Seminarvortrag EMMI 7.4.08

Thermodynamics of ideal quantum gases

- We consider the th. dyn. of ideal non-relativistic Bose & Fermi gases.
- Choose grandcanonical formulation:

given: particles with mass m
dispersion $\epsilon(p) = p^2/2m$
in volume

assume: mean particle number $\langle \hat{N} \rangle = \bar{N}$
mean energy $\langle \hat{H} \rangle = \bar{E}$

$\Rightarrow$ many body state $\hat{\rho} \Leftrightarrow$ distributions of part. #, energy etc.

0. Fock space formulation

(uniform V, $L = \sqrt[3]{V}$, periodic boundary cond.'s
$k_{\nu_j} = 2\pi \nu_j / L$; $j = 1, 2, 3$)

- states $|\{n_i; \vec{k}_i\}\rangle$

$$= \prod_i (n_i!)^{-\frac{1}{2}} (\hat{a}_i^+)^{n_i} |0\rangle \qquad (1)$$

- creation $\hat{a}_i^+$ & annihilation $\hat{a}_j$ operators

$$\hat{a}_j^+ \,|n_j, \vec{k}_j\rangle = \sqrt{\pm n_j + 1}\; |n_j + 1; \vec{k}_j\rangle \qquad (F: n_j \leq 1)$$
$$B: + \quad F: -$$
$$\hat{a}_j \,|n_j, \vec{k}_j\rangle = \sqrt{n_j}\; |n_j - 1; \vec{k}_j\rangle \tag{2}$$
$$\hat{a}_j \,|0\rangle = 0$$

$\boxed{\text{BOSONS}}$	$\boxed{\text{FERMIONS}}$	
$[\hat{a}_i, \hat{a}_j^+] = \delta_{ij}$	$[\hat{a}_i, \hat{a}_j^+]_+ = \delta_{ij}$	(3)

includes spin

(all other (anti) commutators vanish

- particle # op.: $\hat{N}_j = \hat{a}_j^+ \hat{a}_j$ mode j (4)

Hamiltonian : $\hat{H} = \sum_j \epsilon(j)\, \hat{N}_j$ (5)

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1. Partition sum, thermostatic quantities

We introduce the

- partition sum $Z_{gc}(\beta, \mu) = \mathrm{Tr}\left[e^{-\beta(\hat{H} - \mu\hat{N})}\right]$ (6)
- gc density op. $\hat{\rho}_{gc}(\beta, \mu) = Z_{gc}^{-1}\, e^{-\beta(\hat{H} - \mu\hat{N})}$ (7)

→ $\bar{E}$ and $\bar{N}$ as derivatives

$\beta = (k_B T)^{-1}$; μ = chemical potential (8)

We use the Fock basis to determine $Z_{gc}(\beta,\mu)$:

$\boxed{B}$

$$Z_{gc} = \prod_{k_i} \left(\sum_{n_i=0}^{\infty} \left[e^{-\beta(\epsilon_i-\mu)} \right]^{n_i} \right)$$

$$= \prod_{k_i} \frac{1}{1-e^{-\beta(\epsilon_i-\mu)}}$$

if all $\epsilon_i > \mu \Leftrightarrow$ if $\epsilon_0 = 0 > \mu$

$\boxed{F}$

$$Z_{gc} = \prod_{k_i, s} \left(1 + e^{-\beta(\epsilon_i-\mu)} \right) \quad (9)$$

Now we introduce the gc. potential through $Z_{gc} = e^{-\beta\Omega_{gc}}$

$$\Omega_{gc} = \beta^{-1} \sum_i \ln\left(1-e^{-\beta(\epsilon_i-\mu)}\right) \quad \Bigg| \quad \Omega_{gc} = -\beta^{-1} \sum_{\vec{k}_i, s} \ln\left\{1+e^{-\beta(\epsilon_i-\mu)}\right\} \quad (10)$$

Use it as generating function for $\bar{N} = -\partial\Omega_{gc}/\partial\mu$ etc.: (11)

$$\bar{N} = \sum_i \bar{n}_i, \quad \bar{E} = \sum_i \epsilon_i \bar{n}_i \quad (12)$$

$$\bar{n}_i = \frac{1}{e^{\beta(\epsilon_i-\mu)}-1} \quad \Bigg| \quad \bar{n}_i = \frac{1}{e^{\beta(\epsilon_i-\mu)}+1} \quad (13)$$

We introduce the <u>fugacity</u> $z = \exp(\beta\mu)$ such that

$$\bar{n}_i = \frac{1}{z^{-1} e^{\beta\epsilon_i} - 1} \qquad \Big| \qquad \bar{n}_i = \frac{1}{z^{-1} e^{\beta\epsilon_i} + 1} \tag{14}$$

as well as the Bose and Fermi functions

$$g_p(z) = \frac{1}{\Gamma(p)} \int_0^\infty \frac{dx\, x^{p-1}}{z^{-1} e^{x} - 1} = \sum_{\nu=1}^\infty \frac{z^\nu}{\nu^p}$$

$$f_p(z) = \frac{1}{\Gamma(p)} \int_0^\infty \frac{dx\, x^{p-1}}{z^{-1} e^{x} + 1} = \sum_{\nu=1}^\infty (-1)^{\nu-1} \frac{z^\nu}{\nu^p} = -g_p(-z) \tag{15}$$

with

$$g_p(z) \begin{cases} \cong z & z \ll 1 \\ = \zeta(p) & z = 1 \end{cases}$$

$$g_p(z) = \frac{d\, g_{p+1}(z)}{d \ln z}$$

$$f_p(z) \begin{cases} \cong z & |z| \ll 1 \quad (16) \\ = (1 - 2^{1-p})\zeta(p), & z = 1 \quad (17) \\ = \dfrac{(\ln z)^p}{\Gamma(p+1)} \Big\{ 1 + \sum_{l=1}^\infty (\ln z)^{-l} \times \dfrac{\Gamma(p+1)}{\Gamma(p-2l+1)} 2(1-2^{1-2l})\zeta(2l) + \mathcal{O}(1/z) \Big\}; & z \to \infty \quad (18) \end{cases}$$

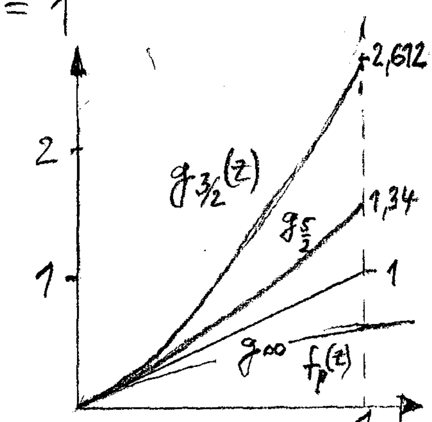


Fig. 1

Using these results we find ?

$$\Omega_{gc}(V,T,\mu) = -k_BT\,\frac{V}{\lambda_T^3}\,g_{5/2}\left(e^{\frac{\mu}{k_BT}}\right) \qquad\Bigg|\qquad \Omega_{gc}(V,T,\mu) = -2k_BT\,\frac{V}{\lambda_T^3}\,f_{5/2}\left(e^{\frac{\mu}{k_BT}}\right) \tag{19}$$

where the <u>thermal deBroglie wave length</u> is $\lambda_T := \sqrt{\frac{2\pi\hbar^2}{m k_BT}}$. (20)

From this let us, for the first, derive the population of the excited modes of the system (excluding the zero-mode for bosons)

$$\bar N_{ex}(V,T,\mu) = \sum_{\vec k \neq 0} \bar n_{\vec k} = \frac{V}{\lambda_T^3}\,g_{\frac{3}{2}}(z) \qquad\Bigg|\qquad \bar N(V,T,\mu) = -\frac{\partial \Omega_{gc}}{\partial\mu} = 2\,\frac{V}{\lambda_T^3}\,f_{\frac{3}{2}}(z) \tag{21}$$

With (16) it follows that for $z \to 0$

$$\frac{\bar N_{ex}}{V}\lambda_T^3 \equiv \bar\rho_{ex}\lambda_T^3 \cong z \qquad\Bigg|\qquad \bar\rho\,\lambda_T^3 \cong 2z \tag{22}$$

Hence, the limit $z \to 0$ corresponds to a small density $\bar\rho$ and/or a large temperature, i.e. $\lambda_T \to 0$. This gives nothing but the <u>classical limit</u>, see (14), the <u>Maxwell-Boltzmann</u> distr.:

$$\bar n_i \to z\,e^{-\beta\epsilon_i} \qquad\Bigg|\qquad \bar n_i \to z\,e^{-\beta\epsilon_i} \tag{23}$$

In the same way we derive the mean pressure $\bar{p} = -\partial\Omega_{gc}/\partial V$:

$$\bar{p} = \frac{k_B T}{\lambda_T^3}\, g_{5/2}(z) \qquad\Big|\qquad \bar{p}_1 = 2\,\frac{k_B T}{\lambda_T^3}\, f_{5/2}(z) \tag{24}$$

and entropy $\bar{S} = -\partial\Omega_{gc}/\partial T$:

$$\bar{S} = k_B \frac{V}{\lambda_T^3}\Big[\frac{5}{2} g_{5/2}(z) - g_{3/2}(z)\ln z\Big] \qquad\Big|\qquad \bar{S} = 2 k_B \frac{V}{\lambda_T^3}\Big[\frac{5}{2} f_{5/2}(z) - f_{3/2}(z)\ln z\Big] \tag{25}$$

From (16), (21), (24) we obtain the equation of state in the class. limit:

$$\bar{p} V \cong \bar{N} k_B T \tag{26}$$

We can also recover the exact relations for $\bar{E} = \Omega_{gc} + TS + \mu\bar{N}$:

$$\bar{E} = \frac{3}{2} k_B T \frac{V}{\lambda_T^3}\, g_{5/2}(z) \qquad\Big|\qquad \bar{E} = 3 k_B T \frac{V}{\lambda_T^3}\, f_{5/2}(z) \tag{27}$$

and thus for both: $\bar{E} = \frac{3}{2}\bar{p}V \longrightarrow \frac{3}{2}\bar{N}k_B T$ in the class. limit.

Similarly one recovers the class. Entropy and chemical pot.:

$$S \xrightarrow{c.l.} k_B \bar{N}\Big\{\frac{3}{2}\ln\frac{m\bar{E}}{3\bar{N}\pi\hbar^2} - \ln\bar{\rho} + \frac{5}{2} + \ln 2\Big\} \quad \text{only fermions (spin)} \tag{28}$$

$$\mu = \beta^{-1}\ln z \longrightarrow k_B T\Big\{\ln\bar{\rho} + 3\ln\lambda_T - \ln 2\Big\} \tag{29}$$

2. Degenerate quantum gas

We now consider the nonclassical regimes of the Bose & Fermi gases.

(17) and FIG. 1 show that there is a maximum phase space density $\tilde{\omega} = \bar{\rho}\,\lambda_T^3 \leq \zeta(3/2) = 2{,}612\ldots$

In the $\bar{\rho}$ - T - plane:

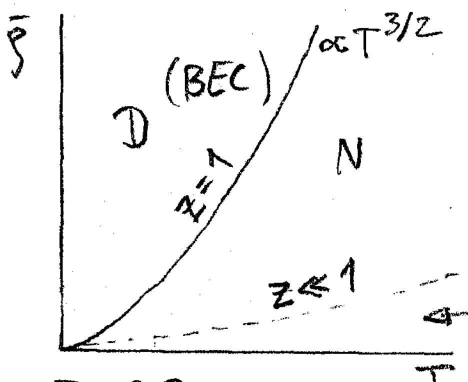


FIG. 2B

For fixed T: max. density $\bar{\rho}_c = \frac{\zeta(3/2)}{\lambda_T^3}$

—"— $\bar{\rho}$: crit. temperature

$$T_c = \frac{2\pi\hbar^2}{m k_B}\left(\frac{\bar{\rho}}{\zeta(3/2)}\right)^{\frac{2}{3}} \qquad (30)$$

below which one has BEC (Bose - Einstein condensation)

For Fermions there is no such restriction:

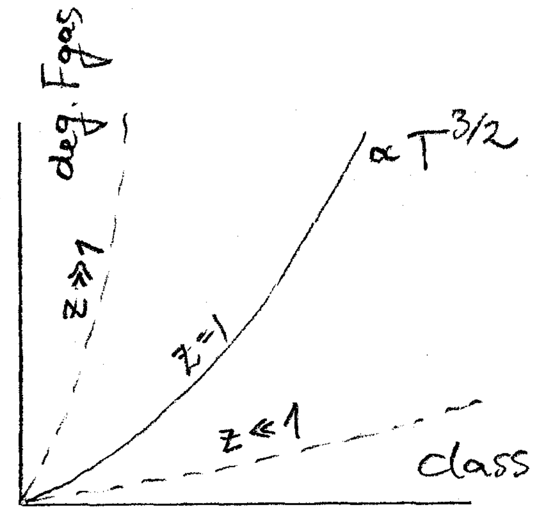
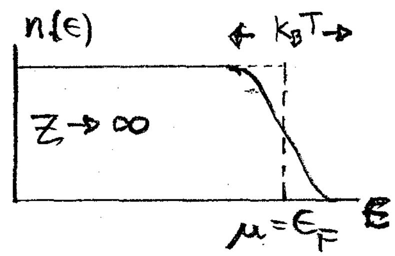
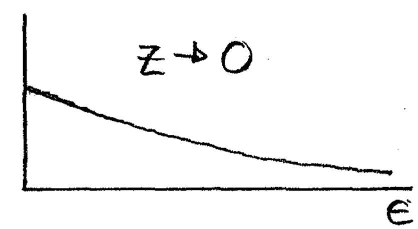


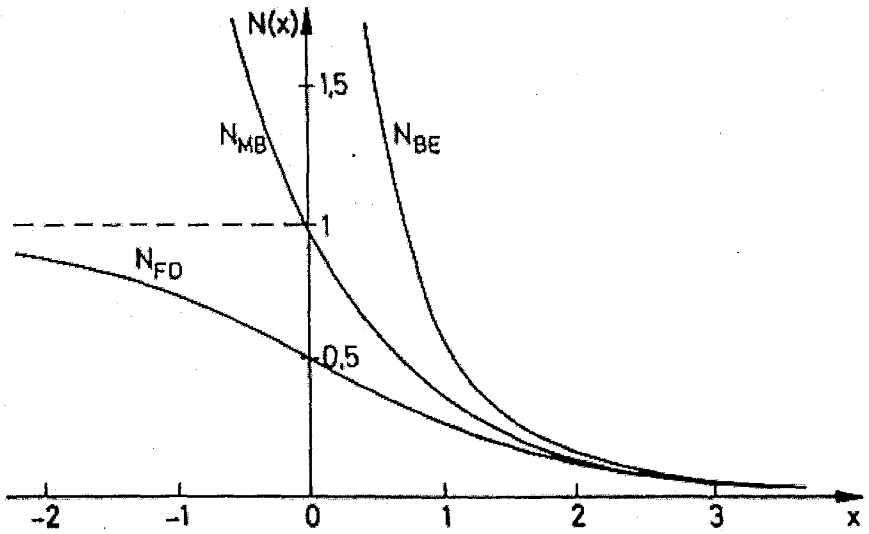
FIG. 2F

We first have a look at the FD distribution (14 F)





The Bose-Einstein, Fermi-Dirac, and Maxwell-Boltzmann-distributions



$$\sigma = e^{\frac{\mu}{kT}}, \quad x := \frac{\epsilon - \mu}{kT},$$

$$N_{iBE} = \frac{1}{\sigma^{-1} e^{\epsilon_i/kT} - 1} \longrightarrow N_{BE}(x) = \frac{1}{e^x - 1}$$

$$N_{iMB} = \frac{1}{\sigma^{-1} e^{\epsilon_i/kT}} \longrightarrow N_{MB}(x) = \frac{1}{e^x}$$

$$N_{iFD} = \frac{1}{\sigma^{-1} e^{\epsilon_i/kT} + 1} \longrightarrow N_{FD}(x) = \frac{1}{e^x + 1}$$

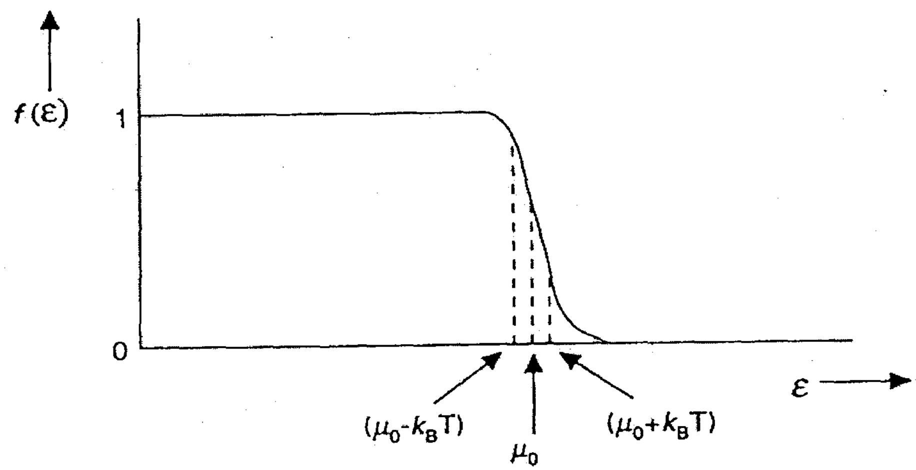


Abb. 7.2: Fermi-Funktion für eine Temperatur T mit $T_F \gg T > 0$.

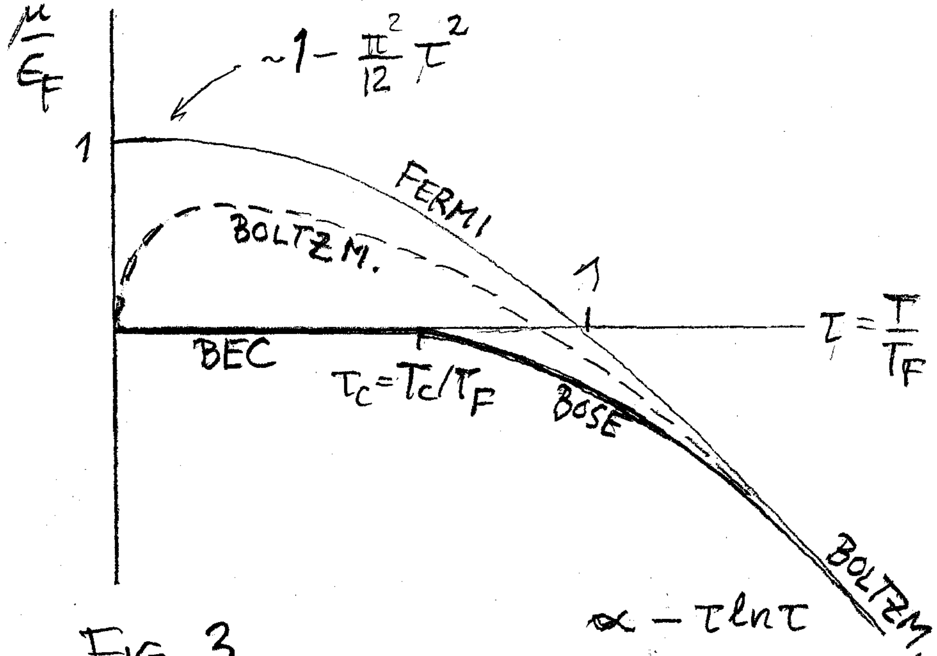


FIG. 3

$$\frac{T_F}{T_c} = \left(\frac{3\pi^2\bar\rho}{8\pi^{3/2}\bar\rho}\,\zeta\!\left(\tfrac{3}{2}\right)\right)^{\frac{2}{3}} = \frac{\pi^{\frac{1}{3}}}{4}\left[3\,\zeta\!\left(\tfrac{3}{2}\right)\right]^{\frac{2}{3}} \cong 1.4 \qquad (36)$$

Also

The Fermi energy (edge) is defined as $\epsilon_F = \mu(z \to \infty)$.

With (18,19) one obtains for fixed $\bar\rho$

$$\mu = \epsilon_F \left\{ 1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2 + O(T^4) \right\} \qquad (31)$$

$$\epsilon_F = \mu(T=0) = \frac{\hbar^2}{2m}\left(3\pi^2\bar\rho\right)^{\frac{2}{3}} \qquad (32)$$

We define a Fermi temperature through $k_B T_F = \epsilon_F$ (33)

One obtains, for $\bar\rho$ = const. the T-dependence of μ as shown in FIG. 3

One finds $\mu = 0$ if

$$\bar\rho\,\lambda_T^3 = 2\left(1 - \frac{1}{\sqrt{2}}\right)\zeta\!\left(\tfrac{3}{2}\right) \qquad (34)$$

$$\Rightarrow T = \frac{\hbar^2}{2mk_B}\,4\pi\left(\frac{\bar\rho}{2-\sqrt{2}}\cdot\frac{1}{\zeta(3/2)}\right)^{\frac{2}{3}}$$

$$= 4\pi\left(3\pi^2[2-\sqrt{2}]\,\zeta(3/2)\right)^{-\frac{2}{3}} T_F \cong 0.989\, T_F$$

$$\simeq 1.43\, T_c \qquad (35)$$

Let's now turn to BEC:

Remember: $\bar{n}_i = (z^{-1}e^{\beta\epsilon_i}-1)^{-1} \Rightarrow \bar{n}_0 = \frac{z}{1-z}$ (37)

$\Rightarrow z = \frac{\bar{n}_0}{1+\bar{n}_0}$ (38)

Consider $\lim_{\text{thdyn}}$: $V \to \infty$ and $\bar{\varrho}_0 = \bar{n}_0/V < \infty$

$\Rightarrow \boxed{z \to 1}$

($\bar{\varrho}_0 \to 0$: $z<1$; $\bar{\varrho}_0 \to \infty$: unphysical)

$\Rightarrow \bar{n}_i = \left(e^{\beta\epsilon_i} - 1 + \cancel{\frac{1}{\bar{\varrho}_0 V} e^{\beta\epsilon_i}}\right)^{-1}$

$\Rightarrow \bar{\varrho}_i \to 0$ for $i \neq 0$, $V \to \infty$, $\bar{\varrho}_0 < \infty$

We need to treat the 0-mode separately:

$$\Omega_{gc} = -k_BT\left[\ln(1-z) - \frac{V}{\lambda_T^3} g_{5/2}(z)\right]$$

$$\xrightarrow{V\to\infty} -k_BT\left[\ln \bar{\varrho}_0 V + \frac{V}{\lambda_T^3}\zeta\left(\tfrac{5}{2}\right)\right] \quad (39)$$

$$\bar{\varrho} = \bar{\varrho}_0 + \frac{1}{\lambda_T^3} g_{3/2}(z) \xrightarrow{V\to\infty} \bar{\varrho}_0 + \bar{\varrho}_c \quad (40)$$

$$\bar{p} = -\partial_V\left[\beta^{-1}\ln(1-z)\right]\Big|_{z,\beta=\text{const.}} + \bar{p}[i\neq 0]$$

$= \bar{p}[i \neq 0]$, see (34B) (41)

$$S = -k_B\left[\ln(1-z) - \frac{z}{1-z}\ln z\right] + S[i\neq 0]$$

$$\xrightarrow{V\to\infty} k_B \ln(\bar{\varrho}_0 V) + \frac{5}{2}k_B\frac{V}{\lambda_T^3}\zeta(5/2) \quad (42)$$

$s = S/V \to \frac{5}{2}k_B\frac{\zeta(5/2)}{\lambda_T^3} \to 0$ $\;T \to 0$ (3. HS) (43)

We consider the V-p-diagram: (44)

(40) $\Rightarrow V = \bar{N}/[\bar{\varrho}_0 + g_{3/2}(z)/\lambda_T^3]$

(24): $\bar{p} = k_BT\, g_{5/2}(z)/\lambda_T^3$

phase boundary: $\bar{p}\,V^{5/3} = \frac{2\pi\hbar^2}{m}\frac{\zeta(5/2)}{[\zeta(3/2)]^{5/3}}\bar{N}^{5/3}$; $z=1$ (45)

$\bar{p}$ = const. in BEC phase

$V \to 0$ for $\bar{\varrho}_0 \to \bar{\varrho}$

$\Rightarrow$ 1st-order phase transition (unphysical! ($V=\infty$, ideal gas)

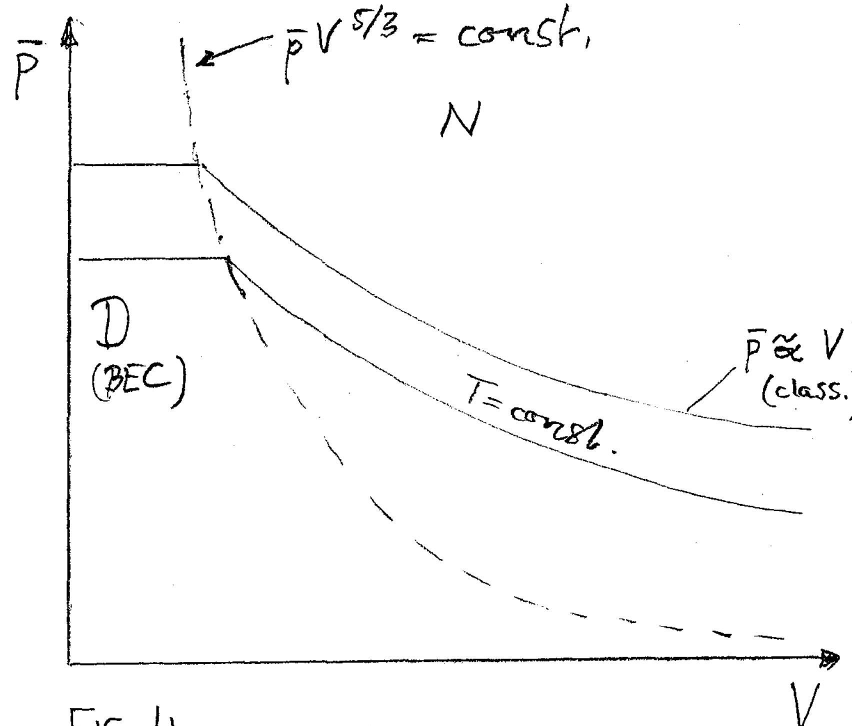


FIG. 4

Isothermic curves below and above the critical line for Bose-Einstein condensation

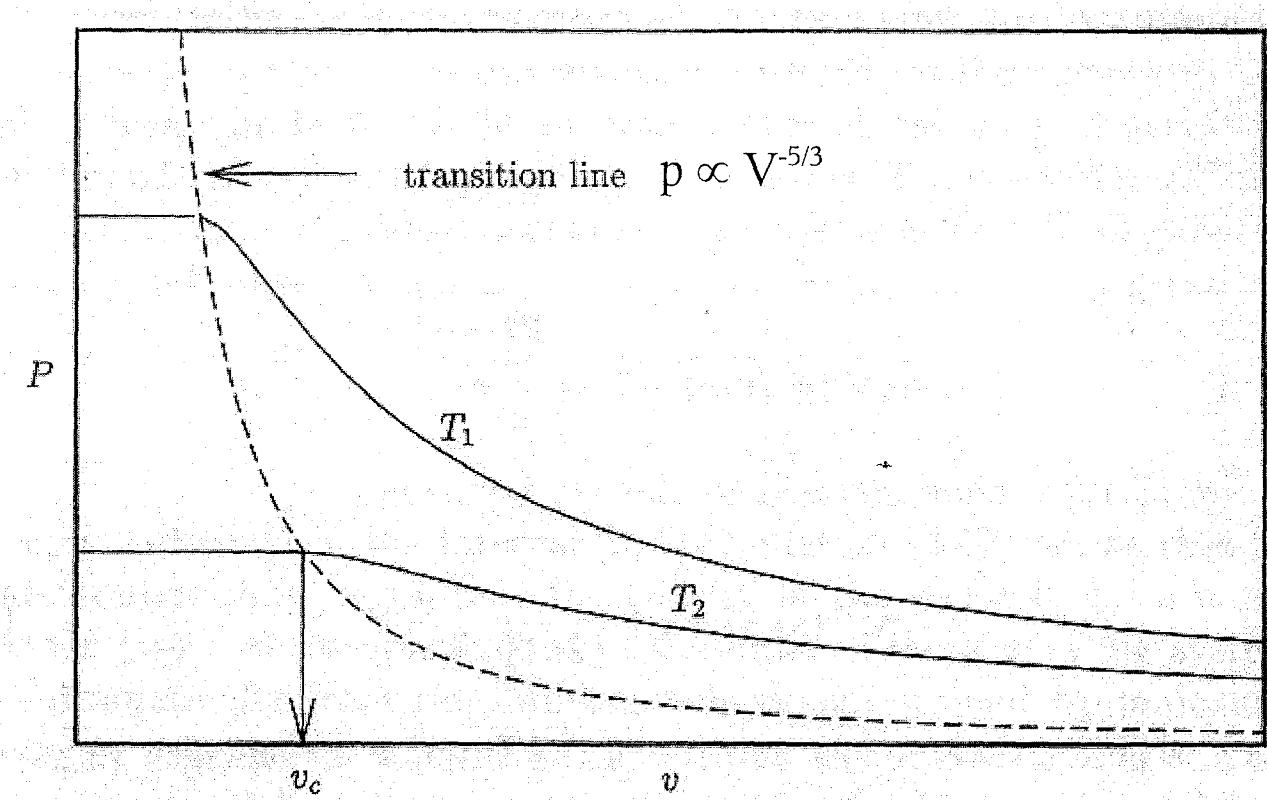


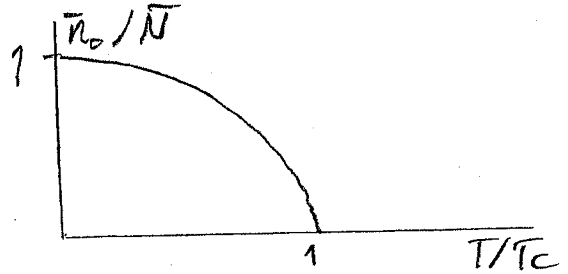
FIG. 3.4. Pressure of the ideal Bose gas as a function of $v = V/N$ for two different temperatures. Below v_c the pressure is constant. The transition line (dashed curve) is also shown (see text)

Fig. 1.7 [Pitaevskii, Stringari: BEC]

[B]

Now: constant $\bar{N}$, V

$$\overset{(30)}{(40)} \Rightarrow \quad \frac{\bar{n}_0}{\bar{N}} = 1 - \left(\frac{T}{T_c}\right)^{3/2}, \quad 0 \le T \le T_c \qquad (46)$$



- specific heat / V = const:

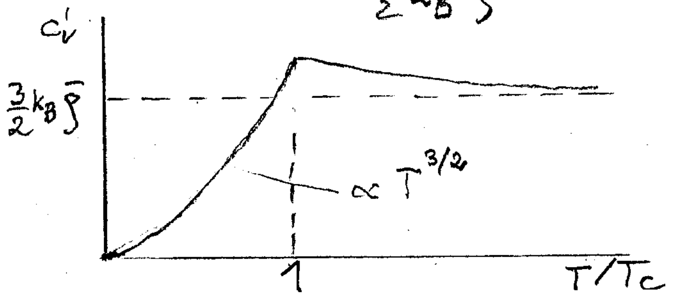
$$c_V' = \frac{1}{V} T \left(\frac{\partial S}{\partial T}\right)_{V,\bar{N}} \qquad (47)$$

BEC phase: from (42):

$$c_V' = \frac{15}{4} k_B \zeta(5/2)/\lambda_T^3 \xrightarrow{T = T_c} \frac{15}{4} \bar{\rho} k_B \zeta(5/2)/\zeta(3/2) \qquad (48)$$

normal phase: from (21B), (25B):

$$c_V' = k_B \bar{\rho} \left[\frac{15}{4} \frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4} \frac{g_{3/2}(z)}{g_{1/2}(z)}\right] \xrightarrow{T \to \infty\ (z \to 0)} \frac{3}{2} k_B \bar{\rho} \qquad (49)$$



[F]

From (18) one finds

$$f_{5/2}(z) = \frac{8}{15\sqrt{\pi}} (\beta\mu)^{5/2} \left\{1 + \frac{1}{(\beta\mu)^2} \frac{5\pi^2}{8} + \dots\right\} \qquad (50)$$

$$f_{3/2}(z) = \frac{4}{3\sqrt{\pi}} (\beta\mu)^{3/2} \left\{1 + (\beta\mu)^{-2} \frac{\pi^2}{8} + \dots\right\}$$

and hence, near T = 0; from (25F):

$$S = \frac{\pi^2}{2} k_B \bar{N} \frac{k_B T}{\epsilon_F} \{1 + \dots\} \qquad (51)$$

$$c_V' = \frac{\pi^2}{2} k_B \bar{\rho} \frac{k_B T}{\epsilon_F} \{1 + \dots\} \qquad (52)$$

Below we compare this to the bosonic case.

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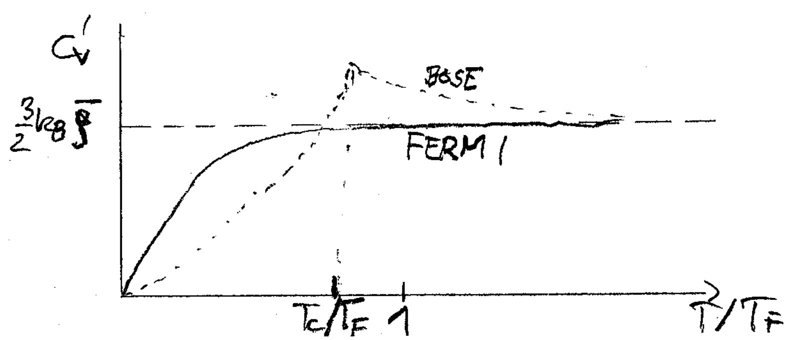


Condensate fraction of the homogeneous ideal gas

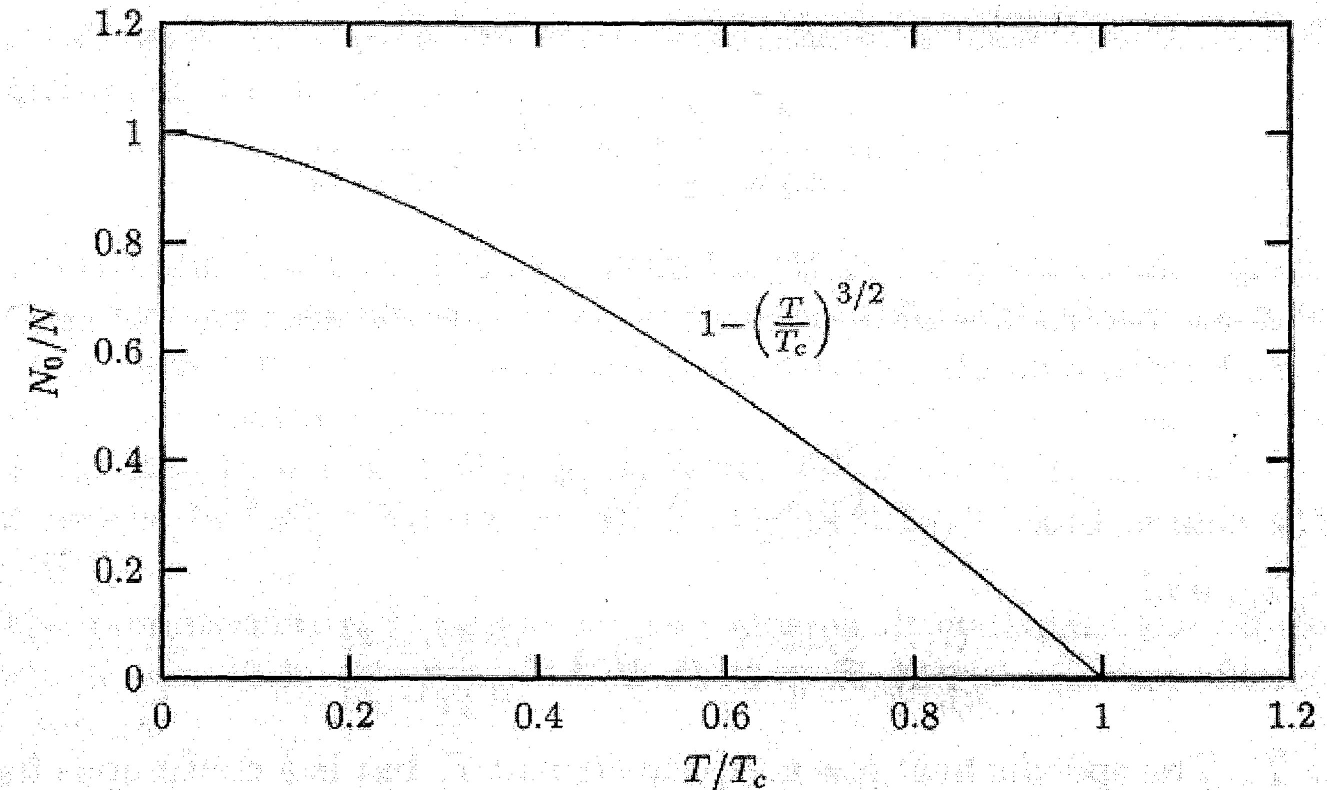


FIG. 3.2. Condensate fraction N_0/N versus temperature for a uniform ideal Bose gas. The condensate fraction is different from zero below T_c where the system exhibits Bose–Einstein condensation

Fig. 1.2

$\boxed{B}$

Pressure ($\bar{p}$ - T - diagram):

$$(24B),(21) \Rightarrow \quad \bar{p} = \bar{\rho} k_B T \frac{g_{5/2}(z)}{g_{3/2}(z)} < \bar{\rho} k_B T$$

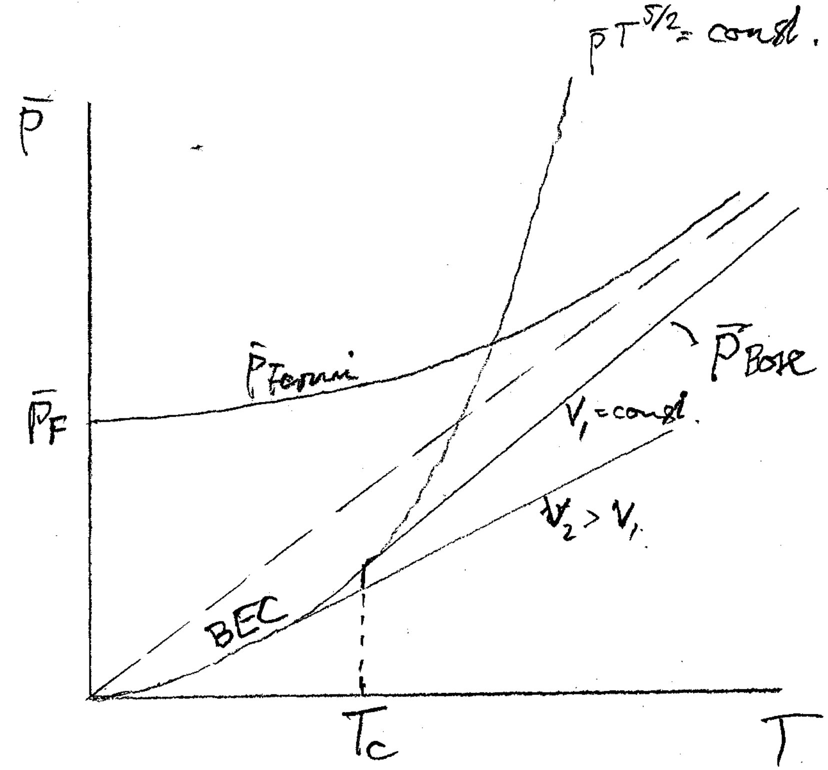
(normal phase) (class. e.o.s)

phase boundary: $\bar{p}\, T^{5/2} = \text{const.}$, see (45)

$\boxed{F}$

near $T = 0$:

$$\bar{p} = \frac{2}{5}\left(3\pi^2\bar{\rho}\right)^{\frac{2}{3}} \bar{\rho}\frac{\hbar^2}{2m}\left\{1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \dots\right\}$$



3. Trapped gases

The trap shows up in the density of states, i.e., # of states in $[\epsilon, \epsilon + d\epsilon[$, divided by $d\epsilon$:

Examples:

(a) homogeneous gas (Sect. 0.-2.) in 3D

$$G(\epsilon) = \frac{V \cdot (\text{momentum} \leq p\text{-vol.})}{\text{phase space vol. per state}} = \frac{V \cdot \frac{4}{3}\pi p^3}{(2\pi\hbar)^3} \qquad (53)$$

$$\Rightarrow g(\epsilon) = \frac{dG}{d\epsilon} = \frac{V m^{3/2}}{\sqrt{2}\,\pi^2 \hbar^3}\, \epsilon^{1/2} \qquad (p = \sqrt{2m\epsilon}) \qquad (54)$$

(b) in d dimensions:

$$g(\epsilon) \propto \epsilon^{d/2-1} \qquad (55)$$

(c) HO: $V = \frac{1}{2} m \sum_i \omega_i^2 r_i^2$, $\bar{\omega} = (\prod_i \omega_i)^{1/3}$

$$G(\epsilon) = \frac{1}{(\hbar\bar{\omega})^3} \int_0^{\epsilon} d\epsilon_1 \int_0^{\epsilon-\epsilon_1} d\epsilon_2 \int_0^{\epsilon-\epsilon_1-\epsilon_2} d\epsilon_3 = \frac{\epsilon^3}{6(\hbar\bar{\omega})^3} \qquad (56)$$

$$\Rightarrow g(\epsilon) = \epsilon^2 / 2(\hbar\bar{\omega})^3 \qquad (57)$$

(d) in d dim.:

$$g(\epsilon) = \epsilon^{d-1} / \left[(d-1)!\, \prod_i \hbar\omega_i\right] \qquad (58)$$

Consider general

$$g(\epsilon) = C_\alpha\, \epsilon^{\alpha-1} \qquad (59)$$

((a): $C_{3/2} = \frac{V}{\Gamma(3/2)} \left(\frac{m}{2\pi\hbar^2}\right)$ (c):

Thermodynamic quantities (only BOSE)

- grandcanonical potential from

$$\bar{N}_{ex} = \int_0^\infty \frac{d\epsilon\, g(\epsilon)}{z^{-1} e^{\beta\epsilon} - 1}$$

$$= C_\alpha \Gamma(\alpha) (k_B T)^\alpha g_\alpha(z) \quad (\text{cf. } (22B))$$

$$= -\partial_\mu \Omega_{gc} \tag{60}$$

$$\Rightarrow \Omega_{gc} = - C_\alpha \Gamma(\alpha) (k_B T)^{\alpha+1} g_{\alpha+1}(z) \quad (\text{cf. } (19B)) \tag{61}$$

- Critical point and BEC:

$$\bar{N} = C_\alpha \Gamma(\alpha) (k_B T_c)^\alpha \zeta(\alpha) \tag{62}$$

$$\Rightarrow T_c = \frac{1}{k_B} \left[\frac{\bar{N}}{C_\alpha \Gamma(\alpha) \zeta(\alpha)} \right]^{1/\alpha} \tag{63}$$

$$\Rightarrow \frac{\bar{N}_0}{\bar{N}} (T \leq T_c) = 1 - \left(\frac{T}{T_c} \right)^\alpha \tag{64}$$

Cases of interest:

Consider* $V_{trap} = \sum_i^3 \varepsilon_i |r_i|^{p_i}$ (65)

$$\Rightarrow \alpha = \frac{3}{2} + \sum_i^3 \frac{1}{p_i}$$

e.g. $\alpha = 3\left(\frac{1}{2} + \frac{1}{p}\right)$ (spherical) (66)

*) Bagnato et al. PRA 35 (87) 4354

Condensate fraction of the ideal/real Bose gas in a harmonic trap

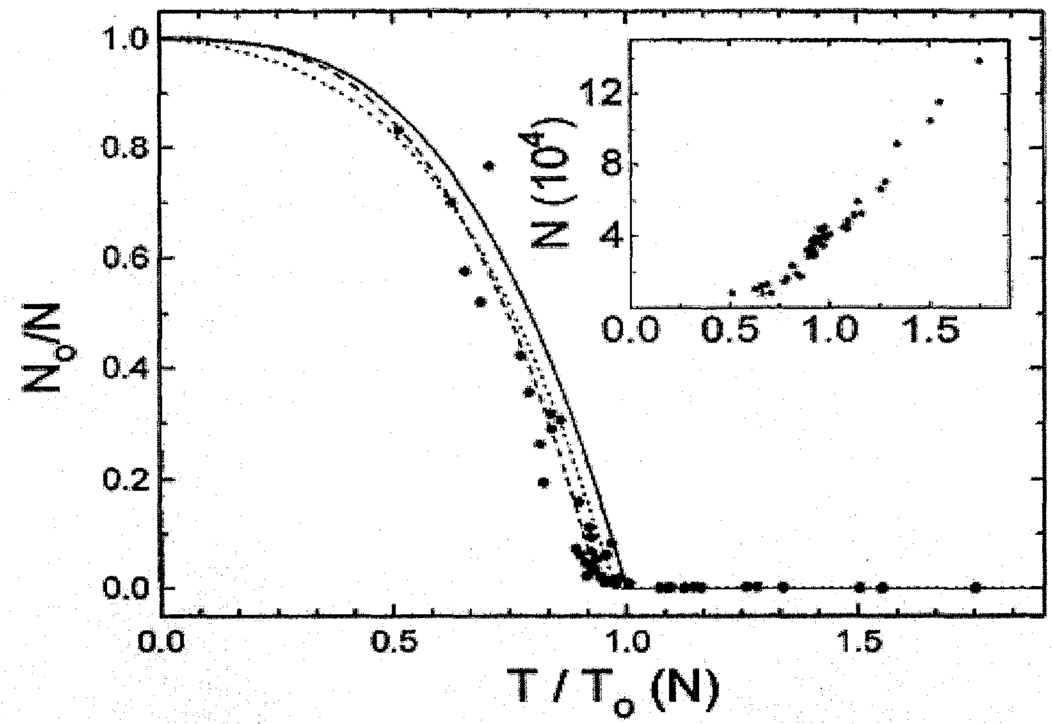


FIG. 1. Total number N (inset) and ground-state fraction N_o/N as a function of scaled temperature T/T_o. The scale temperature $T_o(N)$ is the predicted critical temperature, in the thermodynamic (infinite N) limit, for an ideal gas in a harmonic potential. The solid (dotted) line shows the infinite (finite) N theory curves. At the transition, the cloud consists of 40 000 atoms at 280 nK. The dashed line is a least-squares fit to the form $N_o/N = 1 - (T/T_c)^3$ which gives $T_c = 0.94(5)T_o$. Each point represents the average of three separate images.

Fig. 1.3 [J.R. Ensher et al., PRL 77, 4984 (1996)]

Cases of practical interest

Trap	d	α	$\Gamma(\alpha)$	$\zeta(\alpha)$
HO	1	1	1	∞
box	3	$3/2$	$0,886 = \frac{\sqrt{\pi}}{2}$	2,612
HO	2	2	1	$1.645 = \pi^2/6$
x^3	3	$5/2$	$1.329 = \frac{3}{4}\sqrt{\pi}$	1,341
HO	3	3	2	1,202
$x^{3/2}$	3	$7/2$	$3.323 = \frac{15}{8}\sqrt{\pi}$	1.127
$x^{6/5}$	3	4	6	$1.082 = \pi^4/90$
x	3	$9/2$	$11.632 = \frac{105}{16}\sqrt{\pi}$	1.055

Note:
- For a 2d homogeneous gas one has $\alpha = d/2 = 1$, i.e. $\zeta(\alpha) \to \infty$, and thus $T_c = 0$.
- If there is, however, harmonic confinement in the 3rd direction, one has $p_3 = 2$, $p_1 = p_2 \to \infty$, i.e.

  $\alpha = \frac{3}{2} + \frac{1}{2} = 2 \quad \Rightarrow$ BEC at $T_c > 0$
- in ~1D tube: $p_1 = p_2 = 2$, $p_3 \to \infty$

  $\Rightarrow \alpha = 5/2$

- entropy (see (25B)) from (61):

$$\bar{S} = -\frac{\partial}{\partial T}\Omega_{gc}$$

$$= k_B C_\alpha \Gamma(\alpha)(k_B T)^\alpha \left[(\alpha+1)\, g_{\alpha+1}(z) - g_\alpha(z)\ln z\right]$$

- specific heat (see (49f)):

$$C_V' = T\left(\frac{\partial S}{\partial T}\right)_{\text{const. trap}, N_{ex}} = \dots$$

$$= k_B \bar{N}_{ex}\, \alpha\left[(\alpha+1)\frac{g_{\alpha+1}(z)}{g_\alpha(z)} - \alpha\frac{g_\alpha(z)}{g_{\alpha-1}(z)}\right]$$

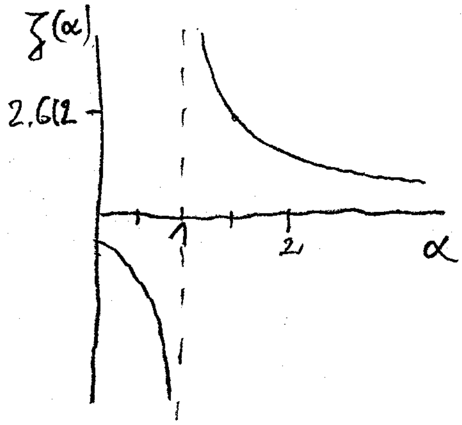
for $T \geq T_c$. For $T < T_c$, the second term is absent.

Consider (a) $T \to \infty \Rightarrow z \to 0$

$$\Rightarrow C_V' \to \alpha k_B \bar{N}$$

(b) $T \leq T_c$:

$$C_V' = (\alpha+1)\frac{\zeta(\alpha+1)}{\zeta(\alpha)}\left(\frac{T}{T_c}\right)^\alpha \alpha k_B \bar{N}$$



$\Rightarrow$ $\alpha > 2$: discontinuity at $z = 1$

$\alpha \leq 2$: $g_{\alpha-1}(1) = \infty$

$\Rightarrow$ disc. disappears

Specific heat of the ideal Bose gas for different densities of states

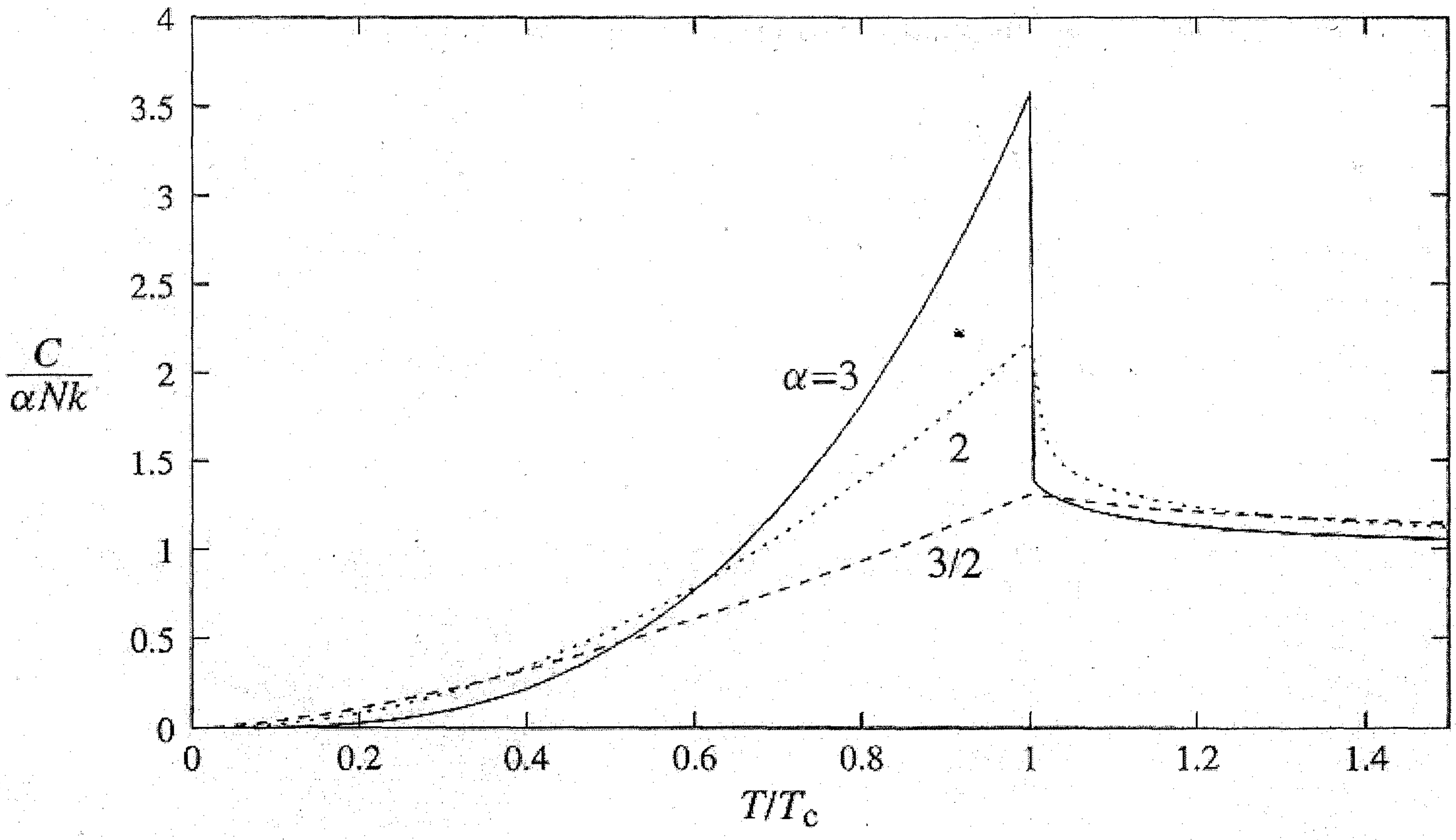


Fig. 2.3. The specific heat C, in units of $\alpha N k$, as a function of the reduced temperature T/T_c for different values of α.

Fig. 1.4 (Density of states: $g(\epsilon) = C_\alpha\, \epsilon^\alpha$)

Specific heat of a ^{87}Rb-BEC in a 3-dimensional harmonic trap

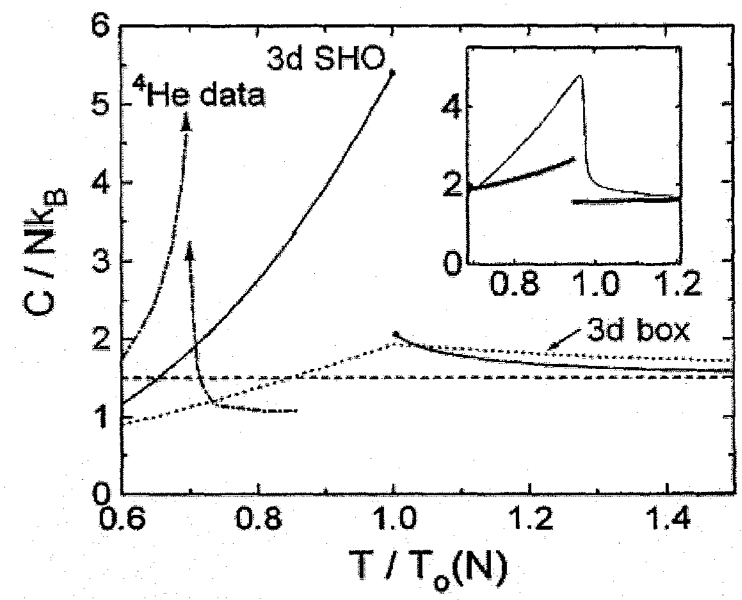


FIG. 3. Specific heat, at constant external potential, vs scaled temperature T/T_o is plotted for various theories and experiment: theoretical curves for bosons in a anisotropic 3D harmonic oscillator and a 3D square well potential, and the data curve for liquid ^{4}He [25]. The flat dashed line is the specific heat for a classical ideal gas. (inset) The derivative (bold line) of the polynomial fits to our energy data is compared to the predicted specific heat (fine line) for a finite number of ideal bosons in a harmonic potential.

Fig. 1.6 [J.R. Ensher et al., PRL 77, 4984 (1996)]

Release energy of a degenerate Fermi gas, normalized to classical limit

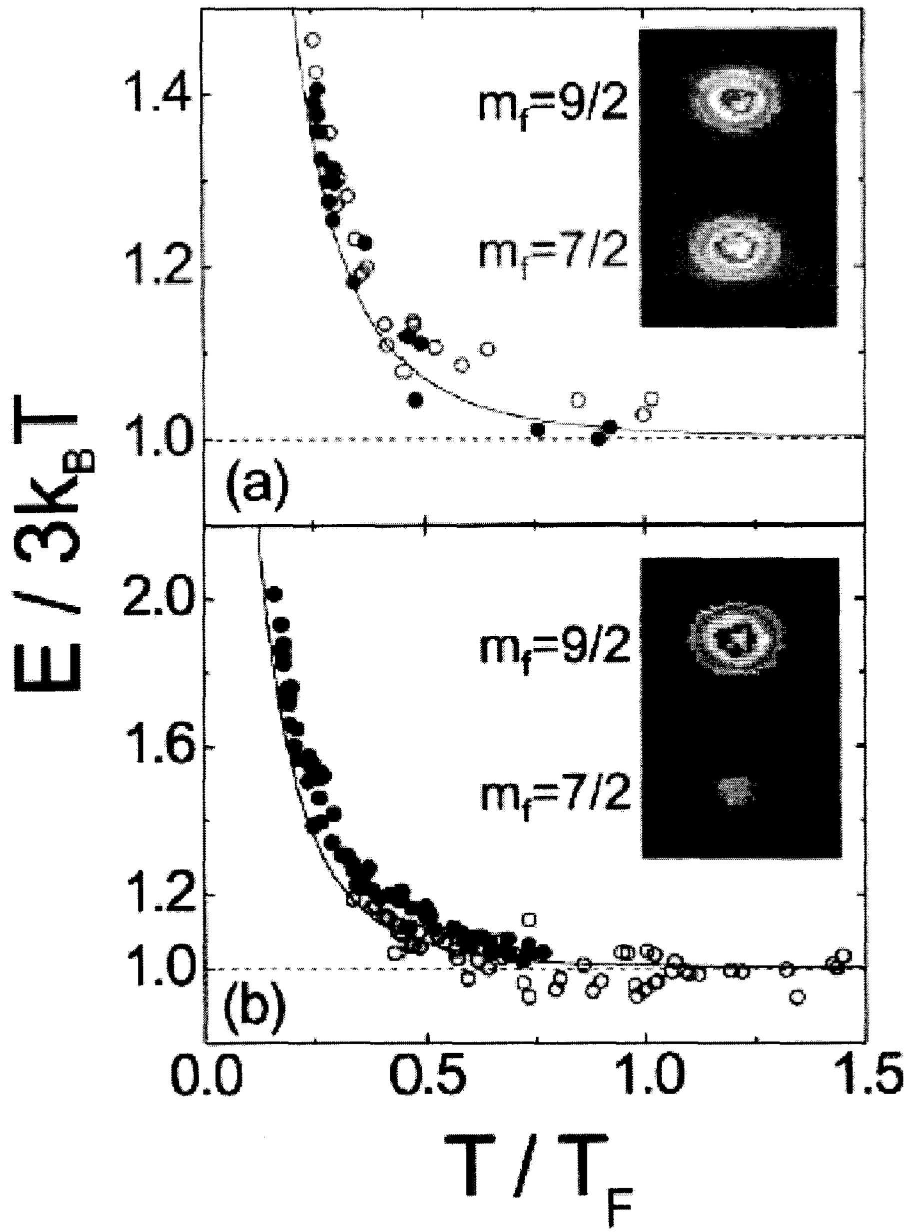


FIG. 1 (color). Thermodynamics of the interacting gas. The average energy per particle E, extracted from absorption images such as the examples shown in the insets, is displayed for two spin mixtures, 46% $m_f = 9/2$ (a) and 86% $m_f = 9/2$ (b). In the quantum degenerate regime, the data deviate from the classical expectation (dashed line) as the atoms form a Fermi sea arrangement in the energy levels of the harmonic trapping potential. The data in (a) represent the spin mixture used for evaporation, where we reach $T/T_F \sim 0.25$ at 90 nK and $N = 2.8 \times 10^5$ atoms. The data agree with the ideal Fermi gas prediction for a harmonic trap, shown by the solid line. The shift between corresponding $m_f = 9/2$ and $m_f = 7/2$ points on the T/T_F axis reflects a difference in the Fermi energies for the two components.

Fig. 1.9 [B. DeMarco et al., PRL 86, 5409 (2001)]