Hydrodynamical Model and Shear Viscosity from Black Holes ($\eta/s$ from AdS/CFT)

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Space-time evolution

- **QGP life time**
  
  \[ 10 \text{ fm/c} \approx 3 \cdot 10^{-23} \text{ s} \]

- **thermalization time**
  
  \[ 0.2 \text{ fm/c} \approx 7 \cdot 10^{-25} \text{ s} \]

  → hydrodynamical expansion until freeze-out

  simplest model: only longitudinal expansion, 1d

  → Bjorken model

- **collision time**
  
  \[ \frac{2R}{\gamma} = 0.005 \text{ fm/c} \approx 2 \cdot 10^{-26} \text{ s} \]

Plot: courtesy of R. Stock.
Hydrodynamical model description

Some basic concepts
Relativistic Hydrodynamics (I)

The energy-momentum tensor $T^{\mu \nu}$ is the four-momentum component in the $\mu$ direction per three-dimensional surface area perpendicular to the $\nu$ direction.

\[
\Delta p = (\Delta E, \Delta p_x, \Delta p_y, \Delta p_z)
\]

\[
\Delta x = (\Delta t, \Delta x, \Delta y, \Delta z)
\]

\[
\mu = \nu = 0 : \quad T^{00}_R = \frac{\Delta E}{\Delta x \Delta y \Delta z} = \frac{\Delta E}{\Delta V} = \varepsilon
\]

\[
\mu = \nu = 1 : \quad T^{11}_R = \frac{\Delta p_x}{\Delta t \Delta y \Delta z} \text{ force in } x \text{ direction acting on a surface } \Delta y \Delta z \text{ perpendicular to the force } \rightarrow \text{ pressure}
\]

\[
T^{\mu \nu} = \begin{pmatrix}
\text{energy density} & \text{energy flux density} \\
\text{momentum density} & \text{momentum flux density}
\end{pmatrix}
= \begin{pmatrix}
\varepsilon & \vec{j}_\varepsilon \\
\vec{g} & \Pi
\end{pmatrix}
\]
Relativistic Hydrodynamics (II)

Isotropy in the fluid rest implies that
the energy flux $T^0_j$ and the momentum density $T^j_0$ vanish
and that $\Pi^{ij} = P \delta_{ij}$

$$T^{\mu\nu}_{R} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

Off-diagonal elements $\neq 0$ in case of viscous
hydrodynamics, not considered here
$\rightarrow$ ideal (perfect) fluid.

See also Ollitrault, arXiv:0708.2433.
Relativistic Hydrodynamics (III)

Energy-momentum tensor (in case of local thermalization) after Lorentz transformation to the lab frame:

\[ T^{\mu \nu} = (\varepsilon + P) u^\mu u^\nu - P g^{\mu \nu} \]

Energy density and pressure in the co-moving system:

<table>
<thead>
<tr>
<th>Energy density</th>
<th>4-velocity: ( u^\mu = \frac{dx^\mu}{d\tau} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon )</td>
<td>( \gamma(1, \vec{v}) )</td>
</tr>
</tbody>
</table>

Energy and momentum conservation:

\[ \partial_\mu T^{\mu \nu} = 0, \quad \nu = 0, \ldots, 3 \]

in components:

\[ \begin{cases} 
\frac{\partial}{\partial t} \varepsilon + \vec{\nabla} j_\varepsilon = 0 \quad (\text{energy conservation}) \\
\frac{\partial}{\partial t} g_i + \nabla_j \Pi_{ij} = 0 \quad (\text{momentum conservation}) 
\end{cases} \]

Conserved quantities, e.g., baryon number:

\[ j_B^\mu(x) = n_B(x) u^\mu(x), \quad \partial_\mu j_B^\mu(x) = 0 \iff \frac{\partial}{\partial t} N_B + \vec{\nabla}(N_B \vec{v}) = 0 \]

continuity equation

\[ N_B = \gamma n_B \]
Ingredients of Hydro - models

- Equation of motion and baryon number conservation:
  \( \partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu j^\mu_B(x) = 0 \)

- 5 equations for 6 unknowns:
  \( (u_x, u_y, u_z, \varepsilon, P, n_B) \)

- Equation of state: \( P(\varepsilon, n_B) \)

- (needed to close the system)

- Initial conditions,
  e.g., from Glauber calculation

- Freeze-out condition, fluid \( \rightarrow \) hadrons
  (Cooper-Frye formalism)

**Equation of state (EOS):**
- **EOS I:** ultra-relativistic gas \( P = \varepsilon/3 \)
- **EOS H:** resonance gas, \( P \approx 0.15 \varepsilon \)
- **EOS Q:** phase transition, QGP \( \leftrightarrow \) resonance gas
LHC: Identified particle spectra

Initial conditions fixed by pion abundance

Protons overestimated

Annihilation of protons and anti-protons in the hadron phase?
Elliptic flow in Hydro - models

Elliptic flow is “self-quenching”: The cause of elliptic flow, the initial spacial anisotropy, decreases as the momentum anisotropy increases.
In hydrodynamic models the momentum anisotropy develops in the early (QGP) phase of the collision. Thermalization times of less than 1 fm/c are needed to describe the data.
Cold atomic gases

200,000 Li-6 atoms in an highly anisotropic trap (aspect ratio 29:1)

Very strong interactions between atoms (Feshbach resonance)

Once the atoms are released, the one observed a flow pattern similar to elliptic flow in heavy-ion collisions
Lesson III

- **First results** from ALICE at LHC show large *increase* in energy density *(factor 2-3 compared to RHIC)*

- **longer life-time** of qgp

- **larger collective flow** effects

- **anisotropic flow** comparable to *ultra-low viscosity*

- triangular flow sensitive to initial energy density fluctuations and viscosity/entropy ratio

- Hydrodynamical model provides framework to characterize QGP, i.e. equation of state, viscosity/entropy ratio
Shear Viscosity from Black Holes
($\eta/s$ from Ads/CFT)

What is this all about?
General Considerations

• Strong coupling $\rightarrow$ quantum effects large

• Use AdS/CFT correspondence

• Holographic duality: relate string theory of higher dimension to 4-d gauge theory on the boundary

• Limit of strong coupling: string theory $\rightarrow$ classical gravity (GR)
Parallel Plate Capacitor

• Bulk: 3-d space between plates
• Fluctuations of the field in the bulk induce fluctuations of electric charges on the surface (boundary)
• Correlations of surface charges correlated to bulk field

Source: http://www.britannica.com
AdS/CFT correspondence

- Maldacena conjecture: string theory and conformal QFT mathematically equivalent
- String theory: 10 dimensions
- E.g. Anti-de-Sitter Space (AdS) in 5dim + 5dim background
- Conformal field theory lives on 4dim boundary of 5dim AdS
- String theory becomes classical GR at boundary

J.J. Friess et al., arXiv:0607022 [hep-th] (2006);
Viscosity

- Viscosity is a measure of a fluid resisting to flow
- ‘Fluid with smaller viscosity makes bigger splash’
- Due to friction between neighbouring particle of a fluid moving at different velocity
- Temperature dependent !
- Shear viscosity, bulk viscosity, ...
- Symbol: $\eta$
- Unit: Pa s

Source: wikipedia
## Viscosity: some numbers

<table>
<thead>
<tr>
<th></th>
<th>$T$(K)</th>
<th>$\mu$(Pa s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>291.15</td>
<td>$18.27 \times 10^{-6}$</td>
</tr>
<tr>
<td>Water</td>
<td>293</td>
<td>$1 \times 10^{-3}$</td>
</tr>
<tr>
<td>Honey</td>
<td>293</td>
<td>$2 - 10$</td>
</tr>
<tr>
<td>Peanut butter</td>
<td>293</td>
<td>250</td>
</tr>
<tr>
<td>Pitch</td>
<td>293</td>
<td>$2.3 \times 10^{8}$</td>
</tr>
<tr>
<td>QGP</td>
<td>2 000 000 000 000</td>
<td>gargantuan</td>
</tr>
</tbody>
</table>

Source: wikipedia
Pitch drop experiment
University of Queensland

- Running since 83 years
- After 3 years of consolidation
- 8 drops fell so far
- No-one ever saw a drop falling
- 9th drop is about to fall
Black Hole

- Black hole, mass $M$
- Temp. $T = \frac{\hbar c^3}{8\pi GMk_B}$
- Entropy $S = A/4 \cdot (k_B c^3/G\hbar)$
  A: area of horizon of boundary
- Physics of the interior region projected onto boundary: hologram

Source: http://media.photobucket.com
Holographic Principle

- Conjectured by ‘t Hooft
- Quantum gravity in (d+1) dimensions ⇔
  equivalent theory living on d-dimensional boundary
  ⇒ holographic dual
AdS/CFT Correspondence

- Fields that propagate in the bulk have well defined values at asymptotic infinity (boundary)
- Asymptotic values behave like field and coupling at the boundary
- Anti-de Sitter spacetime: negative curvature
- Holographic duals are sometimes gauge theories
- E.g. AdS\(_5\) ⇔ N=4 Super Yang-Mills
AdS$_5 \times S_5$ Geometry

- AdS$_5$: 5 dimensional Anti-de-Sitter space
- Infinitesimal line element

\[ ds^2 = \frac{r^2}{L^2}(-dt^2 + dx^2) + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5^2 \]

- S$_5$: 5 dimensional sphere, neglect
- $r$: radial coordinate
- $R = \text{const.}$: 3+1 dim. flat Minkowski space
- $R \to \infty$: boundary
- $L$: curvature radius
**AdS$_5 \times S_5$ Geometry, cont’ed**

\[ ds^2 = \frac{r^2}{L^2} (-dt^2 + dx^2) + \frac{L^2}{r^2} dr^2 \]

- Require \( L \gg l_s \), (classical approx.)
- ‘t Hooft coupling: \( \lambda = g_{YM}^2 N_c \)
- \( (L/l_s)^4 = \lambda \)
- Classical approx. works at strong coupling
AdS$_5 \times S_5$ Geometry, cont’ed

- Rewrite for AdS$_5$ black hole

  \[ ds^2 = \frac{(\pi TL)^2}{u} (-(1-u^2)dt^2 + dx^2) + \frac{L^2}{4u^2(1-u^2)} du^2 \]

- \( u = (r_0/r), \ r_0: \) Schwarzschild (horizon) radius

- Horizon at \( u = 1 \)

- Boundary limit: \( u = \varepsilon, \ \text{then} \ \varepsilon \to 0 \)
Ask the AdS/CFT Dictionary...

- $\eta$ from $T_{\mu\nu}$ (Kubo’s formula)
- $T_{\mu\nu}$ corresponds to graviton $h_{\mu\nu}$
- Graviton is disturbance in $g_{\mu\nu}$
- Graviton at boundary propagates in the bulk and is scattered back
- Cross section $\propto$ surface $A$
- Entropy $s \propto$ surface $A$
- $\eta/s$ does not depend on $A$

Source: Physics Today, p29, May 2010
KSS bound on $\eta/s$

$$\frac{\eta}{s} \geq \frac{1}{4\pi} \cdot \frac{\hbar}{k_B} \left\{ 1 + \frac{15\zeta(3)}{\lambda^{3/2}} + \ldots \right\}$$

Conjectured upper bound from string theory

$$\frac{\eta}{s} = \frac{1}{4\pi} \cdot \frac{\hbar}{k_B} \left( 1 - \frac{1}{2N_C} \right)$$

Potentially lower bound from SU(2)
Some remarks

• Relativistic fluid, but bound does not depend on speed of light
• $N=4$ Super Yang-Mills is not QCD
• $N_c = 3$, not large
• No confinement
• Quarks are massless
• However, details might not matter too much, system driven by temperature and degrees of freedom
Non-ideal Hydro-dynamics

- Spectra and flow reproduced by ideal hydrodynamics calcs.
- Shear viscosity to entropy density ratio close to AdS/CFT bound
- Viscosity leads to decrease in $v_2$, ultra-low viscosity sufficient to describe data
- Hydro-limit exceeded at LHC?
\( \eta \) is from experiments

- \( \eta \) huge, but also entropy \( s \) huge
- QGP close to conjectured bound \( \rightarrow \) ‘prefect liquid’
- Cold atom gases \( (T = 10^{-7} \text{ K}) \) have very similar properties, e.g., collective flow, etc.
References

• T. Schaefer and D. Teaney,

• P.K. Kovtun, D.T. Son, and A.O. Starinets,


• C.V. Johnson and P. Steinberg,
Lesson IV

• shear viscosity / entropy density ($\eta/s$) ultralow in QGP

• however, shear viscosity AND entropy density large in QGP, only ratio becomes small!

• close to the conjectured bound from AdS/CFT correspondence

• common features of many-body systems (QGP vs ultra-cold atomic gases) over 20 orders of magnitude in temperature