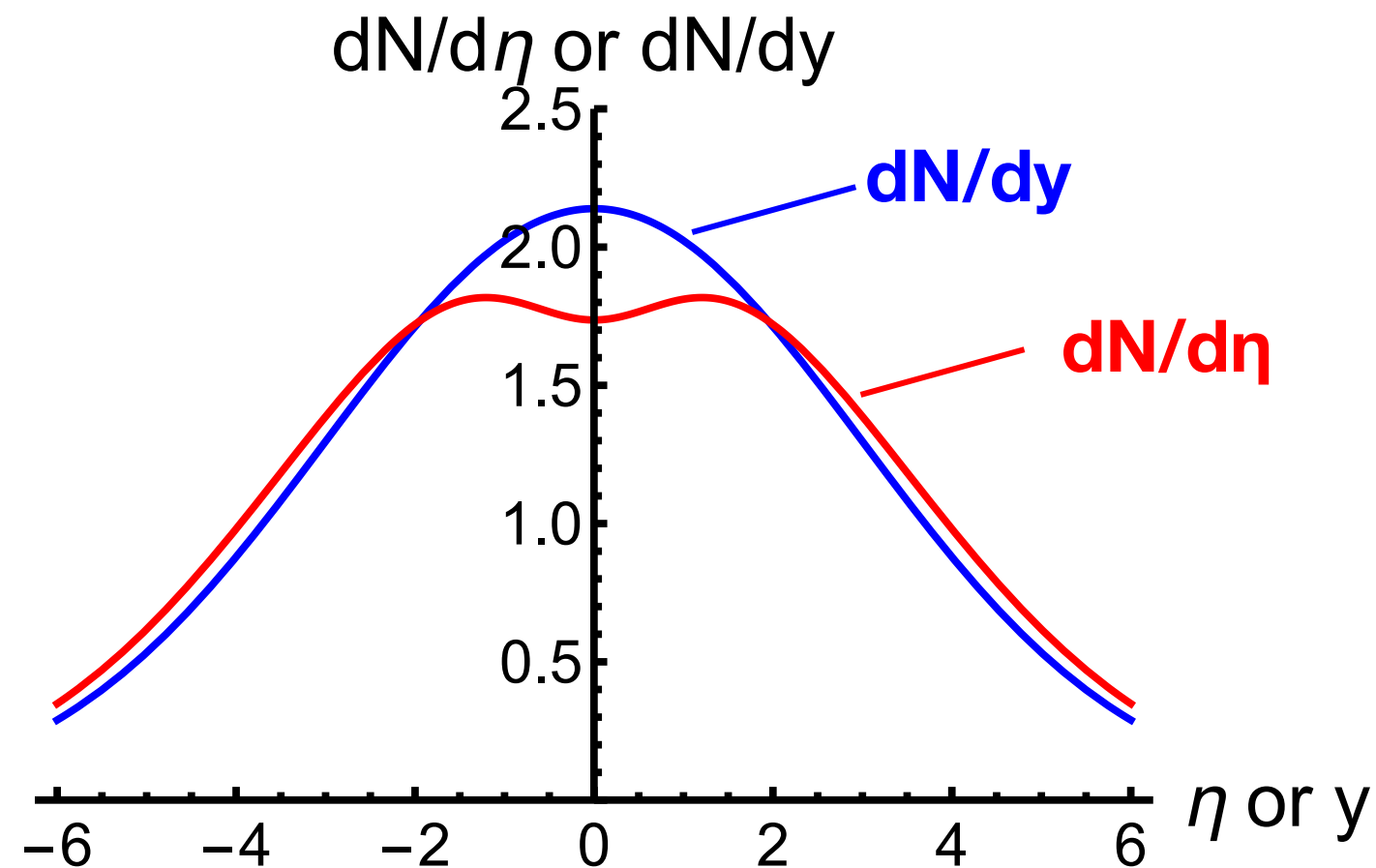
$$\frac{dN}{d\eta} = \sqrt{1 - \frac{m^2}{m_T^2 \cosh^2 y}} \frac{dN}{dy} \quad y(\eta) = \frac{1}{2} \log \left(\frac{\sqrt{p_T^2 \cosh^2 \eta + m^2} + p_T \sinh \eta}{\sqrt{p_T^2 \cosh^2 \eta + m^2} - p_T \sinh \eta} \right)$$

Difference between dN/dy and $dN/d\eta$ in the CMS at $y = 0$:

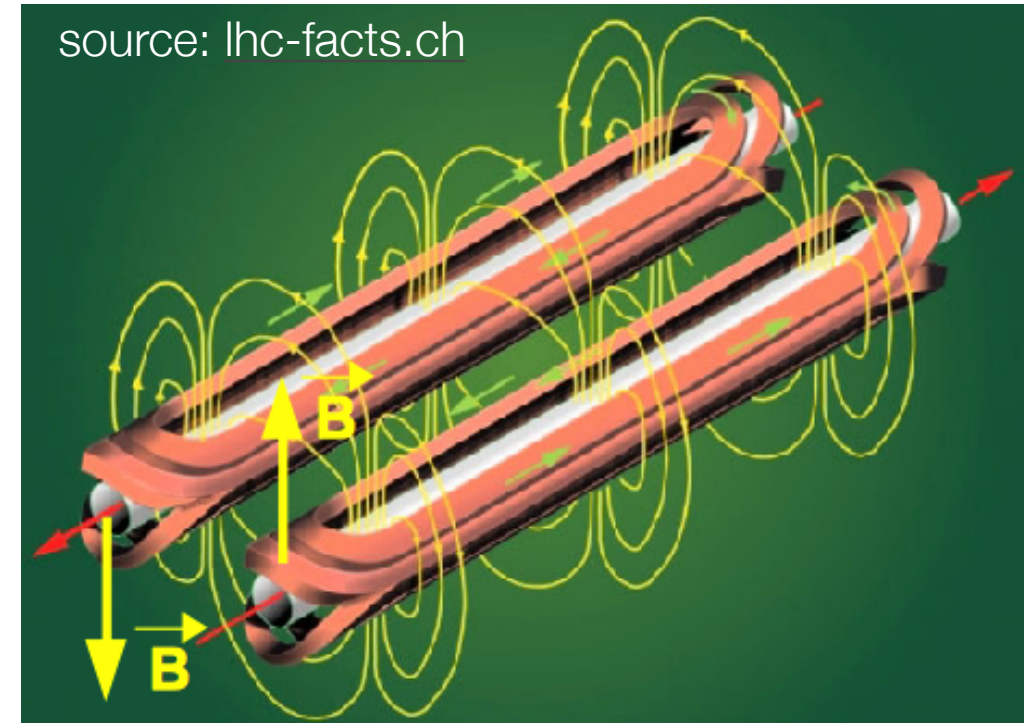
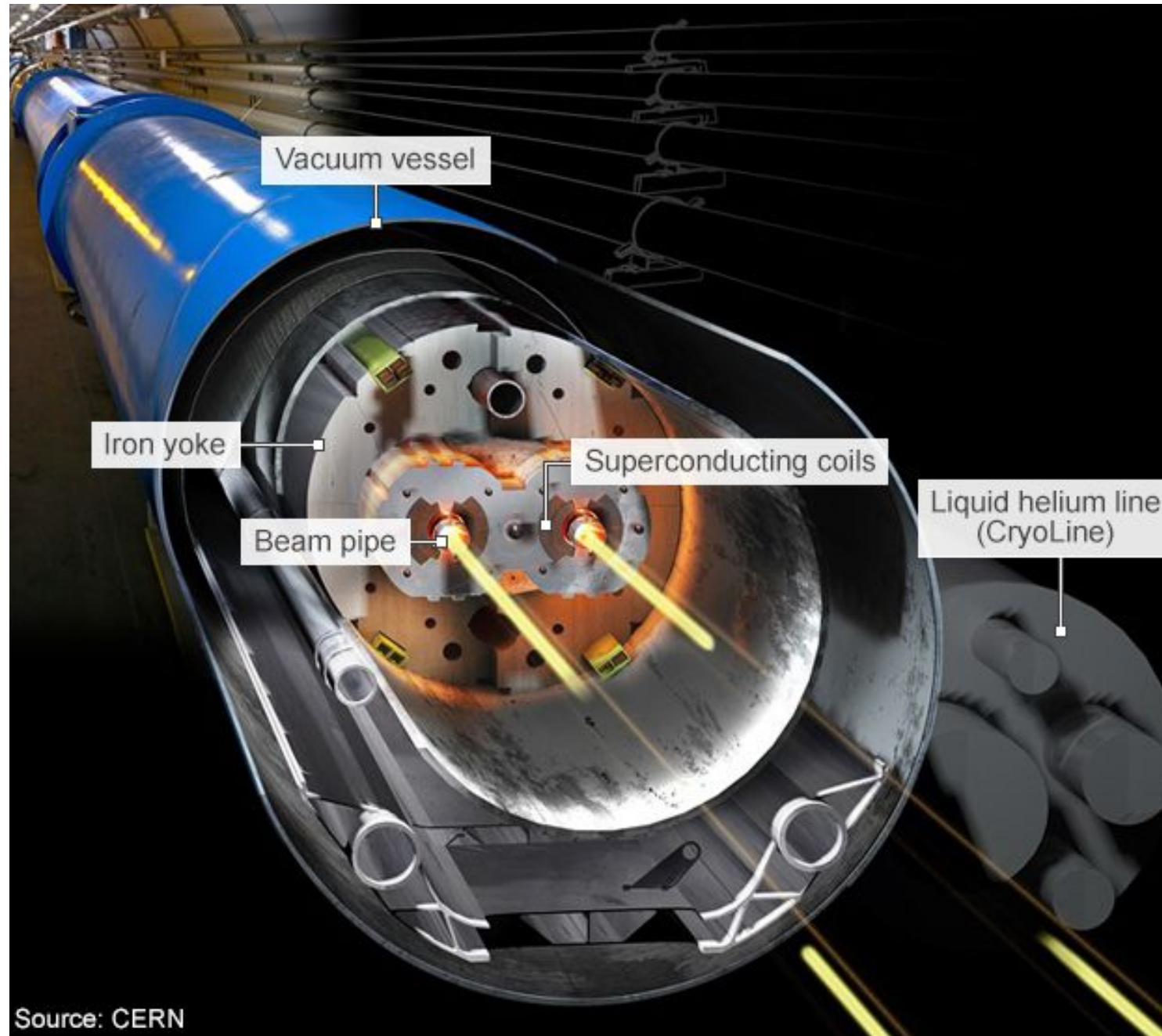
Simple example:
Pions distributed according to

$$\frac{1}{2\pi p_T} \frac{d^2 N}{dp_T dy} = \underbrace{G(y)}_{\text{Gaussian with } \sigma = 3} \cdot \exp(-p_T/0.16)$$

p_T in GeV



LHC dipole



LHC parameters

transverse beam radius: about 20 μm

	pp 2011	Pb-Pb 2011
Beam energy (per nucleon)	3.5 TeV	3.5 TeV·82/208
Particles/bunch	$1.35 \cdot 10^{11}$	$1.2 \cdot 10^8$
#bunches per beam	1380	358
Bunch spacing	50 ns (= 15 m)	200 ns
RMS bunch length	7.6 cm	9.8 cm
peak luminosity	$3.65 \cdot 10^{33} \text{ cm}^{-2} \text{ s}^{-1}$	$0.5 \cdot 10^{27} \text{ cm}^{-2} \text{ s}^{-1}$

Luminosity and cross section

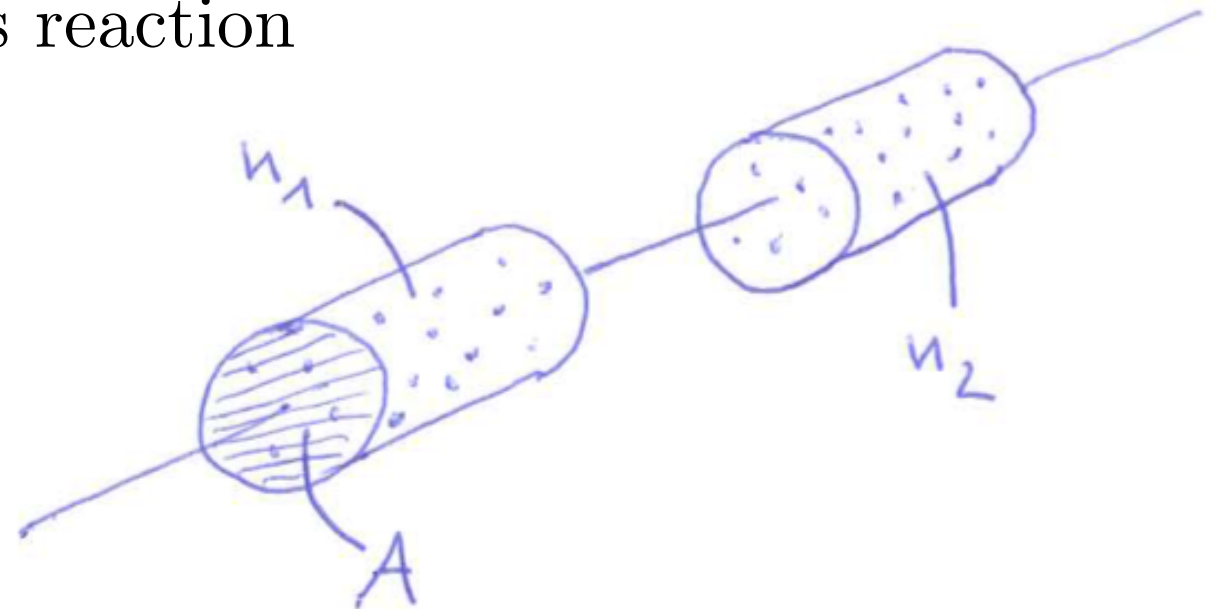
$$\frac{dN_{\text{int}}}{dt} = \sigma \cdot L$$

L = luminosity (in $\text{s}^{-1}\text{cm}^{-2}$)

dN_{int}/dt = Number of interactions of a certain type per second

σ = cross section for this reaction

$$L = \frac{n_1 n_2 f_{\text{coll}}}{A}$$



n_1, n_2 = numbers of particles per bunch in the two beams

f_{coll} = bunch collision frequency at a given crossing point

A = beam crossing area ($A \approx 4\pi\sigma_x\sigma_y$)

Lorentz invariant Phase Space Element

Observable: Average density of produced particles in momentum space

$$\frac{1}{L_{\text{int}}} \frac{d^3 N_A}{d^3 \vec{p}} = \frac{1}{L_{\text{int}}} \frac{d^3 N_A}{dp_x dp_y dp_z}$$

However, the phase space density would then not be Lorentz invariant (see next slides for details):

$$\frac{d^3 N}{dp'_x dp'_y dp'_z} = \frac{\partial(p_x, p_y, p_z)}{\partial(p'_x, p'_y, p'_z)} \cdot \frac{d^3 N}{dp_x dp_y dp_z} = \frac{E}{E'} \cdot \frac{d^3 N}{dp_x dp_y dp_z}$$

Lorentz invariant phase space element: $\frac{d^3 \vec{p}}{E} = \frac{dp_x dp_y dp_z}{E}$

The corresponding observable is called Lorentz invariant cross section:

$$E \frac{d^3 \sigma}{d^3 \vec{p}} = \frac{1}{L_{\text{int}}} E \frac{d^3 N}{d^3 \vec{p}} = \underbrace{\frac{1}{N_{\text{evt,tot}}} E \frac{d^3 N}{d^3 \vec{p}}}_{\text{this is called the invariant yield}} \sigma_{\text{tot}}$$

Lorentz invariant Phase Space Element: Proof of invariance

Lorentz boost along the z axis:

$$\begin{aligned}
 p'_x &= p_x \\
 p'_y &= p_y \\
 p'_z &= \gamma(p_z - \beta E), & p_z &= \gamma(p'_z + \beta E') \\
 E' &= \gamma(E - \beta p_z), & E &= \gamma(E' + \beta p'_z)
 \end{aligned}$$

Jacobian:

$$\frac{\partial(p_x, p_y, p_z)}{\partial(p'_x, p'_y, p'_z)} = \begin{vmatrix} \frac{\partial p_x}{\partial p'_x} & 0 & 0 \\ 0 & \frac{\partial p_y}{\partial p'_y} & 0 \\ 0 & 0 & \frac{\partial p_z}{\partial p'_z} \end{vmatrix}$$

$$\frac{\partial p_x}{\partial p'_x} = 1, \quad \frac{\partial p_y}{\partial p'_y} = 1, \quad \frac{\partial p_z}{\partial p'_z} = \frac{\partial}{\partial p'_z} [\gamma(p'_z + \beta E')] = \gamma \left(1 + \beta \frac{\partial E'}{\partial p'_z} \right)$$

$$\frac{\partial E'}{\partial p'_z} = \frac{\partial}{\partial p'_z} \left[(m^2 + p'^2_x + p'^2_y + p'^2_z)^{1/2} \right] = \frac{p'_z}{E'} \quad \rightsquigarrow \quad \frac{\partial p_z}{\partial p'_z} = \gamma \left(1 + \beta \frac{p'_z}{E'} \right) = \frac{E}{E'}$$

And so we finally obtain:

$$\frac{\partial(p_x, p_y, p_z)}{\partial(p'_x, p'_y, p'_z)} = \frac{E}{E'}$$

Invariant Cross Section

Calculation of the invariant cross section:

$$E \frac{d^3\sigma}{d^3p} = E \frac{1}{p_T} \frac{d^3\sigma}{dp_T dp_z d\varphi}$$

$$\stackrel{dp_z/dy = m_T \cosh y = E}{=} \frac{1}{p_T} \frac{d^3\sigma}{dp_T dy d\varphi}$$

$$\stackrel{\text{symmetry in } \varphi}{=} \frac{1}{2\pi p_T} \frac{d^2\sigma}{dp_T dy}$$

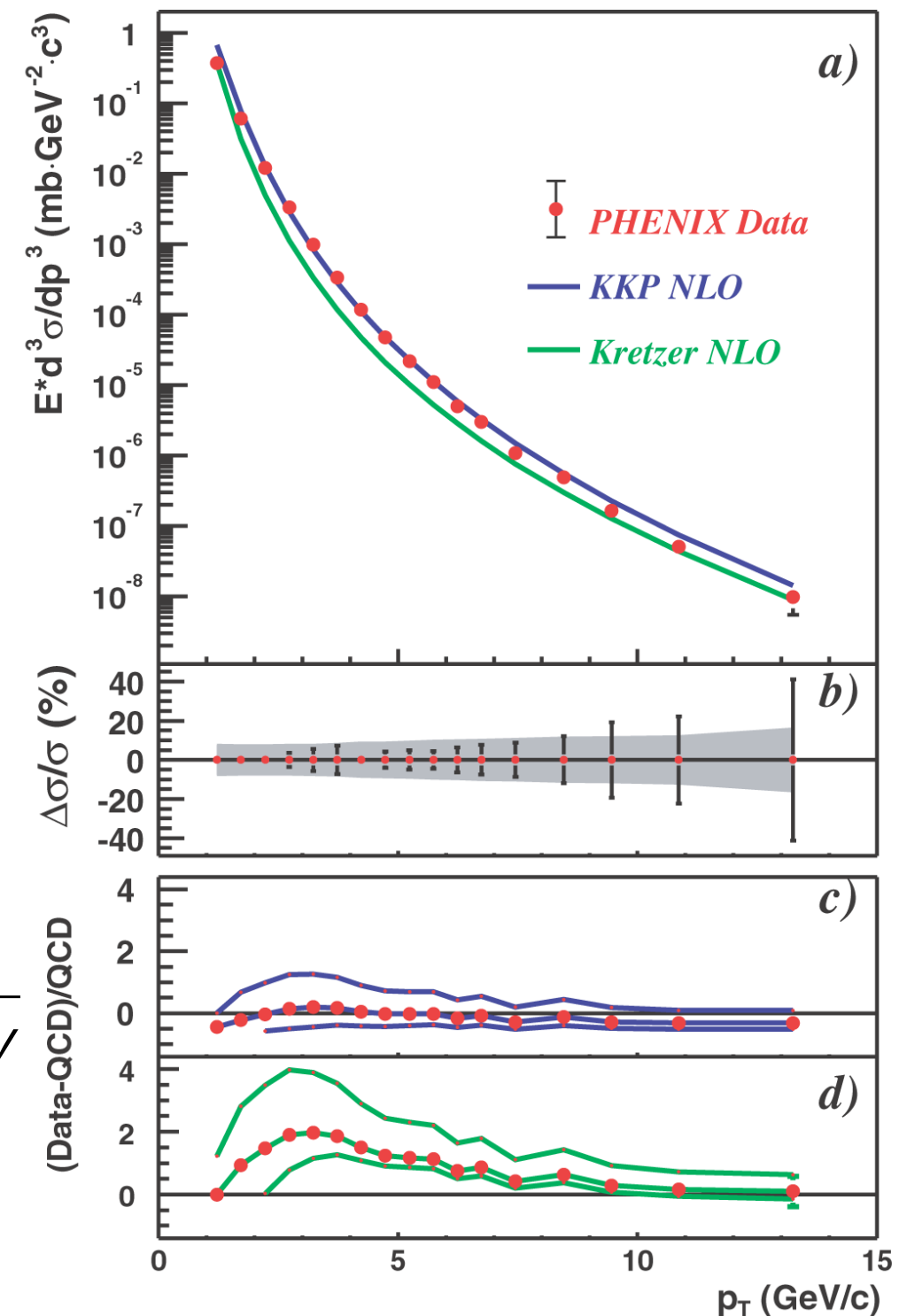
Sometimes also measured as a function of m_T :

$$\frac{1}{2\pi m_T} \frac{d^2\sigma}{dm_T dy} = \frac{1}{2\pi m_T} \frac{d^2\sigma}{dp_T dy} \frac{dp_T}{dm_T} = \frac{1}{2\pi p_T} \frac{d^2\sigma}{dp_T dy}$$

Integral of the inv. cross section

$$\int E \frac{d^3\sigma}{d^3p} d^3p / E = \langle \overset{\text{Average yield of particle X per event}}{N_x} \rangle \cdot \sigma_{\text{tot}}$$

Example: Invariant cross section for neutral pion production in p+p at $\sqrt{s} = 200$ GeV



Average path length of produced particles before decay

$$L_{\text{lab}} = v \cdot \gamma \cdot \tau = \beta \cdot \gamma \cdot \tau \cdot c = \frac{p}{mc} \cdot \tau \cdot c$$

	mass (MeV)	mean life τ	$c \tau$	$L_{\text{lab}} (p = 1 \text{ GeV}/c)$
π^+, π^-	139.6	$2.6 \cdot 10^{-8} \text{ s}$	7.80 m	56 m
π^0	135	$8.4 \cdot 10^{-17} \text{ s}$	25 nm	185 nm
K^+, K^-	494	$1.23 \cdot 10^{-8} \text{ s}$	3.70 m	7.49 m
K_s^0	497	$0.89 \cdot 10^{-10} \text{ s}$	2.67 cm	5.37 cm
K_L^0	497	$5.2 \cdot 10^{-8} \text{ s}$	15.50 m	31.19 m
D^+, D^-	1870	$1.04 \cdot 10^{-12} \text{ s}$	312 μm	167 μm
B^+, B^-	5279	$1.64 \cdot 10^{-12} \text{ s}$	491 μm	93 μm

Reconstruction of unstable particle via the invariant mass calculated from daughter particles

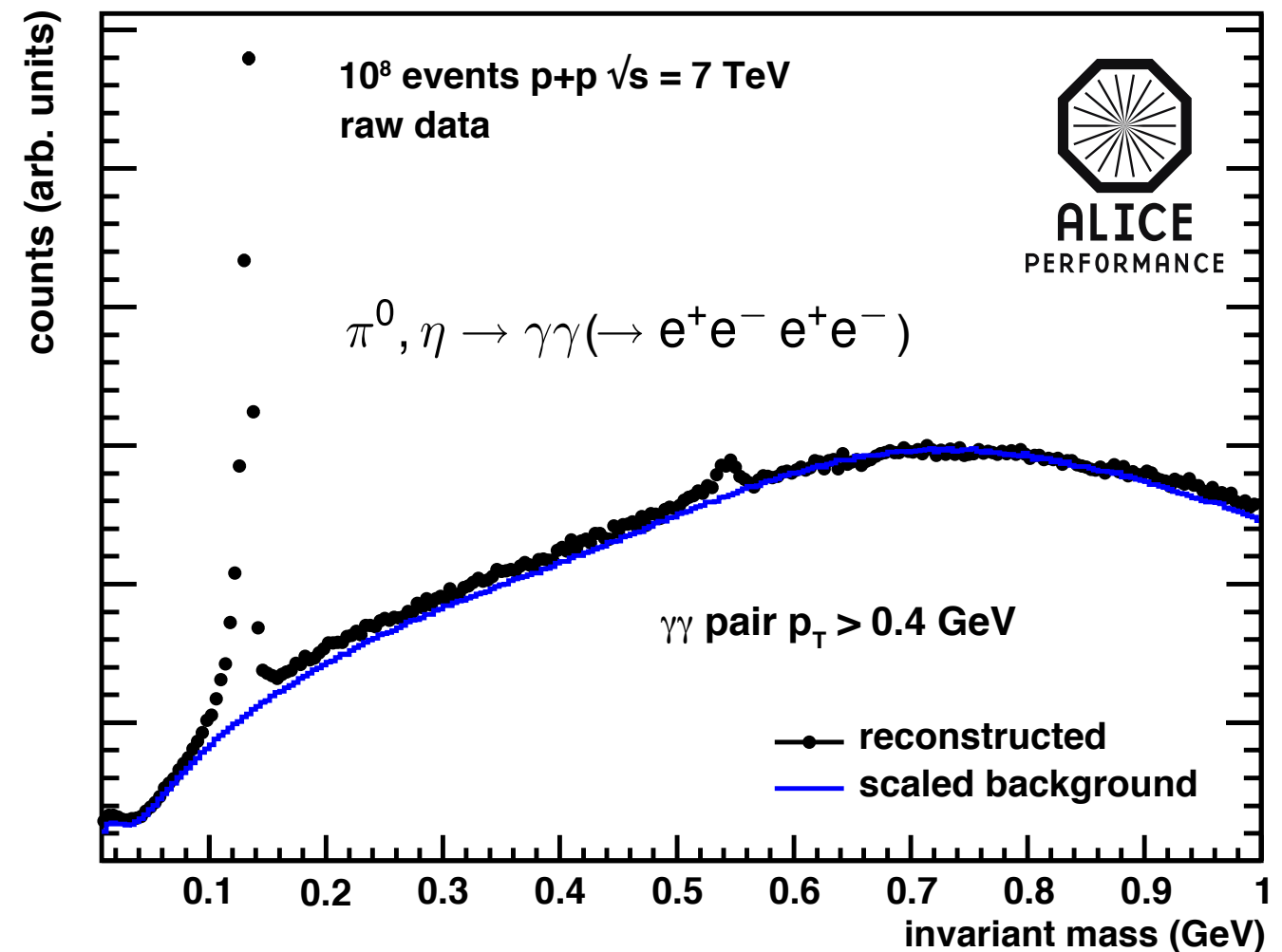
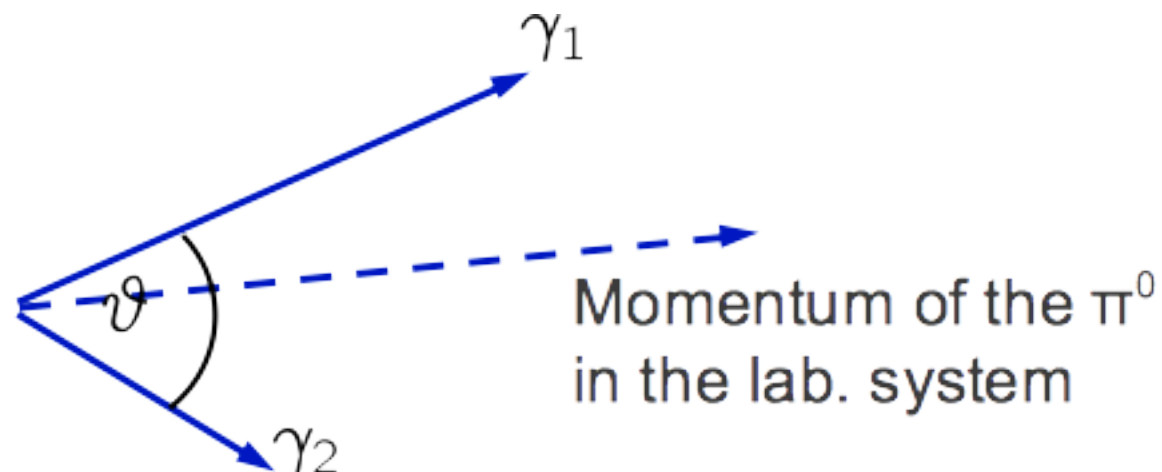
Consider the decay of a particle in two daughter particles. The mass of the mother particle is given by (“invariant mass”):

$$\begin{aligned} M^2 &= \left[\begin{pmatrix} E_1 \\ \vec{p}_1 \end{pmatrix} + \begin{pmatrix} E_2 \\ \vec{p}_2 \end{pmatrix} \right]^2 \\ &= m_1^2 + m_2^2 + 2E_1 E_2 - 2\vec{p}_1 \cdot \vec{p}_2 \\ &= m_1^2 + m_2^2 + 2E_1 E_2 - 2p_1 p_2 \cos \vartheta \end{aligned}$$

Example: π^0 decay:

$$\pi^0 \rightarrow \gamma + \gamma, \quad m_1 = m_2 = 0, \quad E_i = p_i$$

$$\Rightarrow M = \sqrt{2E_1 E_2 (1 - \cos \vartheta)}$$



Summary

- Center-of-mass energy $\sqrt{s}$:
Total energy in the center-of-mass system (rest mass + kinetic energy)
- Observables: Transverse momentum p_T and rapidity y
- Pseudorapidity $\eta \approx y$ for $E \gg m$ ($\eta = y$ for $m = 0$. e.g.. for photons)
- Production rates of particles described by the Lorentz invariant cross section:

$$E \frac{d^3\sigma}{d^3p} = \frac{1}{2\pi p_T} \frac{d^2\sigma}{dp_T dy}$$