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Symmetries in Particle Physics

- Emmy Noether Theorem: any symmetry results in a conserved observable

- Symmetry of isospin SU(2):
  
  „u and d quark are not distinguishable in strong IA“
  
  [this is only an approximative symmetry e.g. m(u)~m(d)]
  
  ![](conservation_of_isospin_in_strong_IA)

- Flavour symmetry SU(3):
  
  „u, d and s quark are not distinguishable in strong IA“
  
  [this is an even more approximative symmetry: m(s) ~ 150 MeV, m(u)~ few MeV]
  
  ![](conservation_of_hypercharge_in_strong_IA)

- Colour symmetry SU(3):
  
  „quarks of different colour are not distinguishable among each other“
  
  this is an exact symmetry
  
  ![](conservation_of_colour_in_all_IA)

Experimental evidence for Colour
Suppose physics is invariant under (linear) transformation: \( \psi \rightarrow \psi' = U \psi \)

Ensure normalization:
\[
\int \psi^\dagger \psi \, d^4x = 1 \quad \Rightarrow \quad \int \psi^\dagger U^\dagger U \psi \, d^4x = 1
\]
\[\Rightarrow \quad U^\dagger U = 1 \quad \Rightarrow \quad U \text{ has to be unitary}
\]

To ensure physics is invariant under transformation:
\[
\int \psi^\dagger H \psi \, d^4x = \int \psi'^\dagger H \psi' \, d^4x = \int \psi^\dagger U^\dagger H U \psi \, d^4x
\]
\[\Rightarrow \quad H = U^\dagger H U \quad \Rightarrow \quad UH = HU \quad \Rightarrow \quad [H, U] = 0
\]

Now consider infinitesimal transformation: \( U = 1 + i \varepsilon G \) \( G \): generator of transformation

\[
U^\dagger U = (1 - i \varepsilon G^\dagger)(1 + i \varepsilon G) = 1 + i \varepsilon (G - G^\dagger) + O(\varepsilon^2)
\]
\[
U^\dagger U = 1 \quad \Rightarrow \quad G = G^\dagger
\]
\[\Rightarrow \quad G \text{ is hermitian, thus corresponds to an observable quantity } g
\]
\[
[H,U] = 0 \quad \Rightarrow \quad [H,G] = 0 \quad \Rightarrow \quad g \text{ is a conserved observable quantity!}
\]

For each infinite symmetry transformation, there is an conserved observable quantity!
Reminder: Emmy-Noether Theorem

Example:

infinite 1D spatial translation: $x \rightarrow x + \delta x$

$\Psi(x) \rightarrow \Psi'(x) = \Psi(x + \delta x) = \Psi(x) + \frac{\partial \Psi(x)}{\partial x} \delta x = (1 + \delta x \frac{\partial}{\partial x}) \Psi(x)$

$P_x = -i \frac{\partial}{\partial x}$

$\Psi'(x) = (1 - i \delta x P_x)\Psi(x)$

$x$ component of momentum is conserved!

In general the symmetry operation may depend on more than one parameters:

$U = 1 + i \varepsilon \hat{G}$

in our example: $\vec{r} \rightarrow \vec{r} + \delta \vec{x}$

$U = 1 - i \delta \vec{x} \hat{P}$

A finite transformation ($\bar{\alpha}$) can be expressed as a series of infinitesimal transformations:

$U(\bar{\alpha}) = \lim_{n \rightarrow \infty} \left(1 + i \frac{\bar{\alpha}}{n} \hat{G}\right)^n = e^{i \bar{\alpha} \hat{G}}$

Emmy Noether Theorem: For each symmetry transformation, there is an conserved quantity!
Symmetries in Particle Physics: Isospin

Proposed by Heisenberg (1932):

Proton and neutron are same particles with respect to strong IA!

Indications: \( m(p) \sim m(n) \) and nuclear binding energies (if one ignores Coulomb IA) are similar

Proton and neutron are two states of single particle: the nucleon \( p=\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \; n=\begin{pmatrix} 0 \\ 1 \end{pmatrix}; \)

In analogy to spin, call this isospin. Expect physics to be invariant under rotation in this space

Later extended idea to u, d quark: \( m(u) \sim m(d) \)

\[
\begin{align*}
  u &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad d &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
  \begin{pmatrix} u' \\ d' \end{pmatrix} &= U \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}
\end{align*}
\]

rotation in 2D u-d quark space

Strong IA is invariant to rotation in isospin space,

U: 2x2 unitary matrix
4 complex matrix elements \( \rightarrow \) 8 real parameters
+ 4 unitary conditions \( \rightarrow \) 4 parameters

In group theory these matrices form U(2) group

unitary matrix 2D
Reminder: Definition of a Group

* is operation which connects elements of a group

There is a neutral element e, that for all group elements a: e*a = a = a*e

**Completeness**: a, b element of the group → a*b as well element of the group

For each element a there is an inverse element a⁻¹, so that a⁻¹*a = a⁻¹*a = e

**Example 1**: entire numbers, “+” is group operation, “0” is neutral element

**Example 2**: rational numbers without 0, “*” is group operation, “1” is neutral element

**Example 3**: element a: rotation around 60°, “*” is group operation,
complete list of group elements: \{a, a², a³, a⁴, a⁵, a⁶ = e\}
→ finite group, dimension 6

Subgroup: element a: rotation around 120°
\{a², a⁴, a⁶ = e\} → finite group, dimension 3
Isospin

\[ U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\phi}; \quad U_1^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{-i\phi}; \quad U_1 U_1^\dagger = 1 \]

\( \Psi' = U_1 \Psi \) for all wave functions not distinguishable from original system

\( U_1 \in U(2) \), however \( U_1 \) has no physical relevance: absolute phases cannot be measured

Left with three independent matrices which form SU(2) subgroup of U(2)

All unitary matrices have \( |\det(U)| = 1 \), all elements of SU(2) have \( \det(U) = 1 \).

\[ U = 1 + i\varepsilon G \quad \det(U) = \det\left( \begin{pmatrix} 1 + i\varepsilon G_{11} & i\varepsilon G_{12} \\ i\varepsilon G_{21} & 1 + i\varepsilon G_{22} \end{pmatrix} \right) = 1 + i\varepsilon(G_{11}+G_{22}) + O(\varepsilon^2) \]

\( G_{11}+G_{22} = \text{Trace}(G) = 0 \)

Generators of rotation in isospin space are traceless 2x2 matrices

A set of linear independent traceless 2x2 matrices are e.g. the Pauli matrices

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

Note: Isospin has NOTHING to do with spin – JUST SAME MATH

2D representation of generator of Isospin symmetry group: \( \vec{I} = \vec{G} = \frac{1}{2} \vec{\sigma} \)
Isospin SU(2) – 2D representation

states: $|u> = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|d> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

symmetry transformation: $U = e^{i\vec{\alpha} \cdot \vec{\sigma}}$

generators: $\sigma_1, \sigma_2, \sigma_3$

isospin operators: $\vec{T} = \frac{\vec{\sigma}}{2}$; $T_i = \frac{\sigma_i}{2}$, $i=1,2,3$

3. component: $T_3 |u> = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} |u>$
$T_3 |d> = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2} |d>$

Casimir operator: $\vec{T}^2 |u> = \frac{1}{4} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{4} |u>$
$\vec{T}^2 |d> = \frac{1}{4} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3}{4} |d>$

Commutation properties: $[\vec{T}^2, T_i] = 0$ for $i=1,2,3$  $[T_j, T_i] \neq 0$ for $i \neq j$ only two observables simultaneously measurable

Choose common eigenstages of $\vec{T}^2$ and $T_3$: $|I, I_3>$

$\vec{T}^2 |I, I_3> = I(I + 1) |I, I_3>$
$T_3 |I, I_3> = I_3 |I, I_3>$
Isospin SU(2) – 2D representation

states:

\[ |u\rangle = |1/2, 1/2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |d\rangle = |1/2, -1/2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

ladder operator:

\[ T_+ = T_1 + i T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad T_- = T_1 - i T_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

\[ T_+ |u\rangle = 0 \quad T_+ |d\rangle = |u\rangle \quad T_- |u\rangle = |d\rangle \quad T_- |d\rangle = 0 \]

Isospin is generator of symmetry of strong IA, thus isospin \((I, I_3)\) are conserved in strong IA!
To describe a set of total isospin = I, need (2I+1) dimensional representation of SU(2)

For any representation of SU(2), following relation are valid:

\[ [T_i, T_j] = 2i \varepsilon_{ijk} T_k \]

\[ T_3 |l, l_3> = l_3 |l, l_3> \]

\[ \bar{T}^2 |l, l_3> = l(l + 1) |l, l_3> \]

\[ T_\pm |l, l_3> = \sqrt{(l \mp l_3)(l \pm l_3 + 1)} |l, l_3 \pm 1> \]

\[ [T_3, \bar{T}^2] = 0 \]
Isospin SU(2) – 3D representation

states: \(|1, 1> = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) \quad |1, 0> = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |1, -1> = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{e.g. } \pi^+, \pi^0, \pi^-

generators: \quad J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}

commutator relations: \([J_i, J_j] = 2i\varepsilon_{ijk}J_k\)

isospin operators: \(\vec{T} = \vec{j}; \quad T_3 = J_3\)

3. component: \(T_3|1,1> = |1,1> \quad T_3|1,0> = 0 \quad T_3|1,-1> = -|1,-1>\)

casimir operator: \(\vec{T}^2 = (J_1^2 + J_2^2 + J_3^2) = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\)

ladder operator: \(T_+ = T_1 + iT_2 = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad T_- = T_1 - iT_2 = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\)
Combining (Iso-)spins (goal: get wave function of proton)

general rule (again identically to angular momentum): \( l_3 \) components are added, \( l \geq l_3 \)

\[ |uu> = |1/2,1/2>, |1/2, -1/2> = |1,-1> \]

\[ |dd> = |1/2,-1/2>, |1/2, -1/2> = |1,-1> \]

How does \( |1,0> \) look like?

\[ T_- |1,1> = \sqrt{2} |1,0> \]

\[ T_- |uu> = (T_- |u>) |u> + |u>(T_- |u>) = |d> |u> + |u> |d> \]

\[ |1,0> = \frac{1}{\sqrt{2}} (|du> + |ud>) \]

Instead of studying ladder operators consider symmetry arguments:

\( |1,0> \) is linear combination of \( |1,1> \) and \( |1,-1> \); \( |1,1> \) and \( |1,-1> \) are symmetric \( |1,0> \) must be symmetric

\[ |0,0> \) orthogonal to \( |1,0> \)

\[ |0,0> = \frac{1}{\sqrt{2}} (|du> - |ud>) \]

**triplett symmetrisch** under exchange of quarks

\[ |1,1> = |uu> \]

\[ |1,0> = \frac{1}{\sqrt{2}} (|ud> + |du>) \]

\[ |1,-1> = |dd> \]

**singlett antisymmetrisch** under exchange of quarks

\[ |0,0> = \frac{1}{\sqrt{2}} (|du> - |ud>) \]
Combining (Iso)spins

From four possible combinations of isospin doublets obtain a triplet of isospin 1 states and a singlet isospin 0 state: \( 2 \otimes 2 = 3 \oplus 1 \)

\[
\begin{align*}
\text{dd} & \quad \frac{1}{\sqrt{2}}(ud+du) & \quad \text{uu} \\
\text{-1} & \quad \text{0} & \quad \text{+1} \\
T_\pm & \quad T_\pm & \quad T_\pm
\end{align*}
\]

Can move around within multiplets using ladder operators.

States with \textbf{different total isospin are physically different}: the isospin 1 \textbf{triplet} is symmetric

Under interchange of quarks where as \textbf{singlet} is anti-symmetric.

Now add additional up or down quark

\[
\begin{align*}
\text{ddd} & \quad \frac{1}{\sqrt{2}}(ud+du)d & \quad \text{ddu} & \quad \frac{1}{\sqrt{2}}(ud+du)u & \quad \text{uud} & \quad \frac{1}{\sqrt{2}}(du-du)d & \quad \frac{1}{\sqrt{2}}(du-du)u \\
-3/2 & \quad -1/2 & \quad +1/2 & \quad +3/2 & \quad +1/2 & \quad -1/2
\end{align*}
\]

Use ladder operators and orthogonality to group the 6 states into isospin multiplets.
Combining (Iso)Spins

\[ T_\pm |I, I_3> = \sqrt{(I + I_3)(I \pm I_3 + 1)} |I, I_3 \pm 1> \]

\[ ddd \equiv |\frac{3}{2}, -\frac{3}{2}> \]

\[ T_+ |\frac{3}{2}, -\frac{3}{2}> = T_+ (ddd) = (T_+ d)dd + d(T_+ d)d + dd(T_+ d) \]

\[ \sqrt{3} |\frac{3}{2}, -\frac{1}{2}> = udd + dud + ddu \]

\[ |\frac{3}{2}, -\frac{1}{2}> = \frac{1}{\sqrt{3}} (udd + dud + ddu) \]

\[ T_+ |\frac{3}{2}, -\frac{1}{2}> = \frac{1}{\sqrt{3}} T_+ (udd + dud + ddu) \]

\[ 2 |\frac{3}{2}, +\frac{1}{2}> = \frac{1}{\sqrt{3}} (uud + udu + uud + duu + udu + duu) \]

\[ |\frac{3}{2}, +\frac{1}{2}> = \frac{1}{\sqrt{3}} (uud + udu + duu) \]

\[ T_+ |\frac{3}{2}, +\frac{1}{2}> = \frac{1}{\sqrt{3}} T_+ (uud + udu + duu) \]

\[ \sqrt{3} |\frac{3}{2}, +\frac{3}{2}> = \frac{1}{\sqrt{3}} (uuu + uuu + uuu) \]

\[ |\frac{3}{2}, +\frac{3}{2}> = uuu \]

★ From the 6 states on previous page, use orthogonality to find \[ |\frac{1}{2}, \pm \frac{1}{2}> \] states

★ The 2 states on the previous page give another \[ |\frac{1}{2}, \pm \frac{1}{2}> \] doublet
Baryon Isospin Wave Function

\[ 2 \otimes 2 \otimes 2 = (3 \oplus 1) \otimes 2 = 4 \oplus 2 \oplus 2 \]

Different multiplets have different symmetry properties:

\[
|3/2,3/2> = |uuu> \\
|3/2,1/2> = \frac{1}{\sqrt{3}}(|uud> + |udu> + |duu>) \\
|3/2,-1/2> = \frac{1}{\sqrt{3}}(|udd> + |dud> + |ddu>) \\
|3/2,-3/2> = |ddd> \\
|1/2,1/2> = \frac{1}{\sqrt{6}}(2|uud>-|udu>-|duu>) \\
|1/2,-1/2> = \frac{1}{\sqrt{6}}(-2|ddu>+|udd>+|dud>) \\
|1/2,1/2> = \frac{1}{\sqrt{2}}(|udu>-|duu>) \\
|1/2,-1/2> = \frac{1}{\sqrt{2}}(|udd>-|ddu>)
\]

S: Sym quartett
Symmetric under interchange
Of any quark

\[ M_s: \text{Mixed sym. sym for } 1 \leftrightarrow 2 \]

\[ M_A: \text{Mixed sym. Anti-sym for } 1 \leftrightarrow 2 \]

Mixed symmetriy states have no definite sym. Under interchange of 1↔3
Can apply exact same mathematics to derive possible combinations of wave functions for three Spin $\frac{1}{2}$ particles

\[ |3/2,3/2> = |\uparrow\uparrow\uparrow> \]
\[ |3/2,1/2> = \frac{1}{\sqrt{3}}(|\uparrow\uparrow\downarrow> + |\uparrow\downarrow\uparrow> + |\downarrow\uparrow\uparrow>) \]
\[ |3/2,-1/2> = \frac{1}{\sqrt{3}}(|\uparrow\downarrow\downarrow> + |\downarrow\uparrow\downarrow> + |\downarrow\downarrow\uparrow>) \]
\[ |3/2,-3/2> = |\downarrow\downarrow\downarrow> \]

S: Sym quartett

\[ |1/2,1/2> = \frac{1}{\sqrt{6}}(2|\uparrow\uparrow\downarrow> - |\uparrow\downarrow\uparrow> - |\downarrow\uparrow\uparrow>) \]
\[ |1/2,-1/2> = \frac{1}{\sqrt{6}}(-2|\downarrow\uparrow\downarrow> + |\uparrow\downarrow\uparrow> + |\downarrow\downarrow\uparrow>) \]

M$_S$: Mixed sym. sym for 1$\leftrightarrow$ 2

\[ |1/2,1/2> = \frac{1}{\sqrt{2}}(|\uparrow\downarrow> - |\downarrow\uparrow>) \]
\[ |1/2,-1/2> = \frac{1}{\sqrt{2}}(|\downarrow\uparrow> - |\uparrow\downarrow>) \]

M$_A$: Mixed sym. Anti-sym for 1$\leftrightarrow$ 2

Can now form total wave function of proton ....
Quarks are fermions so require that the total wave-function is **anti-symmetric** under the interchange of any two quarks.

The total wave-function can be expressed in terms of:

\[ \psi = \phi_{\text{flavour}} \chi_{\text{spin}} \xi_{\text{colour}} \eta_{\text{space}} \]

The colour wave-function for all bound qqq states is **anti-symmetric** (see later in lecture).

- Here we will only consider the lowest mass, **ground state**, baryons where there is no internal orbital angular momentum.
- For \( L=0 \) the spatial wave-function is **symmetric** \((-1)^L\).

**Two ways to form a totally symmetric wave-function from spin and isospin states:**

1. combine totally symmetric spin and isospin wave-functions \( \phi(S) \chi(S) \)

\[
\begin{align*}
\Delta^- & : \quad d & \quad -\frac{3}{2} \\
\Delta^0 & : \quad d & \quad -\frac{1}{2} \\
\Delta^+ & : \quad u & \quad 0 \\
\Delta^{++} & : \quad u & \quad \frac{1}{2}
\end{align*}
\]

Spin \( 3/2 \)

Isospin \( 3/2 \)
Total Baryon Wave Function

2. combine mixed symmetry spin and mixed symmetry isospin states
   - Both $\phi(M_S) \chi(M_S)$ and $\phi(M_A) \chi(M_A)$ are sym. under inter-change of quarks $1 \leftrightarrow 2$
   - Not sufficient, these combinations have no definite symmetry under $1 \leftrightarrow 3, \ldots$
   - However, it is not difficult to show that the (normalised) linear combination:
     $$\frac{1}{\sqrt{2}} \phi(M_S) \chi(M_S) + \frac{3}{\sqrt{2}} \phi(M_A) \chi(M_A)$$

is totally symmetric (i.e. symmetric under $1 \leftrightarrow 2; \ 1 \leftrightarrow 3; \ 2 \leftrightarrow 3$)

![Diagram showing spin and isospin values for proton and neutron](attachment:image.png)

• The spin-up proton wave-function is therefore:
  $$|p \uparrow\rangle = \frac{1}{6\sqrt{2}} (2uud - udu - duu)(2 \uparrow\downarrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) + \frac{3}{2\sqrt{2}} (udu - duu)(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

$$|p \uparrow\rangle = \frac{1}{\sqrt{18}} (2u \uparrow u \uparrow d \downarrow - u \uparrow u \downarrow d \uparrow - u \downarrow u \uparrow d \uparrow + 2u \uparrow d \downarrow u \uparrow - u \uparrow d \uparrow u \downarrow - u \downarrow d \uparrow u \uparrow + 2d \downarrow u \uparrow u \uparrow - d \uparrow u \uparrow u \downarrow - d \downarrow u \uparrow u \uparrow)$$

Wave function e.g. helpfull to determine magnetic moment of proton/neutron (see homeworks).
\[ |J, M > = \sum_{m_1, m_2} |J_1, m_1 > |J_2, m_2 > < J_1, m_1, J_2, m_2 | J, M > \]

\[ M = m_1 + m_2 \]

\[ l \in [ |l_1 - l_2 |, l_1 + l_2 ] \]

Glebsch Gordan Coefficients

Note: A square-root sign is to be understood over every coefficient, e.g., for \(-8/15\) read \(-\sqrt{8/15}\).
Examples for Application of Glebsch-Gordan-Coefficients

\[ |J, M \rangle = \sum_{m_1, m_2} |J_1, m_1 \rangle |J_2, m_2 \rangle <J_1, m_1, J_2, m_2 | J, M \rangle \]

\[ M = m_1 + m_2 \]

\[ J \in [ |J_1 - J_2|, J_1 + J_2] \]

How does

\[ |J, M \rangle = |3/2, 1/2\rangle \] composition look like?

Possible contributions:

\[ |3/2,1/2\rangle = C_1 |1,0\rangle |1/2,1/2\rangle + C_2 |1,1\rangle |1/2,-1/2\rangle \]

\[ C_1 : J=3/2, M=1/2, m_1 = 0, m_2 = 1/2 \rightarrow C_1 = \sqrt{2/3} \]

\[ C_2 : J=3/2, M=1/2, m_1 = 1, m_2 = -1/2 \rightarrow C_2 = \sqrt{1/3} \]

Glebsch-Gordan-Coefficients can be derived by applying ladder operators ....
Flavour SU(3)

Ansatz: $m(s) \sim 150$ MeV – small on nucleon scale ($\sim 1$ GeV)

$\leftrightarrow$ $u, d, s$ are approximately degenerated

Physics is invariant under rotation in 3D flavour space:

$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; $s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; $(u', d', s') = U \begin{pmatrix} u \\ d \\ s \end{pmatrix}$;

$U$ is 3x3 unitary matrix $\rightarrow$ 9 complex entries $\rightarrow$ 18 real parameters

9 fixed due to unitarity condition $UU^\dagger=1$

One matrix is (as for 2D case) just multiplication with phase $\rightarrow$ 8 remaining free parameters describe subgroup SU(3) of group of unitarity matrices U(3)

Generators (in 3D representation) of SU(3) are 8 hermitian matrices $\lambda_i$ $i=1,\ldots,8$

(Gell-Mann matrices)

Define $\hat{T} = \hat{\lambda}/2$; $U = e^{i\hat{\alpha}\hat{T}}$ : group of translation in flavour space
Gell-Mann-Matrices $\lambda_i$

Flavour SU(3): Quark states

\[ u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

\[ \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \]

\[ \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix} \]

\[ \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \]

\[ T_\pm = 1/2(\lambda_1 \pm i\lambda_2) \]

\[ T_3 = \lambda_3/2 \]

(Isospin)

\[ V_\pm = 1/2(\lambda_4 \pm i\lambda_5) \]

\[ U_\pm = 1/2(\lambda_6 \pm i\lambda_7) \]

Hypercharge $Y = \lambda_8/3$; $Y = B + S$

baryon number strangeness

Gell-Mann Matrices are generators of SU(3) flavour symmetry group
Hypercharge

Hypercharge $Y = \lambda_8/3$; $Y = B + S$

Baryon number strangeness

Elementary Charge $Q = I_3 + Y/2$

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>S</th>
<th>Y</th>
<th>Q</th>
<th>$I_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td>1/3</td>
<td>0</td>
<td>1/3</td>
<td>2/3</td>
<td>1/2</td>
</tr>
<tr>
<td>$d$</td>
<td>1/3</td>
<td>0</td>
<td>1/3</td>
<td>-1/3</td>
<td>-1/2</td>
</tr>
<tr>
<td>$s$</td>
<td>1/3</td>
<td>-1</td>
<td>-2/3</td>
<td>-1/3</td>
<td>0</td>
</tr>
<tr>
<td>$\bar{u}$</td>
<td>-1/3</td>
<td>0</td>
<td>-1/3</td>
<td>-2/3</td>
<td>-1/2</td>
</tr>
<tr>
<td>$\bar{d}$</td>
<td>-1/3</td>
<td>0</td>
<td>-1/3</td>
<td>+1/3</td>
<td>1/2</td>
</tr>
<tr>
<td>$\bar{s}$</td>
<td>-1/3</td>
<td>+1</td>
<td>2/3</td>
<td>+1/3</td>
<td>0</td>
</tr>
</tbody>
</table>
What is composition of $Y=0$, $I_3=0$ state?

Exploit ladder operators

$$
\begin{align*}
T_+ |\bar{d} \bar{u}\rangle &= |u \bar{u}\rangle - |d \bar{d}\rangle \\
T_- |u d\rangle &= |d \bar{d}\rangle - |u \bar{u}\rangle \\
V_+ |s \bar{u}\rangle &= |u \bar{u}\rangle - |s \bar{s}\rangle \\
V_- |u \bar{s}\rangle &= |s \bar{s}\rangle - |u \bar{u}\rangle \\
U_+ |s \bar{d}\rangle &= |d \bar{d}\rangle - |s \bar{s}\rangle \\
U_- |d \bar{s}\rangle &= |s \bar{s}\rangle - |d \bar{d}\rangle
\end{align*}
$$

2 independent states, third state not part of multiplet
Combining Quark+Antiquark

Charged and neutral pions are member of same isospin doublet, thus should be in same flavour multiplet as well.

\[ \pi^+ = |u\bar{d}> \quad \pi^0 = \frac{1}{\sqrt{2}}|u\bar{u} - d\bar{d}> \quad \pi^- = |u\bar{d}> \]

\[ \Psi_1 = \frac{1}{\sqrt{2}}|u\bar{u} - d\bar{d}> \]

Singulet state must be symmetric in flavour

\[ \Psi_2 = \frac{1}{3}|u\bar{u} + d\bar{d} + s\bar{s}> \]

You can apply any ladder operator on singulet and it yields 0.

Exploiting orthogonality:

\[ \Psi_3 = \frac{1}{\sqrt{6}}|u\bar{u} + d\bar{d} - 2s\bar{s}> \]
Example: Pseudo-Scalars and Vector-Mesons

Pseudoscalar Mesons \((L=0, S=0, J=0, P=−1)\)

- Because SU(3) flavour is only approximate the physical states with \(I_3 = 0, Y = 0\) can be mixtures of the octet and singlet states.

\[
\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \\
\eta \approx \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \\
\eta' \approx \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})
\]

*Empirically find:

Vector Mesons \((L=0, S=1, J=1, P=−1)\)

- For the vector mesons the physical states are found to be approximately "ideally mixed":

\[
\rho^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \\
\omega \approx \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \\
\phi \approx s\bar{s}
\]

Masses

\[
\begin{align*}
\pi^\pm &: 140 \text{ MeV} & \pi^0 &: 135 \text{ MeV} \\
K^\pm &: 494 \text{ MeV} & K^0/\bar{K}^0 &: 498 \text{ MeV} \\
\eta &: 549 \text{ MeV} & \eta' &: 958 \text{ MeV} \\
\rho^\pm &: 770 \text{ MeV} & \rho^0 &: 770 \text{ MeV} \\
K^{*\pm} &: 892 \text{ MeV} & K^{*0}/\bar{K}^{*0} &: 896 \text{ MeV} \\
\omega &: 782 \text{ MeV} & \phi &: 1020 \text{ MeV}
\end{align*}
\]
Historical Relevance

In history the order was different than our approach up to now:

Physicis started with the „particle zoo“ of more several 100 observed „light“ particles and started to sort them by their mass and properties (branching ratios, ...).
(early days only strong and elm IA was observed)

- they then concluded on the existance of partons/quarks
- they made predictions about not yet observed particles in multipletts