8.5.3 Higher order corrections: Anomalous magnetic moment

1. Magnetic moment of the electron

a) Dirac equation with electron coupling to electro-magnetic field:

\[ \partial_{\mu} \rightarrow D_{\mu} = \partial_{\mu} - ieA_{\mu} \quad \Rightarrow \quad (i\gamma^{\mu} D_{\mu} - m)\psi = 0 \]

\[ \vec{p} \rightarrow \vec{\pi} = \vec{p} - e\vec{A} \quad \text{(canonical momentum)} \]

\[ \text{Ansatz for the solution as for free particle:} \]

\[ \psi = \begin{pmatrix} X \\ \Phi \end{pmatrix} = \begin{pmatrix} \chi e^{-ipx} \\ \varphi e^{-ipx} \end{pmatrix} \]

\[ i \frac{\partial}{\partial t} X = \vec{\sigma}\vec{\pi} \Phi + (eA^0 + m)X \]

\[ i \frac{\partial}{\partial t} \Phi = \vec{\sigma}\vec{\pi} X + (eA^0 - m)\Phi = 0 \]

Reminder:

\[ \vec{\gamma} = \gamma^0 \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \]
Non-relativistic limit: \( E \approx m, \quad eA^0 \ll 2m \)

For this limit it makes sense to separate interaction via charge and magnetic moment

\[ i \frac{\partial}{\partial t} \chi = \overrightarrow{\sigma \pi} \varphi + eA^0 \chi \quad (1) \]
\[ i \frac{\partial}{\partial t} \varphi = \overrightarrow{\sigma \pi} \chi + (eA^0 - 2m)\varphi \quad (2) \]

from (2) we have \( \varphi = \frac{\overrightarrow{\sigma \pi}}{2m} \chi \) inserted in (1):

\[ i \frac{\partial}{\partial t} \chi = \left[ \frac{\overrightarrow{\sigma \pi}}{2m} + eA^0 \right] \chi \]

Pauli equation.

Lower spinor component in non-relativistic limit small.
with \[ \hat{\pi} = \pi^2 + \frac{1}{4} \varepsilon_{ijk} \sigma^i \pi^j = \pi^2 + e \vec{\sigma} \vec{B} \]

\[ i \frac{\partial}{\partial t} \chi = \left[ \frac{\hat{\phi} - e \vec{A}^2}{2m} + \frac{e}{2m} \vec{\sigma} \vec{B} + eA^0 \right] \chi \]

\[ = g \frac{e}{2m} \vec{S} \vec{B} \]

\[ \langle \vec{\mu}_e \rangle = -\frac{e}{2m} \cdot g \cdot \frac{1}{2} \cdot \langle \vec{\sigma} \rangle \]
b) Gordon decomposition for electron current:

\[- e \bar{u}_i \gamma^\mu u_i \cdot A_\mu = \frac{-e}{2m} \bar{u}_f (p_f + p_i)^\mu + i \sigma^{\mu\nu} (p_f - p_i)_\nu \vec{y}_i \cdot A_\mu\]

Interaction of "spinless charge"

"Magnetic interaction" via spin → spin-flip

Non-relativistic limit

\[\chi^+ \left( \frac{e}{2m} \vec{\sigma} \cdot \vec{B} \right) \chi \text{ since. } u = \begin{pmatrix} \chi \\ \phi \end{pmatrix}\]
2. Effect of higher order corrections

\[ -\frac{e}{2m} \bar{u}_f (p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)^\nu \bar{u}_i A_\mu \]

\[ \xrightarrow{g = 2} -\frac{e}{2m} \bar{u}_f \left( (p_f + p_i)^\mu + \left(1 + \frac{\alpha}{2\pi}\right) i\sigma^{\mu\nu}(p_f - p_i)^\nu \right) u_i A_\mu \]

\[ g = 2 \quad g = 2 \quad + \quad \frac{\alpha}{\pi} \]

1st order:

\[ \langle \vec{\mu}_e \rangle = -\frac{e}{2m} \left(2 + \frac{\alpha}{\pi}\right) \cdot \frac{1}{2} \cdot \langle \vec{\sigma} \rangle \]

\[ a = \frac{g - 2}{2} = \frac{\alpha}{2\pi} \]
Higher order corrections to $g$-2

Radiative corrections $g$-2 are calculated to the 4-loop level:

<table>
<thead>
<tr>
<th>Feynman Graphs</th>
</tr>
</thead>
</table>
| $O(\alpha)$    | 1  
| $O(\alpha^2)$  | 7  
| $O(\alpha^3)$  | analytically 72  
| $O(\alpha^4)$  | numerically 891  
| til $O(\alpha^4)$ | 971  

Most precise QED prediction.

*T. Kinoshita et al.*
\[ a = \frac{g - 2}{2} \]

\[ a_e = \frac{\alpha}{2\pi} - 0.328 \left( \frac{\alpha}{\pi} \right)^2 + 1.182 \left( \frac{\alpha}{\pi} \right)^3 - 1.9144 \left( \frac{\alpha}{\pi} \right)^4 \]

*Kinoshita 2007*
3. Electron $g$-2 measurement

Experimental method:
Storage of **single** electrons in a Penning trap (electrical quadrupole + axial B field) ⇒ complicated electron movement (cyclotron and magnetron precessions).

Cyclotron frequency
$$\omega_C = \frac{eB}{2mc}$$

Spin precession frequency
$$\omega_S = g\frac{eB}{2mc}$$

**Idea:** bound electron:

$$E(n, m_s) = \frac{g}{2} h\nu_c m_s + \left( n + \frac{1}{2} \right) h\nu_c - \frac{1}{2} h\delta \left( n + \frac{1}{2} + m_s \right)^2.$$ 

Leading relativistic correction

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H. Dehmelt et al., 1987
G. Gabrielse et al., 2006

Energy levels single electron:

- $\nu_z \approx 200$ MHz
- $\nu \approx 134$ kHz
- $\nu_C \approx 149$ GHz
Excitement of axial oscillation:

Cyclotron levels (n) & Spin orientation

Axial oscillation (E-field)
Magnetron levels (from E-field)

Trigger RF induced transitions $\omega_a$ between different $n$ states or spin flips. (change in cyclotron or spin state revealed by axial oscillation -> feedback driven osc.)

$$\omega_a = \omega_s - \omega_c = (g - 2) \mu_B B$$
$$a = \frac{g - 2}{2} = \frac{\omega_s - \omega_c}{\omega_c}$$

⇒ most precise value of $\alpha$:
$$\alpha^{-1}(a_e) = 137.035999710(96)$$

For comparison $\alpha$ from Quanten Hall
$$\alpha^{-1}(qH) = 137.03600300(270)$$

SEO = single electron oscillation

$$a_{e^-} = 0.0011596521884(43)$$
$$a_{e^+} = 0.0011596521879(43)$$

H. Dehmelt et al. 1987

$$a_e = 0.00115965218085(76)$$

G. Gabrielse et al. 2006

$$a_e = \frac{\alpha}{2\pi} - 0.328...\left(\frac{\alpha}{\pi}\right)^2 + 1.182...\left(\frac{\alpha}{\pi}\right)^3$$

Theory

$$-1.505...\left(\frac{\alpha}{\pi}\right)^4$$

$$a_e = 0.001159652133(290)$$

$$a_e = 0.00115965218085(76)$$

*Phys. Rev. Lett. 97, 030801 (2006)*
*Phys. Rev. Lett. 97, 030802 (2006)*
4. Experimental determination of muon $g-2$

**Principle:**

- store polarized muons in a storage ring; revolution with cyclotron frequency $\omega_c$
- measure spin precession around the magnetic dipole field relative to the direction of cyclotron motion

**Precession:**

$$\vec{\omega}_a = -\frac{e}{m_\mu c} \left[ a_\mu \vec{B} - (a_\mu - \frac{1}{\gamma^2 - 1}) \vec{\beta} \times \vec{E} \right]$$

Difference between Lamor and cyclotron frequency

Effect of electrical focussing fields (relativistic effect).

- $= 0$ for $\gamma = 29.3$
- $\leftrightarrow \rho_\mu = 3.094 \text{ GeV/c}$

First measurements:

CERN 70s

- $a_{\mu^+} = 0.001165937(12)$
- $a_{\mu^-} = 0.001165911(11)$
\[(g-2)_{\mu} \text{ Experiment at BNL}\]

E=24GeV
1 \(\mu\) / \(10^9\) protons on target
6x10^{13} protons / 2.5 sec

"V-A" structure of weak decay:
Use high-energy e\(^+\) from muon decay to measure the muon polarization

*Weak charged current couples to LH fermions (RH anti-fermions)*
Measure electron rate:

\[ N(t) = N_0 \ e^{-\lambda t} + A \cos(\omega_a t + \phi) \]

\[ \frac{\omega_a}{2\pi} = 229023.59(16) \text{ Hz} \]

(0.7ppm)

\[ a_\mu = \frac{\omega_a}{m_\mu c \langle B \rangle} \]
From $\omega_a$ to $a_\mu$ - How to measure the B field

$<B>$ is determined by measuring the proton nuclear magnetic resonance (NMR) frequency $\omega_p$ in the magnetic field.

$$a_\mu = \frac{\omega_a}{e \frac{\hbar \tilde{\omega}_p}{m_\mu c \langle B \rangle}} = \frac{\omega_a}{e \frac{2\mu_p}{m_\mu c}} = \frac{\omega_a}{\frac{4\mu_\mu}{\hbar g_\mu} \frac{\tilde{\omega}_p}{2\mu_p}} = \frac{\omega_a}{\frac{\mu_\mu}{\mu_p} \tilde{\omega}_p} (1 + a_\mu)$$

Frequencies can be measured very precisely

$$a_\mu = \frac{\omega_a}{\frac{\mu_\mu}{\mu_p} - \omega_a}$$

$$\mu_\mu/\mu_p = 3.183\ 345\ 39(10)$$

from hyperfine splitting in muonium

\( \tilde{\omega}_p / 2\pi = 61 \, 791 \, 400(11) \, \text{Hz} \, (0.2 \text{ppm}) \)
About 2.6σ deviation:

- Often interpreted as sign of new physics: SUSY
- But careful: “Theory” has uncertainties … … and sometimes even bugs.
- Quantum loop effects (SM or new physics) are $\sim m^2$ and therefore more important for muons than for electrons.
5. Theoretical prediction of $a_\mu$

Beside pure QED corrections there are weak corrections ($W, Z$) exchange and „hadronic corrections“

$$a_\mu = a_\mu^{QED} + a_\mu^{Had} + a_\mu^{EW}$$

(For the electron with much lower mass the hadronic and weak corrections are suppressed ($\sim m^2$), and can be neglected.)

→ Determination of hadronic corrections is difficult and is in addition based on data: hot discussion amongst theoreticians how to correctly use the data.
Hadronic vacuum polarization:

Hadronic corrections related to virtual intermediate hadronic states \((\pi\pi, \rho, \phi)\) – cannot be calculated.

Use the “optical theorem” to relate the loop corrections to observable cross sections / branching ratios:

\[
\text{Im}[\begin{array}{c}
\text{hadrons}
\end{array}] \propto |\begin{array}{c}
\text{hadrons}
\end{array}|^2
\]

\[
a_\mu (\text{had}; 1) = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{4m_\pi^2}^\infty \frac{ds}{s^2} K(s) R(s)
\]
In 2001 Kinoshita et al. found a sign mistake in their calculation of the light-by-light scattering amplitude:

December 2001
KEK-TH-793
hep-ph/0112102

Comment on the sign
of the pseudoscalar pole contribution
to the muon $g - 2$

Masashi Hayakawa * and Toichiro Kinoshita †

Abstract

We correct the error in the sign of the pseudoscalar pole contribution to the muon $g - 2$, which dominates the $\mathcal{O}(\alpha^3)$ hadronic light-by-light scattering effect. The error originates from our oversight of a feature of the algebraic manipulation program FORM which defines the $\epsilon$-tensor in such a way that it satisfies the relation $\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon_{\nu_1 \nu_2 \nu_3 \nu_4} \eta^{\mu_1 \nu_1} \eta^{\mu_2 \nu_2} \eta^{\mu_3 \nu_3} \eta^{\mu_4 \nu_4} = 24$, irrespective of space-time metric. To circumvent this problem, we replaced the product $\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon_{\nu_1 \nu_2 \nu_3 \nu_4}$ by $-\eta_{\mu_1 \nu_1} \eta_{\mu_2 \nu_2} \eta_{\mu_3 \nu_3} \eta_{\mu_4 \nu_4} \pm \cdots$ in the FORM-formatted program, and obtained a positive value for the pseudoscalar pole contribution, in agreement with the recent result obtained by Knecht et al.
Potential SUSY contribution to muon \((g-2)\)

Potential SUSY contributions:

For muon \(~40000\) times larger than in case of electrons.

\[
a^\text{SUSY}_\mu \simeq \pm 130 \times 10^{-11} \cdot \left(\frac{100 \text{ GeV}}{m_{\text{SUSY}}}\right)^2 \tan \beta,
\]

\[
a_\mu = a_\mu^{\text{QED}} + a_\mu^{\text{Had}} + a_\mu^{\text{EW}} + a_\mu^{\text{SUSY}}
\]

First sign of New Physics ??