An actuarial mathematics approach to option pricing

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Abstract

We derive and test a new option pricing method based on statistics. We show how such a method allows to a) analytically price options with risk measures -such as Value-at-Risk or Expected Shortfall- on assets with stochastic volatility; and b) build a new structural model for the credit spread. We discuss how the method entails a new put-call parity relation, and how the option price is affected by the issuer's credit spread. Finally, we discuss several extensions to the formalism developed here, such as to assets with interdependencies, and to any model for the asset returns distribution.

1 Introduction

Beside the well-known Black and Scholes (B&S) model (Black and Scholes 1973), within the last decades a large variety of option pricing models have been presented in the literature. Among these, we may mention constant-elasticity-of-variance models (Cox and Ross 1976; Hagan and Woodward 1999), jump-diffusion models (Bates 1991; Madan, Carr, and Chang 1998; Merton 1976), and stochastic volatility models (Christoffersen, Heston, and Jacobs 2009; Benhamou, Gobet, and Mohammed 2010). This list is by no means exhaustive, yet it already shows that the choice of the option pricing model to adopt is not an easy one (Bakshi, Cao, and Chen 1997). Beside binary trees (Cox, Ross, and Rubinstein 1979; Rendleman and Bartter 1979), option pricing models are usually built by specifying a basic stochastic infinitesimal step which then yields a stochastic partial differential equation to be solved either analytically or numerically. We shall refer to such an approach by 'financial-mathematics option pricing approach' (FOPA). This way of pricing options has its roots in Bachelier's works at the beginning of the last century (Bachelier 1900). There are several drawbacks of FOPAs, among which we may mention the liquidity and market completeness assumptions, which stem from dealing with an infinitesimal time step.

In the present article, we define a new method of option pricing that is based on statistics rather than stochastic differential calculus. In more practical terms, this means that instead of starting with a stochastic infinitesimal step, we shall take a totally different route, and thereby we shall base the option price on the expected value (ExV) or the quantiles of the option payoff distribution. Loosely speaking, we shall use an actuarial mathematics approach to option pricing. We shall denote these methods by Statistical Option Pricing Approaches (SOPAs). One of the aims of this article is to demonstrate that SOPAs are better than FOPAs. We aim to show that SOPAs permit the bank to price options without hedging requirements, by just working with the quantiles of the payoff distribution. Similarly, we shall show that some SOPAs have the advantage of providing additional risk measures by which to price the option, such as the Value-at-Risk (VaR) or the Expected Shortfall (ESF). Pricing options with risk assessment, beside guaranteeing a more comprehensive and transparent analysis of the option trade, also goes in the direction of regulators, and it is thus considered of utmost importance in financial institutions. We shall also show how SOPAs allow to exactly price options for assets with non-normality features or with stochastic volatility, such as assets whose returns are heteroskedastically or autoregressively distributed, with no hedging requirements. Finally, we shall rigorously demonstrate that the B&S option pricing formula is an approximation that stems from neglecting second order partial derivatives of the option price on the asset value.

We shall furthermore discuss how FOPA implies a new put-call parity relation, and how it may explain the volatility skew. We derive several new structural models for the credit spread, along the lines of the Merton model. By using those structural models, we shall show how the option price is affected by the issuer's credit spread. Finally, we shall briefly outline and discuss further analyses that could be performed within the formalism developed here, such as pricing options for assets with strong interdependencies which are modeled by copulas, or extension to any model for the return distribution.

The article is structured as follows. In the remaining part of the present section, we shall briefly re-visit the B&S model, for the sake of completeness, and because of the need of a reference to the B&S work in subsequent sections. In Section 2 we shall introduce three SOPAs for assets whose continuous returns are normally distributed. These three approaches are based on ExV, VaR and ESF of the option payoff, and thus directly allow for risk assessment. In Section 3, we will then compare those three approaches against the B&S model. We shall compare prices and payoffs, both with simulated and real data. We will then proceed via introducing assets whose mean return or volatility are not constant: Auto Regressive (AR) and Auto Regressive Conditional Heteroskedastic (ARCH) assets. We shall describe in detail how to analytically price options for those kinds of assets, within SOPAs. This will be covered in sections 4 and 5. In section 6, we shall test the SOPAs for assets whose returns are ARCH distributed, with both simulated and real data. This concludes the exposition of SOPAs and their performances. The work will proceed, in section 7, with describing how the put/call parity relation is defined for SOPAs. Here, we shall stress the comparison with the put/call parity relation that is valid within the B&S model. In section 8, a few new structural models, based on the Merton idea, will be built, described in detail, and compared to the original Merton model. We shall find that the classical Merton model underestimates the firms' creditwothiness, when compared to SOPAs structural models. In section 9 we shall describe in detail how to include the credit spread of the option issuer into the option price. We shall study the case of an option issuer with no market value. Possible further studies are outlined in Sec. 10. There, we shall highlight that the presented analysis could be extended to virtually any return distribution for which the joint probability distribution of different days is known. In the same section, we shall also touch the possibility to include asset dependencies within the presented formalism, so to strengthen the idea that dependencies within assets should be priced in, when trading options on several assets simultaneously. Finally, conclusions are drawn in section 11.

As we are dealing only with european options, the terms 'option' and 'european option' are to be considered interchangeable. Moreover, we shall not consider options on dividend paying stocks, and we shall consider a flat structure for the risk-free interest rate.

1.1 The Black and Scholes model

The Black and Scholes (B&S) model is based upon a replicating portfolio, no-arbitrage requirement, framed in a multidimensional stochastic partial differential equation, which is a typical FOPA framework (Black and Scholes 1973). Several extensions have been made to the original B&S model, relaxing some of the basic assumptions (see, for example, (Merton 1973; Merton 1976; Cox and Ross 1976; Yang, Gao, and Yang 2016)). B&S option pricing theory can still be considered the standard technique by which to price put and call european options. As a matter of fact, the B&S model is currently very much used in many financial institutions, together with more complex numerical and Monte Carlo methods.

The call and put option prices that are solution of the B&S differential equation are, respectively,

$$\begin{aligned} \mathcal{C}^{\text{B\&S}}_{\sigma,r_f}(T, S_0, K) &= S_0\mathcal{N}(d_1) - Ke^{-r_f T}\mathcal{N}(d_2)\ , \\ \mathcal{P}^{\text{B\&S}}_{\sigma,r_f}(T, S_0, K) &= Ke^{-r_f T}\mathcal{N}(-d_2) - S_0\mathcal{N}(-d_1)\ , \end{aligned} \tag{1}$$

where

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + (r_f + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\ , \qquad d_2 = d_1 - \sigma\sqrt{T}\ , \tag{2}$$

and where $\mathcal{N}$ is the cumulative normal distribution function (CDF), with vanishing mean and unitary standard deviation, T is the option maturity, r_f is the risk-free interest rate, S_0 is the asset price at time $t = 0$, K is the strike price, σ is the return volatility (i.e., the return standard deviation). These are the B&S option pricing formulae.

One may immediately notice that the B&S formulae depend on only one asset-related characteristic, that is the standard deviation σ. Thus, the bank may not directly steer the option price with any risk measure. Rather, the bank may just take or derive the asset standard deviation from the market, or from financial providers (Corsi, Fusari, and La Vecchia 2013; Mixon 2009; Calypso .), and simply compute the option price. The risk assessment within the B&S is normally carried out in financial institutions by charging the customer the effective amount of money required to hedge the position, plus additional costs and profits. In principle, the B&S option price theoretically already represents the cost required to hedge the position. However, it is often the case that the market price differs from the B&S price, which is the same as saying that the hedging costs are somewhat different in practice to what the B&S formulae postulate. To adjust for this discrepancy, it is common practice to feed the B&S formulae with the so-called implied volatility, which is used as a fitting parameter to match the market price.

In the following sections, we shall construct option pricing methods that are solely based on statistics. Therein, we shall use B&S as benchmark to compare.

2 Statistically based option pricing methods for assets whose returns are normally distributed

In this section, we shall build SOPAs, i.e. option pricing methods that are solely based on statistics. Within this section we shall consider assets whose returns are normally distributed. At the end of this section, we shall make a close comparison between SOPAs and B&S approaches.

While this section focuses on theory, Sec. 3 presents the performances of all investigated option pricing approaches on both empirical and simulated data.

2.1 Statistical tools to be used

2.1.1 Convolution

The key tool that we are going to use out of statistics for our purposes is *convolution* (Bracewell and Bracewell 1986). Let us briefly recall what convolution means and how it is rigorously defined. Suppose there exist one event that may happen (e.g., an insurance claim), and that would bring the profit x to the company if it happens. It might bring also a loss, in which case x is negative. Say the probability density function (PDF) of such an event is $g_{N_1}(x)$. Analogously, suppose there exist another event with PDF $g_{N_2}(y)$ that would bring the profit y to the company, if it happens. Suppose that the events are independent of each other. Then, the probability that the profit s will be brought to the company is the convolution of the two functions:

$$G_2(s) = g_{N_1} * g_{N_2} = \int_{-\infty}^{+\infty} g_{N_1}(x) g_{N_2}(s-x)\,\mathrm{d}x\ . \qquad (3)$$

If the PDFs of events 1 and 2 happen to be gaussians (i.e. normal)

$$g_{N_1}(x) = \frac{1}{\sigma_1\sqrt{2\pi}}\,e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}\ , \qquad g_{N_2}(x) = \frac{1}{\sigma_2\sqrt{2\pi}}\,e^{-\frac{(x-\mu_2)^2}{2\sigma_2^2}}\ , \qquad (4)$$

then the convolution takes a very simple form:

$$G_2(s) = g_{N_1} * g_{N_2} = \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}\, e^{-\frac{\left(s-(\mu_1+\mu_2)\right)^2}{2(\sigma_1^2+\sigma_2^2)}} \,. \tag{5}$$

As we shall suppose that the PDF of assets' returns is gaussian (i.e. normal) throughout this section, we shall make use of Eq. (5) very often.

2.1.2 Time series of continuous returns

Let us take a stock whose daily continuous returns ($r_{\mathrm{d}i}$, for any day i) are normally distributed. After T trading days, the overall continuous return would be

$$r_T = r_{\mathrm{d}1} + ... + r_{\mathrm{d}T} = \sum_{i=1}^{T} r_{\mathrm{d}i} \,. \tag{6}$$

This means that occurrences of each day are summed up to find the overall amount of return, exactly like a sum of insurance claims is the total claim amount.

Now, let us suppose that the return of any i-day has a normal distribution with mean μ_i and standard deviation σ_i, and that it is independent of any other j-day, where $j \neq i$. These are in fact common hypotheses for modeling stock returns. After T trading days, the probability density to obtain the overall continuous return r_T equal to s would be, using (5),

$$G_T(s) = g_{N_1} * g_{N_2} * ... * g_{N_T} = \frac{1}{\sqrt{2\pi \sum_{i=1}^{T} \sigma_i^2}} \,\exp\left[-\frac{\left(s-\sum_{i=1}^{T}\mu_i\right)^2}{2\sum_{i=1}^{T}\sigma_i^2}\right], \tag{7}$$

where μ_i and σ_i are the daily means and standard deviations. Since all occurrences are returns of the same asset in different days, they have the same mean μ and standard deviation σ. We may therefore write

$$G_T(s) = \frac{1}{\sqrt{2\pi T\sigma^2}} \,\exp\left[-\frac{(s-T\mu)^2}{2T\sigma^2}\right]. \tag{8}$$

Equation (8) is our key quantity: $G_T(s)$ is the PDF of the overall continuous return r_T. That means that $G_T(s)\mathrm{d}s$ is the probability that the overall continuous return r_T, at maturity T, will be within s and $s + \mathrm{d}s$. A thorough analysis of the function $G_T(s)$ will be presented in Secs. 4.3 and 5.3, and is therefore skipped here.

Without restriction of generality, in this section we chose the time frame to be one day. Likewise it could have been chosen to be five minutes, one week, or anything else.

2.2 Statistical Expected Value (ExV) approach

A call option is a financial product that pays when the stock price at maturity (S_T) is *above* a certain threshold that is called *strike price* (K). The payoff function of such a product is

$$\textit{call payoff} \equiv \mathrm{cp} = \mathrm{Max}[0, S_T - K] = \mathrm{Max}[0, S_0 e^{r_T} - K] \,, \tag{9}$$

where r_T is the continuous return that has been achieved at maturity T: $-\infty < r_T < +\infty$.

In efficient markets (Malkiel 2003), we may set the option price equal to the expected option payoff, i.e. we may price the option with the average money that we (the bank) expect to pay the client. This is the way banks price options within Monte Carlo simulations (Glasserman 2013), before adding profit and cost loading. By following this prescription, as bank we would not earn anything from the client, in average, which is expected in efficient capital markets. Equivalently, we may say that this approach is suitable for risk neutral issuers (Bingham and Kiesel 2013). We shall call this approach the **Expected Value (ExV)** approach, and we shall investigate it in the following subsection.

2.2.1 Call option pricing within ExV approach

The probability weighted call payoff (WCP), which is the expected payoff for a call option, is

$$
\begin{aligned}
WCP &= \int_{-\infty}^{+\infty} \mathrm{Max}[0, S_0 e^{r_T} - K]\, G_T(r_T)\, \mathrm{d}r_T \\
&= \int_{\substack{\text{where} \\ S_0 e^{r_T} - K < 0}} \mathrm{d}r_T\, G_T(r_T) \cdot 0 + \int_{\substack{\text{where} \\ S_0 e^{r_T} - K \geq 0}} \mathrm{d}r_T\, G_T(r_T) \cdot (S_0 e^{r_T} - K) \\
&= \int_{\log \frac{K}{S_0}}^{+\infty} \mathrm{d}r_T\, G_T(r_T) \cdot (S_0 e^{r_T} - K)
\end{aligned}
\tag{10}
$$

where $G_T(r_T)$ is the PDF of the quantity r_T, as defined in Eq. (8). By considering that $\mathcal{N}(-x) = 1 - \mathcal{N}(x)$, with simple algebraic manipulation we can cast the previous equation into the form

$$
WCP = S_0 e^{T(\mu+\sigma^2/2)}\, \mathcal{N}(c_1) - K\mathcal{N}(c_2)\,, \tag{11}
$$

where

$$
c_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(\mu + \sigma^2\right) T}{\sigma\sqrt{T}}\,, \qquad c_2 = c_1 - \sigma\sqrt{T}\,. \tag{12}
$$

We shall set the option price equal to the discounted WCP, that is

$$
C^{\mathrm{ExV}}_{\mu,\sigma,r_f}(T, S_0, K) \equiv \left[S_0 e^{T(\mu+\sigma^2/2)}\, \mathcal{N}(c_1) - K\mathcal{N}(c_2)\right] e^{-r_f T}\,, \tag{13}
$$

where r_f is the risk-free interest rate. Equation (13) is the final result for the call option price within this approach. The stock characteristic quantities are the mean return, μ, and the standard deviation, σ. Thus, the mean return has emerged as (new) quantity that determines the option price together with σ, in contrast to the B&S pricing model where the option price is completely determined by σ (Ronnie Sircar and Papanicolaou 1998). Both ExV and B&S depend on r_f, K, S_0 and T, which are stock independent quantities. Furthermore, we notice that the risk free interest rate is not present in the argument of the cumulative distribution function, while μ is present instead. This is again in stark contrast with B&S formulae. We shall study such a dependence more in detail in Sec. 2.6. Aside the mentioned differences, one may also remark that B&S and ExV formulae look quite similar, which is not obvious considering that they are obtained within a completely different framework.

As a final comment, we remark that in principle the right member of Eq. (13) must be multiplied by the probability that the bank does not default, which probability can be cast as $e^{-\mathrm{CS}\,T}$, where CS is the bank credit spread. We shall not consider this until Sec. 9.

2.2.2 Put option pricing within ExV approach

A put option is a financial product that pays when the stock price at maturity (S_T) is *below* a certain threshold that is called *strike price* (K). The payoff function of such a product is

$$\textit{put payoff} \equiv \mathrm{pp} = \mathrm{Max}[0, K - S_T] = \mathrm{Max}[0, K - S_0 e^{r_T T}]\,. \tag{14}$$

Similarly to what done for call options, we shall price a put option by using the expected payoff. The probability weighted put payoff (WPP), which is the expected payoff of a put option, is

$$\begin{aligned} WPP &= \int_{\substack{\text{where}\\ K-S_0 e^{r_T}<0}} \mathrm{d}r_T\, G_T(r_T)\cdot 0 + \int_{\substack{\text{where}\\ K-S_0 e^{r_T}\geq 0}} \mathrm{d}r_T\, G_T(r_T)\cdot (K - S_0 e^{r_T}) \\ &= \int_{-\infty}^{\log\frac{K}{S_0}} \mathrm{d}r_T\, G_T(r_T)\cdot (K - S_0 e^{r_T}) \end{aligned} \tag{15}$$

where $G_T(r_T)$ is defined in Eq. (8). With simple algebraic manipulation we can cast the previous equation into the form

$$WPP = -\Big(S_0 e^{T(\mu+\sigma^2/2)}\,\mathcal{N}(-c_1) - K\mathcal{N}(-c_2)\Big)\,, \tag{16}$$

where

$$c_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(\mu + \sigma^2\right)T}{\sigma\sqrt{T}}\,, \qquad c_2 = c_1 - \sigma\sqrt{T}\,. \tag{17}$$

We shall set the option price equal to the discounted WPP, that is

$$\mathcal{P}^{\text{ExV}}_{\mu,\sigma,r_f}(T, S_0, K) \equiv \left[\left(K\mathcal{N}(-c_2) - S_0 e^{T(\mu+\sigma^2/2)}\, \mathcal{N}(-c_1)\right)\right] e^{-r_f T} \ . \tag{18}$$

This concludes the analysis of put option pricing within ExV approach.

2.3 Statistical Value at Risk (VaR) approach

In this section, we shall introduce the first SOPA that allows for risk assessment. Within financial institutions, and considering Basel II regulations, Value at Risk (VaR) is a widely accepted risk measure (Banking Supervision 2011; Wipplinger 2007; Jorion 1997). In the light of this, VaR is the risk measure we adopt first so as to assess the option price from a risk management point of view.

2.3.1 Call option pricing within VaR approach

Suppose we are a bank that sells a call option to a client. We would like to be confident that the option does not turn out to be in-the-money at maturity. In fact, if it does, we will have to pay the client his earnings, which would often mean that the client will make money and we will lose money. So, in order to price a call option, we would look for the answer to the following question: 'Given the PDF in Eq. (8), what is the value such that we are sure, within α confidence level, that the overall continuous return at maturity T will be *below* that value?' The answer to the question is $r_{c,\alpha}$ such that

$$\int_{-\infty}^{r_{c,\alpha}} G_T(x)\mathrm{d}x = \alpha \ . \tag{19}$$

In other words, $r_{c,\alpha}$ is the α-quantile of the r_T distribution. With simple algebraic calculation, one can cast the previous equation into the form

$$\alpha \ = \ \int_{-\infty}^{y} \frac{\mathrm{d}s}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} = \mathcal{N}(y) \ , \tag{20}$$

where $y = \dfrac{r_{c,\alpha} - \mu T}{\sigma \sqrt{T}}$. In conclusion, $r_{c,\alpha}$ is the solution of Eq. (20) which implies $y = \mathcal{N}^{-1}\left(\alpha\right)$, and therefore

$$r_{c,\alpha} = \sigma \sqrt{T}\ \mathcal{N}^{-1}\left(\alpha\right) + \mu T\ . \tag{21}$$

For example, given a stock with $\mu = 1\% = 0.01$, $\sigma = 0.04 = 4\%$, a confidence level $\alpha = 0.9 = 90\%$ and the maturity $T = 10$ days, the value $r_{c,\alpha}$ solution of Eq. (21) is $r_{c,\alpha} \simeq 0.26 = 26\%$. This means that we are 90% sure that the overall continuous return at maturity will be below 26%.

Now we go to the call option price. The stock price is related to the overall continuous return as $S_T = S_0 e^{r_T}$. S_T is a bijective monotonically increasing function of r_T. Given the α quantile of the overall continuous return distribution ($r_{c,\alpha}$), the α quantile of the stock price distribution is simply

$$S_{T,\alpha} = S_0 e^{r_{c,\alpha}}\ . \tag{22}$$

However, we need some care to identify the α quantile of the option payoff distribution. This is because the probability function of the option payoff, although is monotonic, is not strictly monotonic, and so does not admit a continuous PDF. In turn, this is due to the fact that the payoff is vanishing for the whole region $S_T \in [0, K]$, and so the probability to obtain a payoff precisely equal to 0 is not vanishing, as would be required if a continuous PDF were admitted. We can circumvent this problem by resorting to the original definition of quantile. The quantile of the call option payoff at α confidence level, cp_α, is defined as

$$\mathrm{cp}_\alpha = \inf\left\{\mathrm{cp} \in \mathbb{R}^+\ :\ F_{\mathrm{cp}}(\mathrm{cp}) \geq \alpha\right\}\ , \tag{23}$$

where $F_{\mathrm{cp}}(\mathrm{cp})$ is the cumulative distribution function for the call option payoff. Equation (23) yields $\mathrm{cp}_\alpha = 0$ in case $S_{T,\alpha} \leq K$, while it yields $\mathrm{cp}_\alpha = S_{T,\alpha} - K$ in case $S_{T,\alpha} \geq K$. Joining the two cases, cp_α may be defined as

$$\mathrm{cp}_\alpha = \mathrm{Max}[0, S_{T,\alpha} - K]\ . \tag{24}$$

The meaning of cp_α is clear: Within α confidence level, the option payoff will be below or equal to cp_α. In other words, cp_α is the maximum payoff that the bank expects to pay the option owner, within the confidence level α. Stated again differently, cp_α is the maximum loss that the bank expects to face due to the call option that was sold, within α confidence level, disregarding the money earned due to the option price. We therefore see the similarity to the concept of **Value at Risk (VaR)** (Wipplinger 2007; Jorion 1997). That is why this approach has been denoted by 'VaR approach'. This approach, in contrast to the ExV approach, is also suitable for risk averse issuers or investors, since they can choose a confidence level as high as their risk aversion is (Ross 2004).

We will thus set the discounted cp_α equal to the option price. This will ensure that, within α confidence level, the bank will make money by selling the option. Therefore, the price of the call option in the VaR SOPA will be

$$C^{\mathrm{VaR}}_{\mu,\sigma,\alpha,r_f}(T, S_0, K) = e^{-r_f\,T}\mathrm{Max}[0, S_{T,\alpha} - K]\,, \tag{25}$$

where

$$S_{T,\alpha} = S_0 e^{r_{c,\alpha}}\,, \qquad r_{c,\alpha} = \sigma\sqrt{T}\,\mathcal{N}^{-1}\!\left(\alpha\right) + \mu T\,, \qquad r_f = \text{risk free rate}\,. \tag{26}$$

Equation (25) is the final result for the call option price within the VaR approach. The stock characteristic quantities are μ, σ, while α is a new risk measure that is used to steer the option price. α is the probability that the option payoff will be below the option price, i.e. the confidence level to earn money (as bank) out of the sold option. Consequently, the higher α, the higher the option price. Stated differently, the more the option issuer wants to be sure to earn from the sold option, the more the option price must be.

As already mentioned, Basel II regulations stress the point that a wise risk assessment in financial institutions must be performed through a VaR approach. The presented VaR statistical option pricing approach goes along this direction, as it allows to assess the impact of the option deal on the bank within Basel II compliance. In our opinion, the presented SOPA VaR method has, therefore, a fundamental value in setting up the relation between option issuer and regulators.

In this section, we showed how to price option with a confidence level, i.e. within a VaR-like approach. We showed that, by working with the quantile of the return distribution at maturity, we can set the option price by selecting the maximum loss we (the bank) expect to have by selling the option, within a certain probability α. Now, let us suppose that we choose $\alpha = 0.5 = 50\%$. It means that the option price coincides with the median, so that we will earn money 50% of times. Beware, however, that is not equal to the option expected payoff, which is the average money we will have to pay to the client. The expected payoff is rather equal to the quantity $C^{\mathrm{ExV}}_{\mu,\sigma,r_f}$, which we investigated in detail in the Sec. 2.2. We shall spend more words on this concept later on, at pg. 18.

2.3.2 Put option pricing within VaR approach

In order to price a put option within the VaR approach, analogously to the case of a call option, we look for the answer to the following question: 'Given the PDF in Eq. (8), what is the value such that we are sure, within α confidence level, that the overall continuous return at maturity T will be *above* that value?' The answer to the question is $r_{p,\alpha}$ such that

$$\int_{r_{p,\alpha}}^{+\infty} G_T(x)\mathrm{d}x = \alpha \; . \tag{27}$$

Since $G_T(x)$ is normalized to one by definition of PDF, we can write

$$\int_{r_{p,\alpha}}^{+\infty} G_T(x)\mathrm{d}x = 1 - \int_{-\infty}^{r_{p,\alpha}} G_T(x)\mathrm{d}x \; . \tag{28}$$

Joining with Eq. (27), we have

$$\int_{-\infty}^{r_{p,\alpha}} G_T(x)\mathrm{d}x = 1 - \alpha \; . \tag{29}$$

Now we see that Eq. (29) is just Eq. (19) with $\alpha \to 1-\alpha$. Therefore, the relevant equation for the put option is

$$\mathcal{N}(y) = (1-\alpha)\,, \tag{30}$$

where $y = \dfrac{r_{p,\alpha} - \mu T}{\sigma\sqrt{T}}$. This implies $r_{p,\alpha} = \sigma\sqrt{T}\,\mathcal{N}^{-1}\big(1-\alpha\big) + \mu T$.

Analogously to the case of the call option price, the put price in the VaR SOPA would be

$$\mathcal{P}^{\text{VaR}}_{\mu,\sigma,\alpha,r_f}(T, S_0, K) = e^{-r_f\,T}\text{Max}[0, K - S_{T,\alpha}]\,, \tag{31}$$

where

$$S_{T,\alpha} = S_0 e^{r_{p,\alpha}}\,, \qquad r_{p,\alpha} = \sigma\sqrt{T}\,\mathcal{N}^{-1}\big(1-\alpha\big) + \mu T\,. \tag{32}$$

This concludes the analysis of the VaR pricing approach for put options.

2.4 Statistical Expected Short-Fall (ESF) approach

Similarly to what done in market risk, the second typical risk measure we consider, after VaR, is the **Expected Short-Fall (ESF)** (Yamai and Yoshiba 2005; Acerbi and Tasche 2002), also called Conditional Value at Risk, Average Value at Risk, and Expected Tail Loss. ESF is the average loss above the VaR, and is therefore a stronger (safer) risk measure than VaR. The usage of ESF instead of VaR is particularly beneficial in cases of structured tail dependency, since VaR gives no information on the tail losses. ESF is the risk measure to adopt within revised standards for minimum capital requirements for Market Risk, issued by the Basel Committee on Banking Supervision (Banking Supervision 2016; Banking Supervision 2012), and must be therefore considered of utmost importance for financial institutions.

2.4.1 Call option pricing within ESF approach

The ESF approach for a call option starts with the findings in Sec. 2.3.1. We choose a confidence level α, we find the quantile of the stock distribution $S_{T,\alpha} = S_0 e^{r_{c,\alpha}}$. As discussed in Sec. 2.3.1, the quantile for the option payoff is $\text{cp}_\alpha = \text{Max}[0, S_{T,\alpha} - K]$. The ESF is the average payoff out of the tail events that are beyond the quantile of the distribution, i.e.

$$\begin{aligned}
\text{ESF}_{call} &= \frac{1}{M_{call}} \int_{r_{c,\alpha}}^{+\infty} \text{Max}[0, S_0 e^{r_T} - K]\, G_T(r_T)\, \mathrm{d}r_T \\
&= \frac{1}{M_{call}} \int_{r_{c,\alpha}}^{\text{Max}\left[r_{c,\alpha}, \log\left(\frac{K}{S_0}\right)\right]} 0 \cdot G_T(r_T)\, \mathrm{d}r_T + \frac{1}{M_{call}} \int_{\text{Max}\left[r_{c,\alpha}, \log\left(\frac{K}{S_0}\right)\right]}^{+\infty} (S_0 e^{r_T} - K)\, G_T(r_T)\, \mathrm{d}r_T \\
&= \frac{1}{M_{call}} \int_{\text{Max}\left[r_{c,\alpha}, \log\left(\frac{K}{S_0}\right)\right]}^{+\infty} G_T(r_T)\, (S_0 e^{r_T} - K)\, \mathrm{d}r_T
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
r_{c,\alpha} &= \sigma \sqrt{T}\, \mathcal{N}^{-1}\big(\alpha\big) + \mu T\,, \qquad G_T(r_T) = \frac{1}{\sqrt{2\pi T \sigma^2}}\, \exp\left[-\frac{(r_T - T\mu)^2}{2T\sigma^2}\right], \\
M_{call} &= \text{normalization factor} = \mathcal{P}(r_T \geq r_{c,\alpha}) = \int_{r_{c,\alpha}}^{+\infty} G_T(r_T)\, \mathrm{d}r_T \\
&= 1 - \int_{-\infty}^{r_{c,\alpha}} G_T(r_T)\, \mathrm{d}r_T = 1 - \mathcal{N}\left(\frac{r_{c,\alpha} - T\mu}{\sigma\sqrt{T}}\right) = \mathcal{N}\left(\frac{-r_{c,\alpha} + T\mu}{\sigma\sqrt{T}}\right),
\end{aligned} \tag{34}$$

and where $\mathcal{P}$ indicates the probability. One can see the similarity between Eq. (33) and Eq. (10). Therefore, we can just solve the integral by replacing $\log\left(\frac{K}{S_0}\right) \to \text{Max}[r_{c,\alpha}, \log\left(\frac{K}{S_0}\right)]$ in the solution:

$$\text{ESF}_{call} = \frac{1}{M_{call}} \Big(S_0 e^{T(\mu+\sigma^2/2)}\, \mathcal{N}(b_1) - K\mathcal{N}(b_2)\Big)\,, \tag{35}$$

where

$$b_1 = \frac{-\text{Max}[r_{c,\alpha}, \log\left(\frac{K}{S_0}\right)] + \left(\mu + \sigma^2\right) T}{\sigma\sqrt{T}}\,, \qquad b_2 = b_1 - \sigma\sqrt{T}\,. \tag{36}$$

Finally, the call option price would be the discounted ESF, i.e.

$$C^{\text{ESF}}_{\mu,\sigma,\alpha,r_f}(T, S_0, K) = \frac{e^{-r_f\, T}}{M_{call}} \left[S_0 e^{T(\mu+\sigma^2/2)}\, \mathcal{N}(b_1) - K\mathcal{N}(b_2)\right]\,, \tag{37}$$

where r_f is the risk-free interest rate. Within the ESF approach, the bank is pricing the option as the average loss above the VaR, where this latter is set within a certain confidence level α. Analogously to the VaR approach, the ESF approach is therefore (also) suitable to risk averse issuers and investors, as they can choose their confidence level as high as their risk aversion is (Ross 2004).

2.5 Put option pricing within ESF approach

The reasoning for the put option within the ESF approach goes similarly. After having chosen a confidence level α, and the quantile of the put option payoff is found to be $\mathrm{pp}_\alpha = \mathrm{Max}[0, K - S_0 e^{r_{p,\alpha}}]$ (see Sec. 2.3.1). The ESF is the average payoff out of the tail events that lie beyond the quantile of the payoff distribution, i.e.

$$\begin{aligned}
\mathrm{ESF}_{put} &= \frac{1}{M_{put}} \int_{-\infty}^{r_{p,\alpha}} \mathrm{Max}[0, K - S_0 e^{r_T}]\, G_T(r_T)\, \mathrm{d}r_T \\
&= \frac{1}{M_{put}} \int_{-\infty}^{\mathrm{Min}\left[r_{p,\alpha}, \log\left(\frac{K}{S_0}\right)\right]} (K - S_0 e^{r_T})\, G_T(r_T)\, \mathrm{d}r_T + \frac{1}{M_{put}} \int_{\mathrm{Min}\left[r_{p,\alpha}, \log\left(\frac{K}{S_0}\right)\right]}^{r_{p,\alpha}} 0 \cdot G_T(r_T)\, \mathrm{d}r_T \\
&= \frac{1}{M_{put}} \int_{-\infty}^{\mathrm{Min}\left[r_{p,\alpha}, \log\left(\frac{K}{S_0}\right)\right]} G_T(r_T)\, (K - S_0 e^{r_T})\, \mathrm{d}r_T
\end{aligned} \tag{38}$$

where

$$r_{p,\alpha} = \sigma \sqrt{T}\, \mathcal{N}^{-1}\Big(1 - \alpha\Big) + \mu T\,, \qquad G_T(r_T) = \frac{1}{\sqrt{2\pi T \sigma^2}} \exp\left[-\frac{(r_T - T\mu)^2}{2T\sigma^2}\right], \tag{39}$$

and where

$$M_{put} = \text{normalization factor} = \mathcal{P}(r_T \le r_{p,\alpha}) = \int_{-\infty}^{r_{p,\alpha}} G_T(r_T)\, \mathrm{d}r_T = \mathcal{N}\left(\frac{r_{p,\alpha} - T\mu}{\sigma \sqrt{T}}\right). \tag{40}$$

One can see the similarity between Eq. (38) and Eq. (15). Therefore, we can just solve the integral by replacing $\log\left(\frac{K}{S_0}\right) \to \text{Min}[r_{p,\alpha}, \log\left(\frac{K}{S_0}\right)]$ in the solution:

$$\text{ESF}_{put} = -\frac{1}{M_{put}}\left(S_0 e^{T(\mu+\sigma^2/2)}\, \mathcal{N}(-b_1) - K\mathcal{N}(-b_2)\right), \tag{41}$$

where

$$b_1 = \frac{-\text{Min}[r_{p,\alpha}, \log\left(\frac{K}{S_0}\right)] + \left(\mu+\sigma^2\right)T}{\sigma\sqrt{T}}, \qquad b_2 = b_1 - \sigma\sqrt{T}\,. \tag{42}$$

Finally, the put option price would be the discounted ESF, i.e.

$$\mathcal{P}^{\text{ESF}}_{\mu,\sigma,\alpha,r_f}(T, S_0, K) = -\frac{e^{-r_f\,T}}{M_{put}}\left[S_0 e^{T(\mu+\sigma^2/2)}\, \mathcal{N}(-b_1) - K\mathcal{N}(-b_2)\right]. \tag{43}$$

This concludes the analysis of put option pricing within ESF approach.

2.6 Option pricing approaches compared: B&S vs. ExV vs. VaR vs. ESF

For the sake of brevity, we shall consider only call option prices in this comparison. All SOPAs visited so far, namely ExV, VaR, ESF, and B&S are plotted and compared in Fig. 1, for different settings of the confidence level (α) and mean return (μ). The first thing to remark from the plot is that ESF is always above VaR. This is expected, since ESF is a safer measure than VaR. More precisely, VaR represents the smallest loss above the quantile that the bank would face from selling the option (i.e the quantile itself), while ESF represents the average loss above the quantile. ESF must therefore be higher than, or at least equal to, VaR. Since the loss has been used to set the option price, it follows that ESF price must be higher than, or at least equal to, VaR price.

The second thing to remark is that only VaR price can, for certain settings, be cheaper than B&S, while all other methods turn out to be more expensive (note that this does not always hold for put options). While a 90% confidence level gives a significant guarantee to the bank to make profit out of the option trade, a confidence level of 50% may seem more like a fair price. In this latter case, viz. when $\alpha = 50\%$, the bank has 50% chances to make profit by selling the option. Although this could be seen as the fair price, in general it is not, since it would actually underestimate by far the option price. That is because in those 50% of times that the bank does not make money, the bank pays the customer the option payoff, which might be very high. On the other hand, in those 50% of times that the bank makes money, the bank earns just the option price. In more technical terms, pricing the option with a VaR(50%) method would set the price equal to the median of the payoff distribution, which could be very different from the average payoff.

One may notice that the B&S price is quite similar to the VaR price for the case of 50% confidence level and $\mu = 0$, apart from the region close to the strike price. Therefore, thanks to the present analysis, one may state that pricing along with the B&S model would make the bank earn money only in approximately 50% of cases on assets with $\mu \approx 0$, if we disregard the situation in which the strike price is close to the asset price at inception, and if we suppose that the bank does not hedge the position.

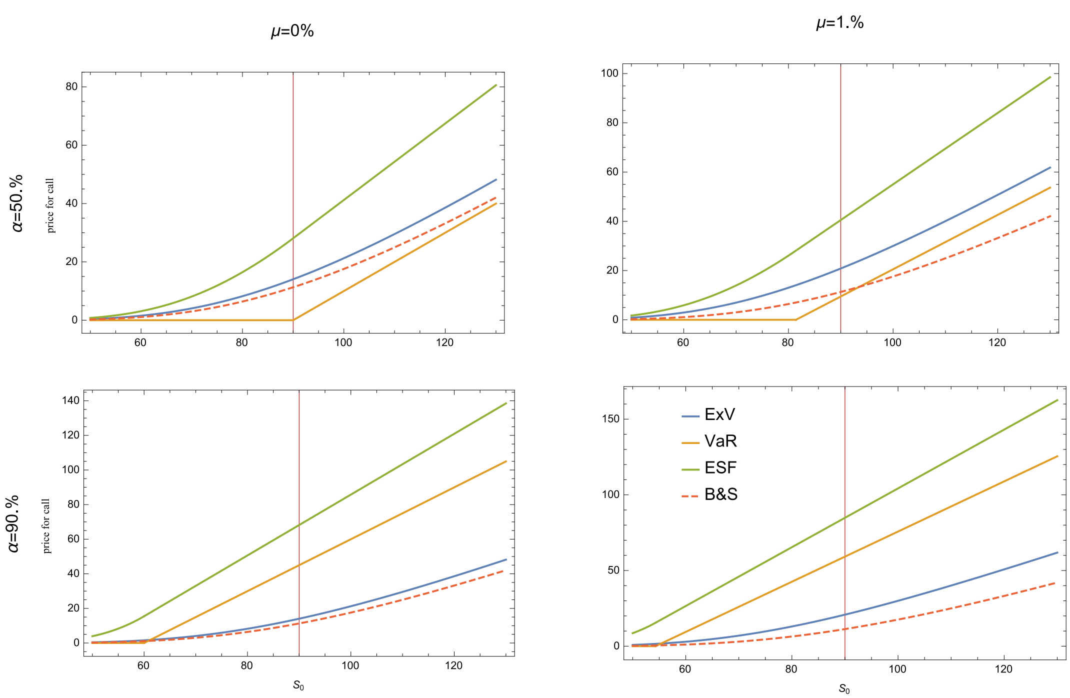

Figure 1: Call price as obtained by B&S pricing model, as well as ExV, VaR, ESF statistical pricing models. All panels have $T = 10$, $\sigma = 10\%$, $r_f = 0$. Other quantities can be read off the graph. The red vertical bar denotes the strike price. Note that ExV and B&S do not use the parameter α.

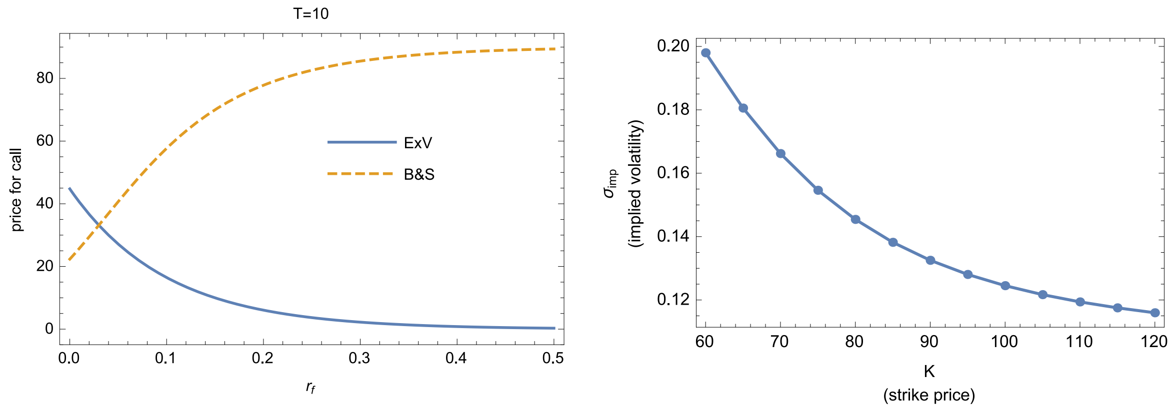

Figure 2: Left panel: B&S and ExV call option price dependence on r_f. Parameters: $K = 90$, $T = 10$, $\mu = 0.01$, $\sigma = 0.2$, $S_0 = 90$. Right panel: The 'volatility skew', viz. the implied volatility as obtained from fitting B&S to ExV call option prices. Parameters: $\mu = 0$, $\sigma = 10\%$, $T = 10$, $S_0 = 100$, $r_f = 0$.

One may also notice that B&S is below ExV, which implies that B&S price will fall below the expected payoff. This is indeed what we shall find in Secs. 3.1 and 3.2, by analyzing performances.

As can be seen in Sec. 1.1, the risk free interest rate (r_f) within the B&S formulae is located beside the standard deviation, in the argument of the cumulative normal distribution function. Due to this, the B&S option price is moderately increasing when r_f is increasing. For very large r_f, the discounting factor $e^{-r_f T}$ keeps the option price low, so to reach an asymptotic constant value. On the other hand, r_f is not present in the argument of the cumulative normal distribution function within ExV approach. Therefore, the dependence of ExV option price on r_f is reversed: The option price decreases when r_f increases. This is driven by the fact that, within ExV, the risk free interest rate plays role only in discounting. The different dependence on r_f of B&S and ExV approaches can be most strikingly observed in Fig. 2, left panel. To this regard, one must however point out that, although μ and r_f are in principle independent parameters, in general one has $\mu \geq r_f$ in the long run. That is because investing in a stock with mean return μ involves a loss risk, which must be generally compensated by higher expected returns with respect to a risk free investment. In the short run, on the other hand, the effective mean return μ can be very different from the long run mean return and one can thus have $\mu < r_f$.

As thoroughly described above, the ExV method yields a price for the option that *exactly* matches the average payoff. One may thus expect the ExV price to be the price the market assigns to the option. On the other hand, by fitting B&S option prices to stock market option prices one expects to obtain the so-called 'volatility' skew (Doran, Peterson, and Tarrant 2007). Then, within this supposition, we expect to find the volatility skew by fitting B&S prices to ExV prices. That is indeed what happens, as we show in Fig. 2, right panel. The curve is obtained by numerically finding the implied volatility (σ_{imp}) out of the equation: $C^{\mathrm{B\&S}}_{\sigma_{imp},r_f} = C^{\mathrm{ExV}}_{\mu,\sigma,r_f}$, where σ, μ are asset characteristics while σ_{imp} is the fitting parameter. The parameters have been set as: $\mu = 0$, $\sigma = 10\%$, $T = 10$, $S_0 = 100$, $r_f = 0$. The volatility skew is anyway obtained for a large set of values of the asset parameters.

3 Testing statistical approaches for assets whose returns are normally distributed

In this section, we shall test the SOPAs that we described in Section 2. While in the first part of the section we shall test the SOPAs with simulated data, in the second part we shall use empirical data. As for the second part, we shall only perform in-sample testing, viz. the test shall be conducted on events that are used to fit the parameters that enter the option pricing formulae. That is because we do not expect a big difference between in-sample and out-of-sample testings for the case under investigation, due to the fact that days and option contracts in the considered pool are many, with not any noticeable change in trend overtime, and with very different payoffs.

3.1 Testing with simulated data

3.1.1 Monte Carlo simulation

We simulated 10000 paths for the asset $S_t = S_0 e^{r_t}$, with $S_0 = 1$, from $t = 0$ to $t = 20$. The continuous return of any day i ($r_{\mathrm{d}i}$) is modeled as the inverse cumulative Normal distributions function, with random argument between 0 and 1. We set mean return $\mu = 0$ and standard deviation

$\sigma = 0.1$. Figure 3, left panel, displays the plot of the path density, as a function of the day and the asset price. Lower and higher quartile, together with median and average values for the asset price are showed. In the right panel of the same figure, the histogram of the simulated prices at maturity is shown, so to see what is the statistic distribution of the asset price at maturity.

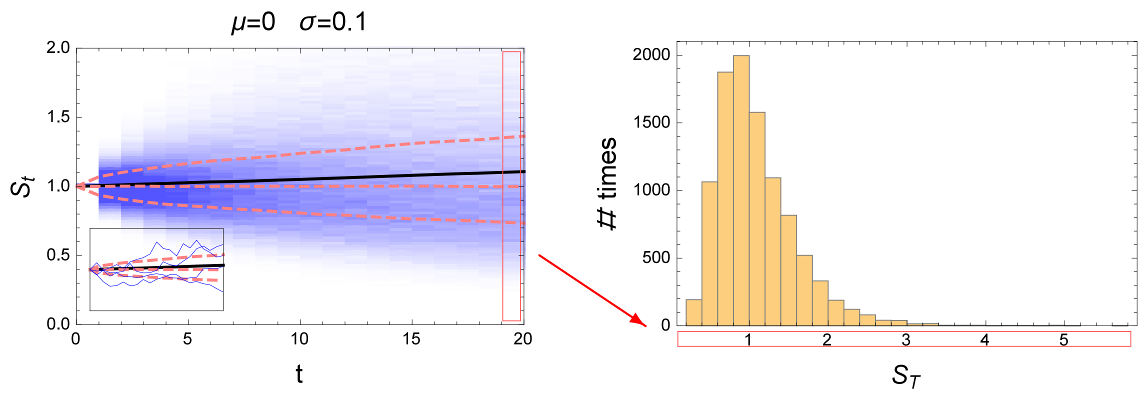

Figure 3: Monte Carlo simulation of 10000 paths for the asset price $S_t = S_0 e^{r_t}$, from t=0 to t=20. The continuous returns r_t are normally distributed with $\mu = 0$, $\sigma = 0.1$, while $S_0 = 1$. Left panel: Density of paths as a function of the day and the asset price, as obtained from the simulated paths. The average price (black solid line), lower and higher quartiles as well as median (pink dashed lines) are showed. In the inset at the bottom left part of the panel, we explicitly show the first four simulated paths. Right Panel: Histogram of the 10000 simulated prices at maturity.

3.1.2 Performance report

Here we shall test prices and performances of B&S, ExV, VaR, ESF option pricing formulae, on the dataset created in the previous subsection. To this purpose, let us suppose that we (the bank) sell a call option to a customer at time $t = 0$. The call option characteristics are: strike price $K = 1.1$, maturity $T = 20$. The strike price has been chosen to approximately coincide with the asset expected value. This latter can be read off Fig. 3, left panel, or alternatively, as we will show in Section 7, can be calculated as $\langle S_{T=20} \rangle = S_0 e^{(\mu + \frac{\sigma^2}{2})T} = e^{0.1} \simeq 1.1$. With this choice for the stike price, the considered situation is realistic: The option at maturity is expected to be at-the-money, and therefore just between being profitable and not profitable. Additionally, for VaR and ESF pricing methods, we set a confidence level $\alpha = 90\%$. We shall test the performances of the

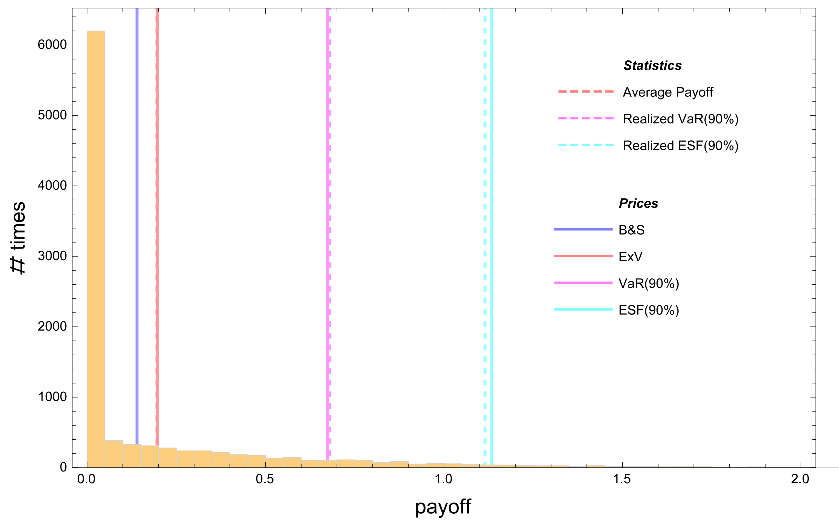

Figure 4: Histogram of option payoffs at maturity of the simulated paths. The average payoff, the realized VaR and the realized ESF are showed, together with the option prices given by B&S, ExV, VaR, ESF approaches.

option pricing methods, by comparing their price with the realized statistics, where these latter are obtained out of the simulation. We shall thus deduce how much money we (the bank) have earned or lost as consequence of the option trades.

As a graphics assessment of the performances related to the various option pricing methods, in Fig. 4 we display the histogram of payoffs at maturity, related to the simulated paths. The realized statistics are displayed as dashed vertical bars. On the other hand, option prices are showed as solid vertical bars. One immediately sees from the figure that B&S price lies below the average option payoff, meaning that pricing with the B&S method would result in a loss for the bank, if the bank does not hedge its position. Furthermore, one can also notice that the option price provided by the ExV method is extremely close to the average option payoff, the difference being $\approx 1\%$ of the average payoff. This is obviously not surprising, since the ExV price, being the (analytical) expected payoff, is supposed to match the average payoff. In turn, this means that by pricing with ExV the bank would neither lose nor gain money from selling the option, in average. This is also in line with ExV pricing method: ExV has been set up such that, in line with the market efficiency hypothesis, there must be no money gain nor loss by the bank. In conclusion, within ExV SOPA, the bank may decide not to hedge the position since its average revenue matches the expected payoff. We continue by analyzing VaR and ESF prices. Both prices lie far higher than the average option payoff. This is because we chose a high (90%) confidence level: If we (the bank) want to profit by selling the option the 90% of times, which is what the VaR(90%) SOPA guarantees, the option price must evidently be quite high. Nevertheless, VaR and ESF price are quite close to their correspondent statistics, thereby showing that both VaR and ESF SOPAs work as expected.

We showed that B&S price is in general lower than the average payoff, and consequently that the bank would lose money by pricing the option via B&S method, if the bank is not hedging the position. Although this statement is correct, one should take into account that the parameter σ that enters the B&S pricing formulae is normally treated as a free parameter to adjust the option price to the market price. In other words, σ entering the B&S formulae is normally not the historical standard deviation, but rather the implied volatility, which in principle should represent the volatility that the market foresees in the next future. Implied volatilities can be bought by external providers, such as Calypso (Calypso .). Even so, here the volatility of the simulated data is fixed at $\sigma = 0.1$ and therefore implied and historical volatilities should coincide.

In terms of numbers, we present a summary in Tab. 1, where realized statistics (outcome of simulation) are listed, together with the option price for any pricing method. The average profit per option is the difference between the price and the average payoff. Numbers that are related to the same physical quantity are colored alike, for an easier comparison. We also show the relative difference price to statistics of reference. As one can see from the table, pricing within the ExV method turns out to be close to the break even point, the difference between ExV price and average

	Statistics	**B&S**	**ExV**	**VaR**	**ESF**
average payoff	0.196				
Realized VaR(90%)	0.679				
Realized ESF(90%)	1.12				
price		0.140	0.198	0.674	1.133
average profit		-0.056	0.002	0.478	0.937
| price-Statistics| /price			1.0%	0.7%	1.1%

Table 1: Statistics and option pricing performances for simulated events.

payoff being close to zero. The difference of 1%, moreover, is totally expected as it is in line with the standard error. This latter can be approximately estimated as $\sim 1/\sqrt{N} = 1/\sqrt{10000} = 1\%$, as standardly done.

The VaR(90%) SOPA fully achieved its goal, since the number of simulated payoffs that were found below the VaR price have been 89.9% of the total, which means that the bank has made profit in 89.9% of cases, which is quite close to the chosen confidence level (90%). This can also be graphically seen directly from Fig. 4. Similarly, for business conducted within ESF pricing method, the risk-management goal is achieved, since the realized average of exceptional payoffs (i.e. the average of those payoffs that lie above the VaR) is located at approximately the ESF price. As seen, both VaR and ESF prices differ by approx 1% from the realized statistics of reference, thereby confirming the accuracy and correctness of the proposed pricing method.

In conclusion: B&S's price was found to be quite far from the average payoff; ExV performance was found to be very good, since ExV profit was close to the break even point (i.e. the average payoff) as wanted; VaR and ESF performances were also good since they almost perfectly estimated the risk margin (i.e. they were very close to the realized statistics). We remark that profits originating from VaR and ESF option pricing were very high due to option overpricing, which in turn is due to the safety measures that we, as bank, wanted to adopt for selling the option, such as 90% confidence level to make profit out of the option business.

3.2 Testing with empirical data

With the purpose to test SOPAs and B&S on empirical data, we collected the daily closing prices of 'Alphabet Inc Class A stock', traded as NASDAQ:GOOGL, from '1st Jan 2010' to '31st Dec 2016'.

3.2.1 Calibration of the normal distribution

From the NASDAQ:GOOGL daily data, we extracted the continuous returns and standard deviation as

$$\mu = \frac{1}{N}\sum_{i=1}^{N} r_{\mathrm{d}i} \simeq 0.053\% , \qquad \sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(\mu - r_{\mathrm{d}i})^2} \simeq 1.556\% . \tag{44}$$

These are the only two parameters which we use to model the asset return distribution. The skewness of the dataset was found to be ≈ 0.8.

3.2.2 Performance report

To test the prices and performances of B&S, ExV, VaR, ESF option pricing methods, we simulated a trade of 10648 call options. The deal's inception day (iday) is picked randomly within '1st Jan 2010' and '30th Dec 2016'. The final settlement day (fday) is picked randomly within iday+1 and the '31st Dec 2016'. The strike price is picked randomly between $0.9S_0e^{-\frac{\sigma^2}{2}T}$ and $1.1S_0e^{+\frac{\sigma^2}{2}T}$, where T= fday - iday = maturity. Additionally, for VaR and ESF pricing methods, we set a confidence level $\alpha = 90\%$.

To have a glance of the performances, we plotted in Fig. 5 the performances for four traded options, randomly picked among the 10648 available. The price of B&S is lower than the option payoff, similarly to what happened in the simulated data (see Sec. 3.1). On the other hand, ExV

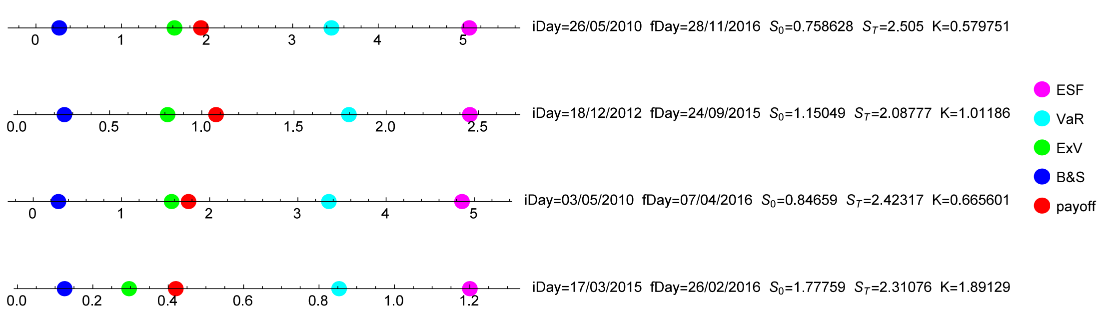

Figure 5: Result of four randomly picked traded options. The option payoff is showed, together with the option price given by B&S, ExV, VaR, and ESF pricing methods.

option price is closer to the option payoff, while VaR and ESF are quite higher than the option payoff, as expected due to the chosen confidence level.

More convincingly, we present in Tab. 2 the average profit within all option pricing methods considered. One can clearly see that B&S option pricing method fell below the average payoff, thus resulting in loss of 0.44569 per traded option for the bank, in case the bank does not hedge. ExV method performed quite well, as its price matches quite precisely the average profit, which is what we aimed at within this approach. The VaR profit turned out to be positive in $\approx$ 96.8% of cases. This result goes slightly above our expectations since, by choosing a confidence level of 90%, it was expected the VaR profit to be positive in just the 90% of cases. Thus the VaR option price based on empirical data was slightly overestimated.

	B&S	**ExV**	**VaR**	**ESF**
average profit	-0.445688	0.0957452	0.616181	1.10792

Table 2: Option pricing performances for empirical events.

We end here the presentation of SOPAs and SOPAs' performances on assets whose returns are normally distributed. We shall proceed in the next sections with considering assets whose returns are AR and ARCH distributed.

4 Assets whose returns are AR distributed

In this section, we shall introduce Auto Regressive (AR) processes, and we shall therewith explore SOPAs. For the sake of brevity, we shall denote by 'AR type assets' those assets whose returns are AR distributed.

4.1 Brief introduction to AR type assets

In AR type assets, the asset mean value is affected by the previous return. For simplicity, we shall restrict ourselves to AR(1) processes. In a AR(1) process, the continuous return for the day t is

(Wei 1994)

$$r_{\mathrm{d}t} = (1-\theta_1)X_t + \theta_1 r_{\mathrm{d}t-1} \, , \tag{45}$$

where $\theta_1 < 1$, while X_t is a white noise with mean value μ and standard deviation σ.

Let us recall some statistics of AR(1) processes. With standard techniques, one can derive that the long term mean return of the asset, μ_r, is equal to mean return of the white noise X_i, i.e. $\mu_r = \mu$. Similarly, the conditional mean, that is the return mean value of day i given the realized return of day $i-1$, can be shown to be $\mu_{\mathrm{d}i} = (1-\theta_1)\mu + \theta_1 r_{\mathrm{d}t-1}$. On the other hand, the long term variance of the asset's return distribution, which we shall call σ_r^2, equals $\sigma_r^2 = \dfrac{(1-\theta_1)^2}{1-\theta_1^2}\,\sigma^2$. Since $\dfrac{(1-\theta_1)^2}{1-\theta_1^2} \leq 1$ within the θ_1 domain, one sees that the variance of the asset is slightly lower than the variance of the white noise X_i. This is expected, since the return of day t, being dependent on the return of day $t-1$, has got some resilience to change trend, whence the reduced volatility. For small θ_1, the long term variance reduces to $\sigma_r^2 \simeq (1-2\theta_1)\sigma^2$. Finally, one can also compute the conditional variance of the asset, i.e. the variance of the $r_{\mathrm{d}i}$ given $r_{\mathrm{d}i-1}$. From Eq. (45), it can be easily calculated as $\sigma_{\mathrm{d}i}^2 = (1-\theta_1)^2\sigma^2$. Thus, the conditional variance of the asset turns out to be independent of the day i considered. If θ_1 is small enough, then $\sigma_{\mathrm{d}i}^2 \simeq (1-2\theta_1)\sigma^2$, and therefore $\sigma_{\mathrm{d}i}^2 \simeq \sigma_r^2$.

The aim of AR(1) model is to catch the market feature for which the return has some 'memory' of last achieved returns, so it partially depends on them. This dependence breaks the Markovianity of the return distribution, and is parametrized by θ_1. One would expect that the older the achieved return, the fainter the 'memory'. This is guaranteed by the fact that a factor θ_1 is added after each day has passed by. More specifically, as seen from Eq. (45), the return of day i depends on the return of day $i-1$ by a factor θ_1, and therefore it (indirectly) depends on the return of day $i-2$ by a factor θ_1^2, and so on. Due to the requirement $\theta_1 < 1$, one immediately sees that $\theta_1^2 < \theta_1$, and therefore the dependence on past days returns goes asymptotically (and exponentially) to zero as time passes by.

To model asset's returns with an AR(1) model, the parameter θ_1 needs to be fitted to the data. To this regard, AR type processes are often calibrated to weekly or monthly return series (see (Fama 1965; Louhelainen 2005) and citations therein) since these latter tend to depend on the previous returns.

4.2 The PDF of the sum of returns in AR(1) type assets

With the aim to find the PDF of the sum of returns in AR(1) type assets, we shall define the function

$$g_R^y(x) = \frac{1}{\sqrt{2\pi(1-\theta_1)^2\sigma^2}}\, e^{-\frac{\left(x-\left[(1-\theta_1)\mu+\theta_1 y\right]\right)^2}{2(1-\theta_1)^2\sigma^2}} . \tag{46}$$

Let us call the continuous return of 0th and 1st day as r_{d0} and r_{d1}, respectively. The PDF of the continuous return of 1st and 2nd day must then be, respectively,

$$\begin{aligned} g_R^{r_{\mathrm{d0}}}(x) &= \frac{1}{\sqrt{2\pi\sigma_{\mathrm{d1}}^2}}\, e^{-\frac{(x-\mu_{\mathrm{d1}})^2}{2\sigma_{\mathrm{d1}}^2}} = \frac{1}{\sqrt{2\pi(1-\theta_1)^2\sigma^2}}\, e^{-\frac{\left(x-\left[(1-\theta_1)\mu+\theta_1 r_{\mathrm{d0}}\right]\right)^2}{2(1-\theta_1)^2\sigma^2}} , \\ g_R^{r_{\mathrm{d1}}}(x) &= \frac{1}{\sqrt{2\pi\sigma_{\mathrm{d2}}^2}}\, e^{-\frac{(x-\mu_{\mathrm{d2}})^2}{2\sigma_{\mathrm{d2}}^2}} = \frac{1}{\sqrt{2\pi(1-\theta_1)^2\sigma^2}}\, e^{-\frac{\left(x-\left[(1-\theta_1)\mu+\theta_1 r_{\mathrm{d1}}\right]\right)^2}{2(1-\theta_1)^2\sigma^2}} . \end{aligned} \tag{47}$$

Then the PDF of the sum of the returns of 1st and 2nd day must be

$$\begin{aligned} G_2^R(s) &= g_R^{r_{\mathrm{d0}}} * g_R^{r_{\mathrm{d1}}} = \int_{-\infty}^{+\infty} \mathrm{d}r_{\mathrm{d1}}\, g_R^{r_{\mathrm{d0}}}(r_{\mathrm{d1}})\; g_R^{r_{\mathrm{d1}}}(s-r_{\mathrm{d1}}) \\ &= \int_{-\infty}^{+\infty} \mathrm{d}r_{\mathrm{d1}} \frac{1}{2\pi(1-\theta_1)^2\sigma^2}\, e^{-\frac{\left(r_{\mathrm{d1}}-\left[(1-\theta_1)\mu+\theta_1 r_{\mathrm{d0}}\right]\right)^2}{2(1-\theta_1)^2\sigma^2}}\; e^{-\frac{\left(s-r_{\mathrm{d1}}-\left[(1-\theta_1)\mu+\theta_1 r_{\mathrm{d1}}\right]\right)^2}{2(1-\theta_1)^2\sigma^2}} . \end{aligned} \tag{48}$$

The subscript $_2$ indicates that this is the PDF of the continuous return of day 2, as projected from day 0. On the other hand, the superscript ^{R} denotes that it is related to AR(1) assets.

The integral in Eq. (48) can be solved analytically by completing the square of the exponential argument. For what follows, we shall suppose to have no information on the return at day 0, so we shall set $r_{\mathrm{d0}} = \mu$, which is our best guess. This supposition is motivated by several facts: i) we would like to have a general PDF for the return after T days, regardless of the particular return r_{d0} that happened at day 0, so to have a wider usability; ii) calculations are slightly easier; iii) the comparison with SOPAs described in Sec. 2, as well as with B&S method, is easier to make; iv) the extension to the case where r_{d0} is left as free parameter is trivial, and so does not add any significant value to the mathematical framework presented. Nevertheless, one must stress that the parameter r_{d0} represents a very useful information for option pricing and should be used when possible. At the end of the day, AR assets differ from assets whose returns are normally distributed because of their dependence on the last achieved return, and therefore it would be natural to incorporate such a dependence in the option price. As we shall mention in Sec. 10, a systematic analysis of the option pricing dependence on past returns within SOPA could be an interesting further study to perform.

Conversely, one must also remark that, although we supposed to have no information on the return of day 0, the resulting PDF of the overall continuous return after T days for AR asset is very different from that one of assets whose returns are normally distributed, where this latter has been presented in Sec. 2. We shall see this in detail in Fig. 6.

Coming back to Eq. (48), with the aforementioned assumption (i.e. with $r_{d0} = \mu$), we obtain the gaussian function

$$G_2^R(s) = \frac{\exp\left(-\frac{(s-2\mu)^2}{2(1-\theta_1)^2\sigma^2\left(\theta_1^2+2\theta_1+2\right)}\right)}{\sqrt{2\pi(1-\theta_1)^2\sigma^2(\theta_1^2+2\theta_1+2)}} \,. \tag{49}$$

The PDF of the sum of the returns until the 3rd day can be analogously obtained, and is found to be also gaussian. Via induction, one can find the distribution of the overall continuous return until any day T. The reasoning goes as follows. Suppose the overall continuous return until day n is of the type

$$G_n^R(s) = \frac{e^{-\frac{(s-\beta_n)^2}{2\gamma_n^2}}}{\sqrt{2\pi\gamma_n^2}} \,, \tag{50}$$

where β_n and γ_n are the mean value and standard deviation of the distribution of the overall continuous return from day 0 **until** day n, where n is arbitrary. These must not be confused with the mean value and standard deviation of the distribution of the continuous return **of** day n, which are $\mu_{dn} = (1-\theta_1)\mu + \theta_1 r_{di-1}$ and $\sigma_{dn} = (1-\theta_1)\sigma$, as we mentioned above. To state a few examples, from above one clearly sees that $\beta_2 = 2\mu$, $\gamma_2 = \sqrt{(1-\theta_1)^2\sigma^2(\theta_1^2 + 2\theta_1 + 2)}$, and also $\beta_1 = \mu$, $\gamma_1^2 = (1-\theta_1)^2\sigma^2$.

Given $G_n^R(s)$ in Eq. (50), the distribution of the overall continuous return until day $n+1$ is easily found as

$$G_{n+1}^R(s) = \int_{-\infty}^{+\infty} \mathrm{d}r_{\mathrm{d}n} G_n^R(r_{\mathrm{d}n})\, g_R^{r_{\mathrm{d}n}}(s - r_{\mathrm{d}n}) = \frac{\exp\left(-\frac{\left(s-(1-\theta_1)\mu-(1+\theta_1)\beta_n\right)^2}{2\left((1-\theta_1)^2\sigma^2+(1+\theta_1)^2\gamma_n^2\right)}\right)}{\sqrt{2\pi\left((1-\theta_1)^2\sigma^2 + (1+\theta_1)^2\gamma_n^2\right)}}\,. \tag{51}$$

This entails $\beta_{n+1} = (1-\theta_1)\mu + (1+\theta_1)\beta_n$, and $\gamma_{n+1}^2 = (1-\theta_1)^2\sigma^2 + (1+\theta_1)^2\gamma_n^2$. Since β_1 and γ_1 are known (see above), one can analytically find β_n and γ_n for any n.

One can check by sight that G_i^R is normalized to 1 for any day i, and that by sending $\theta_1 \to 0$ one obtains G_T as in Eq. (8).

4.3 $G_T(s)$ and $G_T^R(s)$ compared

It is instructive to compare the functions $G_T^R(s)$ and $G_T(s)$, for different T, so to highlight the differences. In Fig. 6, we show the evolution of G_T^R and G_T, as T days have passed by, for $\theta_1 = 0.2$. The function G_T is defined in Eq. (8). One may notice that both distributions widen as time passes by. This is certainly reasonable to expect: As time passes by, there is more uncertainty over the overall continuous return. One may also notice that G_T^R widens faster than G_T, for $T \geq 4$. This is also certainly reasonable to expect: AR(1) distributed returns have memory of past returns, so after a very bad (good) day, returns tend to keep being bad (good) for a while, instead of being back to the mean. This evidently increases the chance to have stark deviations from the mean.

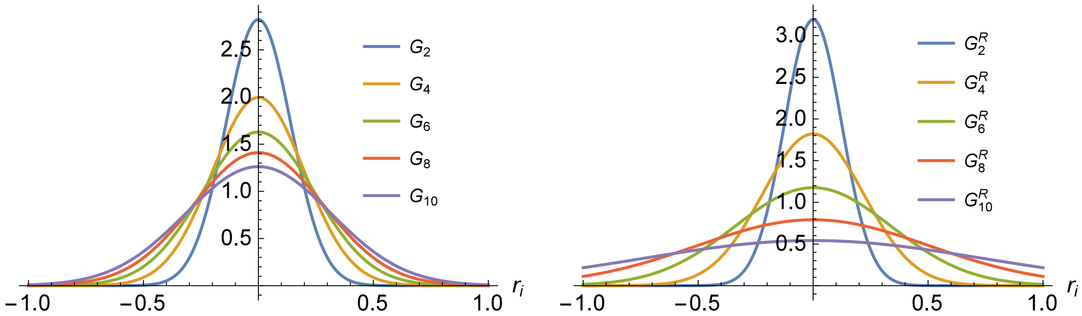

Figure 6: G_T and G_T^R, for different maturities $T \in (2, 4, 6, 8, 10)$. Parameters: $\theta_1 = 0.2$, $\mu = 0$, $\sigma = 0.1$.

4.4 Statistical Expected Value approach for AR(1) type assets (RExV)

By following the same strategy we described in Sec. 2.2, and by using G^R instead, one can investigate call/put option prices within the ExV approach and AR(1) type assets. We shall denote this option pricing method by **RExV**. Within RExV approach, we therefore have

$$\begin{aligned} WCP^R &= \int_{\log \frac{K}{S_0}}^{+\infty} \mathrm{d}r_T \, G_T^R(r_T) \cdot (S_0 e^{r_T} - K) \\ &= S_0 e^{(\beta_T + \gamma_T^2/2)} \, \mathcal{N}(e_1) - K \mathcal{N}(e_2) \end{aligned} \tag{52}$$

where evidently

$$e_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(\beta_T + \gamma_T^2\right)}{\gamma_T} , \qquad e_2 = e_1 - \gamma_T , \tag{53}$$

and where β_T and γ_T have been defined in Eq. (51) and thereafter. Finally, we shall set the option price equal to the discounted WCP^R:

$$C^{\text{RExV}}_{\mu,\sigma,r_f,\theta_1}(T, S_0, K) \equiv \left[S_0 e^{\beta_T + \gamma_T^2/2} \, \mathcal{N}(e_1) - K \mathcal{N}(e_2) \right] e^{-r_f T} . \tag{54}$$

Similarly, one may easily find the put option price

$$\mathcal{P}^{\text{RExV}}_{\mu,\sigma,r_f,\theta_1}(T, S_0, K) \equiv \left[\left(K \mathcal{N}(-e_2) - S_0 e^{\beta_T + \gamma_T^2/2} \, \mathcal{N}(-e_1) \right) \right] e^{-r_f T} . \tag{55}$$

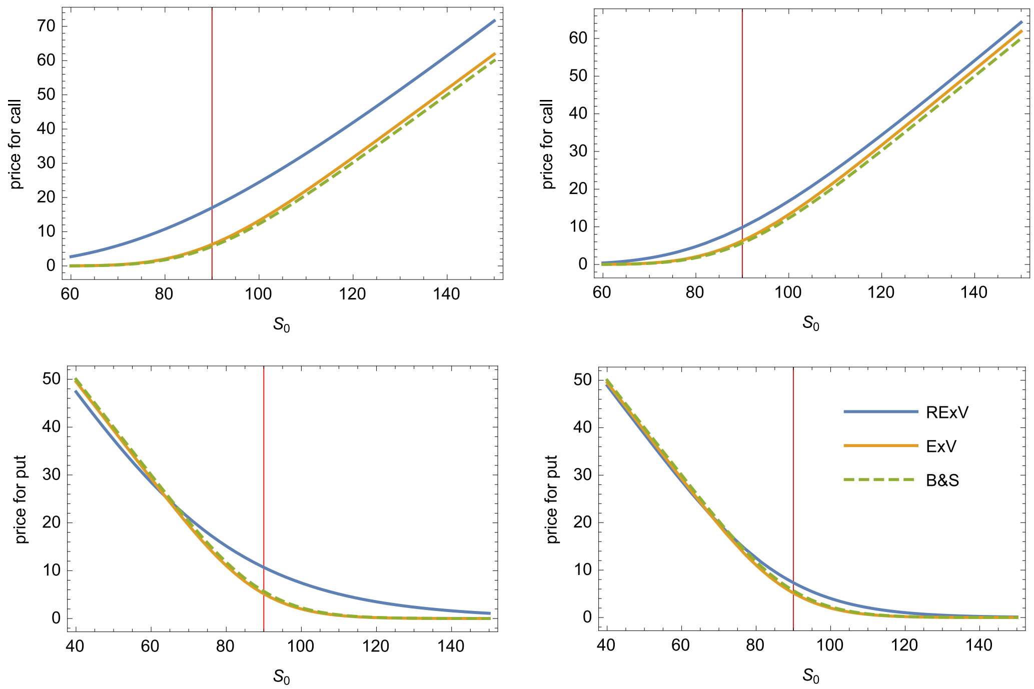

Figure 7: Call (top panels) and Put (bottom panels) option price comparison: RExV, ExV, B&S pricing methods are showed. The AR(1) parameter θ_1 is set as 0.2 (left panels) and 0.1 (right panels). Overall parameters: $T = 10$, $\mu = r_f = 0$, $\sigma = 0.05$. The strike price is denoted with a red vertical bar.

In Fig. 7 a direct comparison between RExV, ExV, B&S call and put option prices is showed. One may notice that RExV turns out to be often (not always) more expensive than ExV and B&S. This is expected due to the analysis we carried out on G_T^R and G_T in Sec. 4.3, where it was showed that AR type assets are expected to have stronger deviations from the mean.

In the following sections, we shall not test the RExV option pricing method that we have here theoretically described. That is due to a twofold reason: i) brevity, ii) we shall test SOPAs for assets whose returns are ARCH distributed in Sec. 6. ARCH processes are more difficult to deal with than AR, although they share some characteristics. They are therefore more challenging to test. For the same reasons, we shall not consider AR type assets in sections 7, 8 and 9.

Finally, the investigation of VaR and ESF SOPAs is also trivially possible within the formalism developed.

5 Assets whose returns are ARCH distributed

In this section, we shall briefly introduce Auto Regressive Conditional Heteroskedastic (ARCH) processes, and we shall therewith explore statistical ExV option pricing approach. For the sake of brevity, we shall denote by 'ARCH type assets' those assets whose returns are ARCH distributed.

5.1 Brief introduction to ARCH type assets

ARCH processes are somewhat similar to AR processes we introduced in Sec. 4, although somewhat more complex. The time series of ARCH type returns does not fulfill the heteroskedasticity requirement, i.e. the return volatility is not constant over time. Hereinafter, we shall focus on ARCH(1) type of processes, although a generalization to ARCH(p) type is possible. For our purposes, it suffices to say that an ARCH(1) asset's daily return ($r_{\mathrm{d}i}$) has its daily variance ($\sigma^2_{\mathrm{d}i}$) that depends on the achieved previous return (Wei 1994):

$$\sigma^2_{\mathrm{d}i} = \sigma^2(1 - \alpha_1) + \alpha_1(r_{\mathrm{d}i-1} - \mu)^2 , \tag{56}$$

where σ, μ are the long term standard deviation and mean return of the asset's return distribution, respectively. These latter are asset specific characteristics. The first term in the right member of Eq. (56) has been chosen so that the average variance of the ARCH process be σ^2. This could be easily showed. On the other hand, α_1 is the ARCH parameter, it ranges within $0 \leq \alpha_1 < 1$, and must be fitted to the data. Such a parameter determines the dependence of the ith day volatility on the deviation of the $(i-1)$th return to the mean value. This term evidently breaks the Markovianity of the return series.

As evident from above, in ARCH(1) type processes, the PDF of the ith day's return depends on the $(i-1)$th day's return, similarly to AR(1) processes. While in AR(1) processes such a dependence was on the PDF's mean value, in ARCH(1) processes the dependence is on the PDF's variance. The aim of the ARCH modeling is to catch the fact that assets undergo periods of high volatility after having deviated substantially from their mean value (Shephard 1996; Baldauf and

Santoni 1991).

For the sake of simplicity, we shall assume $\mu = 0$ for any ARCH asset throughout this section.

5.2 The PDF of the sum of returns in ARCH(1) type assets

Let us define the function

$$g_A^y(x) = \frac{1}{\sqrt{2\pi(\sigma^2(1-\alpha_1)+\alpha_1 y^2)}}\, e^{-\frac{x^2}{2(\sigma^2(1-\alpha_1)+\alpha_1 y^2)}} \quad . \tag{57}$$

Let us further suppose that the continuous return of the 0-th and 1-st day have been equal to $r_{\mathrm{d}0}$ and $r_{\mathrm{d}1}$, respectively. The PDFs of the continuous return of the 1st and 2nd day must then be

$$\begin{aligned} g_A^{r_{\mathrm{d}0}}(x) &= \frac{1}{\sqrt{2\pi\sigma_{\mathrm{d}1}^2}}\, e^{-\frac{x^2}{2\sigma_{\mathrm{d}1}^2}} = \frac{1}{\sqrt{2\pi(\sigma^2(1-\alpha_1)+\alpha_1 r_{\mathrm{d}0}^2)}}\, e^{-\frac{x^2}{2(\sigma^2(1-\alpha_1)+\alpha_1 r_{\mathrm{d}0}^2)}} , \\ g_A^{r_{\mathrm{d}1}}(x) &= \frac{1}{\sqrt{2\pi\sigma_{\mathrm{d}2}^2}}\, e^{-\frac{x^2}{2\sigma_{\mathrm{d}2}^2}} = \frac{1}{\sqrt{2\pi(\sigma^2(1-\alpha_1)+\alpha_1 r_{\mathrm{d}1}^2)}}\, e^{-\frac{x^2}{2(\sigma^2(1-\alpha_1)+\alpha_1 r_{\mathrm{d}1}^2)}} . \end{aligned} \tag{58}$$

The PDF of the sum of the returns is[1]

$$\begin{aligned} G_2^A(s) &= g_A^{r_{\mathrm{d}0}} * g_A^{r_{\mathrm{d}1}} = \int_{-\infty}^{+\infty} \mathrm{d}r_{\mathrm{d}1}\; g_A^{r_{\mathrm{d}0}}(r_{\mathrm{d}1}) g_A^{r_{\mathrm{d}1}}(s - r_{\mathrm{d}1}) \\ &= \int_{-\infty}^{+\infty} \mathrm{d}r_{\mathrm{d}1}\; \frac{e^{-\frac{r_{\mathrm{d}1}^2}{2(\sigma^2(1-\alpha_1)+\alpha_1 r_{\mathrm{d}0}^2)}}}{\sqrt{2\pi(\sigma^2(1-\alpha_1)+\alpha_1 r_{\mathrm{d}0}^2)}} \frac{e^{-\frac{(s-r_{\mathrm{d}1})^2}{2(\sigma^2(1-\alpha_1)+\alpha_1 r_{\mathrm{d}1}^2)}}}{\sqrt{2\pi(\sigma^2(1-\alpha_1)+\alpha_1 r_{\mathrm{d}1}^2)}} \quad . \end{aligned} \tag{59}$$

Unfortunately, we are not aware of any analytical solution of the integral in Eq. (59). The mentioned integral is not present in standard books of reference for integrals (Jeffrey and Hui 2008; Gradshteyn and Ryzhik 2014). Nonetheless, since the integral is well defined, we shall evaluated it numerically in the following.

Equation (59) can be read as follows. $G_2^A(s)\mathrm{d}s$ gives the sum of the probabilities that the two events happen in a row with profit $r_{\mathrm{d}1}$ and $s - r_{\mathrm{d}1}$, respectively, where all possible $r_{\mathrm{d}1}$ are taken

[1] Note that the term ‘convolution’ is properly used only for a set of two independent variables, and therefore it cannot be applied to Eq. (59).

into account. The subscript $_2$ indicates that this is the PDF of the continuous return of day 2, as projected from day 0. On the other hand, the superscript ^{A} denotes that it is related to ARCH(1) assets.

Now, the PDF of the total continuous return until day 3 (i.e. the PDF related to the sum of returns until day 3) can be also built in a similar way:

$$G_3^A(s) \quad = \quad g_A^{r_{\mathrm{d}0}} * g_A^{r_{\mathrm{d}1}} * g_A^{r_{\mathrm{d}2}} = \int_{-\infty}^{+\infty} \mathrm{d}r_{\mathrm{d}2}\, G_2^A(r_{\mathrm{d}2}) g^{r_{\mathrm{d}2}}(s - r_{\mathrm{d}2})\,. \tag{60}$$

The only parameter we need to fix is the starting point of the series: $r_{\mathrm{d}0}$. We shall set it as $r_{\mathrm{d}0} = \mu = 0$, as our best guess. In other words, as already done for AR(1) assets (see Sec. 4.2), we shall disregard any information on the return of day 0. As highlighted before, this choice allows for a more straightforward comparison with methods explored in Sec. 2, as well as with B&S method, beside making calculations slightly easier. As already mentioned, one must nonetheless stress that the parameter $r_{\mathrm{d}0}$ represents a very useful information for option pricing and should be therefore used when possible. As we shall suggest in Sec. 10, a systematic analysis of the option price dependence on past returns in SOPAs could be a possible interesting study to further perform.

Finally, we can build the PDF of the total continuous return until day T as $G_T^A(s) = g_A^{r_{\mathrm{d}0}} * g_A^{r_{\mathrm{d}1}} * ... * g_A^{r_{\mathrm{d}T-1}}$. This is the analogous of Eq. (8) for ARCH(1) assets. We remind the reader that $G_T^A(s)$ is normalized to 1, for each chosen T.

As we now have $G_T^A(s)$ for ARCH(1) processes from Eq. (60) and discussion thereafter, we can in principle perform the same VaR, ExV and ESF analyses on put and call options that we performed for assets whose returns are normally distributed, in Sec. 2. However, for the sake of brevity, after a brief analysis of $G_T^A(s)$ conducted in the next subsection, we shall only analyze how to price call and put options within the ExV approach on ARCH(1) assets. The ExV approach is in fact the most similar approach to the popular B&S model. We shall refer to the ExV approach on ARCH(1) assets as AExV approach.

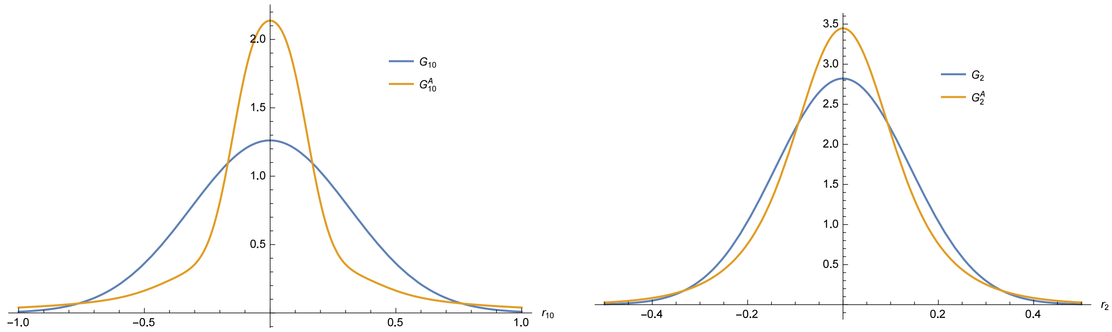

Figure 8: Comparison between G_T^A and G_T, for $T = 10, 2$. Parameters: $\alpha_1 = 0.5$, $\mu = 0$, $\sigma = 0.1$.

5.3 $G_T(s)$ and $G_T^A(s)$ compared

It is instructive to have a look at how the function $G_T^A(s)$ compares to $G_T(s)$, for different maturities T. The function G_T is defined in Eq. (8). In Fig. 8, we show the comparison between G_T^A and G_T, for $T = 2, 10$, $\alpha_1 = 0.5$. As for the computation of G_i^A, each day involves a numerical integration, so the overall precision drops after each day. The error has been kept below 1% for any day. From the figure one may notice that both G_T^A and G_T widen as time passes by. This is certainly reasonable to expect: As time passes by, there is more uncertainty over the overall continuous return.

In the left panel, one can see that G_{10}^A has fatter tails than G_{10}. This is expected, since the non-constant variance can potentially give rise to returns that are far from the mean value. We can also notice that G_{10} has wider width close to the peak than G_{10}^A. Overall, this means that a) ARCH distributed returns tend to stay closer to the mean in general; b) those few times that they differ from the mean, they can do it significantly, that is by a lot more than what normally distributed returns usually do. This feature embeds some psychological market behaviour related to the (non-markovian) momentum that markets possess: When daily returns differ significantly from the mean, the market seems more uncertain as to what the next returns could be. In the right panel, on the other hand, we see that the differences between G_2^A and G_2 are similar to those of the left panel, although much less pronounced. We may also remark that $G_1^A = G_1$ by construction .

Finally, we verified that G_T^A coincides with G_T for any T, when $\alpha_1 = 0$, within the numerical error.

5.4 Statistical Expected Value approach for ARCH(1) type assets (AExV)

Although ARCH models are quite popular (Hamilton and Susmel 1994), they seem to be incompatible with the FOPAs (Kallsen and Taqqu 1998). To the best of our knowledge, there is not a commonly accepted method to price options for ARCH type assets. More generally, option pricing for assets with stochastic volatility (such as ARCH assets) may be still considered an open problem in quantitative finance (Hull and White 1987), although some progress has been made recently (Perelló, Sircar, and Masoliver 2008; Benhamou, Gobet, and Mohammed 2010; Christoffersen, Heston, and Jacobs 2009; Lewis 2000; Lewis 2016). This section is therefore a new attempt to find a solution to this problem.

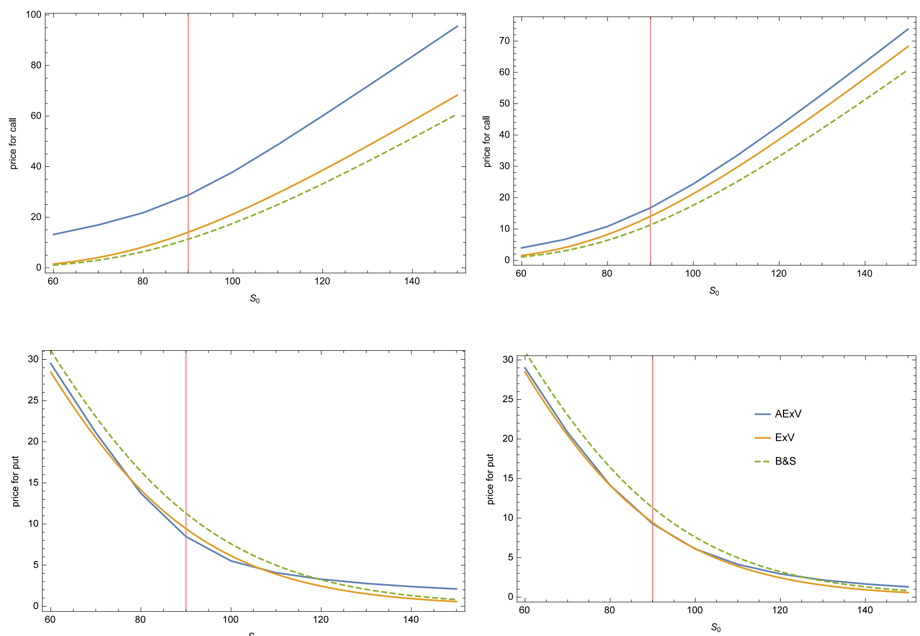

Figure 9: Call (top panels) and Put (bottom panels) option price comparison: AExV, ExV, B&S are showed. The ARCH(1) parameter α_1 is set as 0.5 (left panels) and 0.1 (right panels). Overall parameters: $T = 10$, $\mu = r_f = 0$, $\sigma = 0.1$. The strike price is denoted with a red vertical bar.

5.5 Call and Put options within AExV approach

Here we shall explore ExV SOPA for ARCH(1) assets (AExV). Similarly to what done in Eq. (10), the weighted call payoff for ARCH(1) type assets is

$$\begin{aligned} WCP^A &= \int_{\substack{\text{where} \\ S_0 e^{r_T} - K < 0}} \mathrm{d}r_T \; G_T^A(r_T) \cdot 0 + \int_{\substack{\text{where} \\ S_0 e^{r_T} - K \geq 0}} \mathrm{d}r_T \; G_T^A(r_T) \cdot (S_0 e^{r_T} - K) \\ &= \int_{\log \frac{K}{S_0}}^{+\infty} \mathrm{d}r_T \; G_T^A(r_T) \cdot (S_0 e^{r_T} - K) \ . \end{aligned} \tag{61}$$

As it is not possible to proceed with analytical integration, we shall proceed numerically. The option price will then be the discounted WCP^A, that is

$$C^{\text{AExV}}_{\mu,\sigma,r_f,\alpha_1}(T, S_0, K) \equiv WCP^A \; e^{-r_f T} \ . \tag{62}$$

As for the put option, we shall proceed similarly to what done in Eq. (15). The option price will turns out to be

$$\mathcal{P}^{\text{AExV}}_{\mu,\sigma,r_f,\alpha_1}(T, S_0, K) \equiv e^{-r_f T} \int_{-\infty}^{\log \frac{K}{S_0}} \mathrm{d}r_T \; G_T^A(r_T) \cdot (K - S_0 e^{r_T}) \ . \tag{63}$$

The call and put option prices as obtained by AExV, ExV, B&S, are showed in Fig. 9, for $\alpha_1 = 0.5$ and $\alpha_1 = 0.1$. We easily notice that the AExV call prices turn out to be often (not always) the most expensive. That is expected, to some extent, since, as remarked above, ARCH processes open up the possibilities to deeper tail events, due to time dependent volatility (see Fig. 8).

6 Testing statistical approaches for ARCH type assets

Similarly to Sec. 3, we shall here investigate prices and performances of AExV and B&S option pricing methods, with simulated and empirical data.

6.1 Testing with simulated data

6.1.1 Monte Carlo simulation

We simulated 10000 paths for the asset $S_t = S_0 e^{r_t}$, with $S_0 = 1$, from $t = 0$ to $t = 10$. We chose a shorter maturity with respect to Sec. 3 to reach numerical convergence rapidly. The continuous return of any day i ($r_{\mathrm{d}i}$) is modeled as the inverse cumulative Normal distributions function whose variance is characterized by ARCH(1) modeling, with random argument. We set mean return $\mu = 0$, standard deviation $\sigma = 0.1$, and the ARCH(1) parameter as $\alpha_1 = 0.5$.

In Fig. 10, left panel, we display the plot of the path density, quartiles, average values out of the simulation are showed. In the right panel, the histogram of the 10000 simulated prices at maturity is shown, so to see what is the statistic distribution of the asset price at maturity.

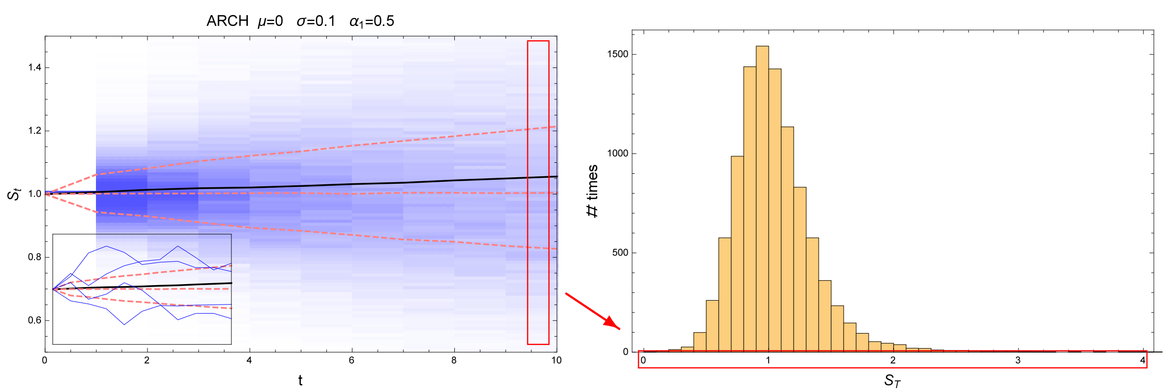

Figure 10: Monte Carlo simulation of 10000 paths for the asset price $S_t = S_0 e^{r_t}$, from t=0 to t=10. The continuous returns r_t are ARCH(1) distributed with $\mu = 0$, $\sigma = 0.1$, $\alpha_1 = 0.5$, while $S_0 = 1$. Left panel: Density of paths as a function of the day and the asset price, as obtained from the simulated paths. The average price (black solid line), lower and higher quartiles as well as median (pink dashed lines) are showed. In the inset at the bottom left part of the panel, we explicitly show the first four simulated paths. Right Panel: Histogram of the 10000 simulated prices at maturity.

6.1.2 Performance report

Here we shall test prices and performances of B&S and AExV option pricing formulae, on the dataset created in the previous subsection. To this purpose, let us suppose that we (the bank) sell a

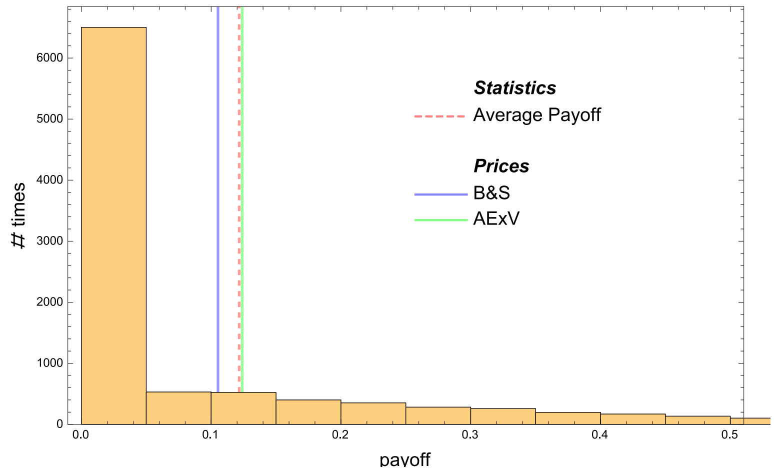

Figure 11: Histogram of payoffs at maturity, related to the simulated paths. The average payoff is also showed, together with the option prices given by B&S and AExV approaches.

call option to a customer at time $t = 0$. The call option characteristics are: strike price $K = 1.05$, maturity $T = 10$. We shall compare B&S and AExV prices with the average payoff, where this latter is obtained out of the simulation. We shall thus deduce how much money we (the bank) have earned or lost by the option trade, in case the we choose not to hedge the position.

As a graphics assessment of the performances related to the option pricing methods we considered, in Fig. 11 we display the histogram of payoffs at maturity, related to the ARCH(1) simulated paths. The average payoff is also displayed as vertical red-dashed bar, together with the option prices given by B&S, and AExV approaches. One may see from the figure that B&S price lies below the average option payoff, meaning that pricing with the B&S method would result in a loss for the bank that does not hedge its position. One may also see that AExV lies very close to the average option payoff, as expected.

In terms of numbers, we present a summary in Tab. 3, where average option payoff (outcome of simulation) is listed, together with the option price for the B&S and AExV pricing methods. The average profit per option is the difference between the price and the average payoff. As one can see from Tab. 3, the business conducted within the B&S pricing method without hedging results in a loss of 170, in units of S_0. On the other hand, pricing within the AExV method turns out to be very convenient, being it very close to the average payoff.

In conclusion: B&S's price was quite far from the average option payoff; AExV performance was high since it was very close to the expected payoff.

	Statistics	**B&S**	**AExV**
average payoff	0.122		
price		0.105	0.124
average profit		-0.017	0.002

Table 3: Option pricing performances for simulated events in the case of ARCH distributed returns.

6.2 Testing with empirical data

We collected the daily closing prices of 'Alphabet Inc Class A stock', traded as NASDAQ:GOOGL, from '1st Jan 2010' to '31st Dec 2016'.

6.2.1 Calibration of the ARCH(1) distribution

Given the time series of daily closing prices, we extracted the continuous returns ($r_{\mathrm{d}i}$, for any day i), and the estimations for the parameters σ and μ. As already reported in Sec. 3.1, the estimations for mean continuous return and long term standard deviation are found to be $\mu \simeq 0.053\%$ and $\sigma \simeq 1.556\%$, respectively. Finally, α_1, defined in Eq. (56) needs also to be estimated. This can be done via maximizing the Log-Likelihood. The Log-Likelihood of a set of N daily asset prices is found to be

$$\log \mathcal{L} = -\sum_{i=1}^{N} \frac{(r_{\mathrm{d}i} - \mu)^2}{2[\sigma^2(1-\alpha_1) - \alpha_1(r_{\mathrm{d}i-1} - \mu)^2]} - \frac{1}{2}\sum_{i=1}^{N} \log\left(2\pi[\sigma^2(1-\alpha_1) + \alpha_1(r_{\mathrm{d}i-1} - \mu)^2]\right), \quad (64)$$

where one may set $r_{\mathrm{d}0} = \mu$ for convenience. Maximizing $\log \mathcal{L}$ yields the estimate for the parameter α_1. By doing this, one finds that the max-likelihood estimate for α_1 must satisfy the following equation

$$0 = \sum_{i=1}^{N} \frac{\left[\sigma^2 - (r_{\mathrm{d}i-1} - \mu)^2\right]\left[\sigma^2(1-\alpha_1) + \alpha_1(r_{\mathrm{d}i-1} - \mu)^2 - (r_{\mathrm{d}i} - \mu)^2\right]}{\left[\sigma^2(1-\alpha_1) + \alpha_1(r_{\mathrm{d}i-1} - \mu)^2\right]^2}, \quad (65)$$

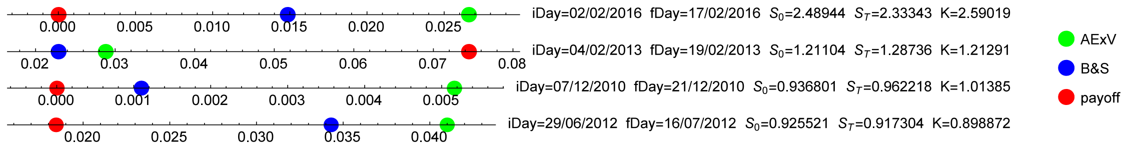

Figure 12: Description of four randomly picked traded options. The option payoff is showed, together with the option price of B&S and AExV pricing methods.

where N is the number of daily continuous returns in the dataset. The equation above needs in general be solved numerically with the constraint $0 \le \alpha_1 < 1$. Within the considered dataset, the α_1 that satisfies Eq. (65) is numerically found to be $\alpha_1 \approx 0.154413$.

6.2.2 Performance reports

To test the performances of B&S against AExV option pricing method, let us suppose that we (the bank) sell 10000 call options to customers. The deal's inception day (iday) is picked randomly within '1st Jan 2010' and '21th Dec 2016'. The final settlement day (fday) is set as iday + 10 working days, which is to say that we choose a maturity of 10 days. This choice is motivated by numerical convergency. The strike price is picked randomly between $0.9S_0e^{-\frac{\sigma^2}{2}T}$ and $1.1S_0e^{+\frac{\sigma^2}{2}T}$, where T= fday - iday = 10 working days = maturity. We must stress the fact that we do not re-calibrate the ARCH model depending on the time window used. Rather, the α_1 parameter, as well as μ and σ, are taken as calculated in Sec. 6.2.1 with the whole dataset. Finally, in measuring the performances we used G_T^A as defined at pg. 36 since $\mu \approx 0$.

To have a glance of the performances, we plotted in Fig. 12 the performances for four traded options, randomly picked among the 10000 available. Furthermore, we present a summary in Table 4, with the average profit of both B&S and AExV option pricing methods.

To summarize, AExV performed twice better than B&S since it lies closer to zero profit, if no hedging is pursued after having sold the option. Matching zero profit is indeed the aim here, as already done in the previous sections, since one expects that the option is traded with zero profit margin, in efficient markets.

	B&S	AExV
average profit	-0.00966	-0.00457

Table 4: Option pricing performances for empirical events in the case of ARCH distributed returns.

7 Put-Call parity relation

In this section we shall analyse the Put-Call Parity Relation (PCPR). We shall show how the PCPR works within B&S model and within ExV SOPAs we described in the previous sections.

Let us suppose that at $t = 0$ the bank A writes a put ($\mathcal{P}$) and buys a call ($\mathcal{C}$) for the same strike price K. The payoff of bank A's position at time $t = T$ is surely $S_T - K$:

$$\left(\mathcal{C} - \mathcal{P}\right)_{\text{payoff at } t=T} = S_T - K \, , \tag{66}$$

where S_T is the underlying's price at time T. This is the starting point of the argumentation for the put-call parity relation.

7.1 Put-Call parity relation in the B&S approach

As mentioned above, the payoff of bank A's position at time T is surely $S_T - K$, if at $t = 0$ the bank A writes a put and buys a call for the same strike price K. Let us now consider bank B that at time $t = 0$ buys the stock share at price S_0 and takes a loan such that the debt to repay at maturity T is K. At time T, bank B sells the share and pays back the loan. The payoff of bank B's position at maturity is again surely $S_T - K$.

Since the payoffs at maturity T of these two positions are with certainty the same, *no-arbitrage argument* requires that the cost of them at initial time (i.e., at $t = 0$) must be the same. The cost of bank B's position at time $t = 0$ is the stock price S_0 minus the discounted debt. As here we consider that bank B never defaults, the loan yield that the bank B is supposed to pay must be just equal to r_f. So this reasoning entails the equation

$$S_0 - Ke^{-r_f T} = \Big(\text{buy call + sell put}\Big)_{\text{cost at } t=0} = \Big(\text{price call - price put}\Big)\,, \tag{67}$$

$$\Rightarrow \Big(\text{price call - price put}\Big)_{\text{B\&S}} = S_0 - Ke^{-r_f T}\ . \tag{68}$$

The B&S option pricing model satisfies Eq. (68). That is why we added the subscript $(...)_{\text{B\&S}}$ to the left member of the equation.

7.2 Put-Call parity relation in statistical ExV approach

7.2.1 Expected value of the Stock price

Before speaking about PCPR in ExV SOPAs, we need to discuss what the expected value of the stock price at some future time T is. Suppose we know the stock price at time $t = 0$, i.e. S_0. The stock price at time $t = T$ will be $S_T = S_0 e^{r_T}$. However, since r_T is not know at time $t = 0$, what we aim to know at time $t = 0$ is just the expected value for the stock price at time $t = T$, which we denote by $\langle S_T \rangle$. By using Eq. (8), the quantity $\langle S_T \rangle$ can be readily calculated

$$\begin{aligned} \langle S_T \rangle &= \int_{-\infty}^{+\infty} \mathrm{d}r_T S_0 e^{r_T} G_T(r_T) = S_0 \int_{-\infty}^{+\infty} \mathrm{d}r_T e^{r_T} \frac{1}{\sqrt{2\pi T\sigma^2}} e^{-\frac{(r_T - T\mu)^2}{2T\sigma^2}} \\ &= S_0 e^{\left(\mu + \frac{\sigma^2}{2}\right)T} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} = S_0 e^{\left(\mu + \frac{\sigma^2}{2}\right)T}\ . \end{aligned} \tag{69}$$

Thus, even though the expected continuous return is μ, the asset price is expected to grow with a factor $e^{(\mu + \sigma^2/2)T}$. This effect is very well known.

7.2.2 To the Put-Call parity relation

Let us once again suppose that at $t = 0$ the bank A writes a put and buys a call for the same strike price K. As stated above, the payoff of bank A's position at time T is surely $S_T - K$. Let us take the expected values of both members in Eq. (66):

$$\langle \left(\mathcal{C} - \mathcal{P}\right)_{\text{payoff at } t=T} \rangle = \langle S_T - K \rangle \ . \tag{70}$$

Considering that the expected value is a linear function, the equation above entails

$$\langle \mathcal{C}_{\text{payoff at } T} \rangle - \langle \mathcal{P}_{\text{payoff at } t=T} \rangle = \langle S_T \rangle - \langle K \rangle \ . \tag{71}$$

Now, we may proceed as

1. $\langle K \rangle = K$ since K is a number,
2. $\langle S_T \rangle = \int_{-\infty}^{+\infty} \mathrm{d}r_T S_0 e^{r_T} G_T(r_T) = S_0 e^{\left(\mu + \frac{\sigma^2}{2}\right)T}$ from Sec. 7.2.1,
3. $\langle \mathcal{C}_{\text{payoff at } t=T} \rangle \, e^{-r_f T}$ is precisely the call option price in the ExV approach,
4. $\langle \mathcal{P}_{\text{payoff at } t=T} \rangle \, e^{-r_f T}$ is precisely the put option price in the ExV approach.

We therefore may conclude

$$\left(\text{price call} - \text{price put}\right)_{\text{ExV}} = S_0 e^{\left(\mu - r_f + \frac{\sigma^2}{2}\right)T} - K e^{-r_f T} \ , \tag{72}$$

where we added the subscript $(...)_{\text{ExV}}$ to the left member, since ExV option pricing model described in Sec. 2.2 satisfies this equation.

As remark, the fact that B&S and ExV approaches lead to two different PCPR might suggest that one of the two must be wrong, or that one of the two leads to arbitrage. However, although they definitely lead to slightly different put-call price relations, they both start from the same put-call parity statement that is universally recognized (Klemkosky and Resnick 1979), which is Eq. (66), and then add some assumptions. Inasmuch as the assumptions are different, the final PCPRs are different. We may also remark that Put-Call parity relations can be tested in financial markets (Nissim and Tchahi 2011), so as to verify which one fits more closely market data.

7.3 Put-Call parity relation in statistical AExV approach

AExV approach is similar to ExV. As a matter of fact, Eq. (71) is valid also in AExV. However, the steps we made thereafter must be reformulated as follows:

1. $\langle K \rangle = K$ since K is a number,
2. $\langle S_T \rangle = \int_{-\infty}^{+\infty} \mathrm{d}r_T S_0 e^{r_T} G_T^A(r_T)$ from Sec. 5.2, which is not analytically reduceable,
3. $\langle C_{\text{payoff at } t=T} \rangle \, e^{-r_f T}$ is precisely the call option price in the AExV approach,
4. $\langle \mathcal{P}_{\text{payoff at } t=T} \rangle \, e^{-r_f T}$ is precisely the put option price in the AExV approach.

We therefore have:

$$\Big(\text{price call} - \text{price put}\Big)_{\text{AExV}} = e^{-r_f T} \int_{-\infty}^{+\infty} \mathrm{d}r_T S_0 e^{r_T} G_T^A(r_T) - K e^{-r_f T} \ , \qquad (73)$$

where we added the subscript $(...)_{\text{AExV}}$ to the left member to make it clear that this equation is satisfied by the option prices as found within the AExV approach. This concludes the PCPR analysis in AExV approach.

8 A few new Merton-type structural models

Structural models, amongst which the Merton Model is probably the most known (Merton 1974), are a type of credit risk models that attempt to model the probability to default (PD) by giving a reason for the default, which is derived from the stochastic random motion of firm's market value. Consequently, the debt and equity of the firm are priced out, so as to find the firm's credit spread.

8.1 The Merton idea and the original Merton model

The original Merton model for the credit spread (CS) can be built in the following way. One may notice that the payoff of a company debt (B) at time T is equal to the payoff of the position of having a written put option on the company market value with strike price equal to the debt at inception (D), plus the discounted debt at inception in cash:

$$\begin{aligned}\Big(\text{company debt}\Big)_{\text{payoff at t=T}} &= \Big(\text{written put with strike } D\Big)_{\text{payoff at t=T}} + D \\ &= -\Big(\text{bought put with strike } D\Big)_{\text{payoff at t=T}} + D\,.\end{aligned} \tag{74}$$

By using of no-arbitrage requirements, the cost of both positions at inception (t=0) must be the same:

$$B \equiv \Big(\text{company debt}\Big)_{\text{price at t=0}} = -\Big(\text{price put with strike } D\Big) + De^{-r_f T}\,. \tag{75}$$

The underlying assumption of the above equations is that the firm's market value, and therefore firm's market continuous return, is a random variable. Such a random variable can be taken normally distributed, or also ARCH(1) type distributed, as we shall do in Sec. 8.4.

One may then model the debt price at inception as $B = De^{-(r_f + CS)T}$, as typically done in financial markets to define the credit spread CS. By doing so, one finds the expression for CS.

8.2 Merton Model within financial B&S approach

Let us define the firm's current market value as V_0, as well as the firm's volatility as σ. By using the B&S formula for the put option price, one finds the following expression for the CS (Merton 1974):

$$CS_{\text{B\&S}} = -\frac{1}{T}\log\left(\mathcal{N}(d_2) + \frac{V_0}{De^{-r_f T}}\mathcal{N}(-d_1)\right)\,, \tag{76}$$

where

$$d_1^M = \frac{\log(V_0/D) + (r_f + \sigma^2/2)T}{\sigma\sqrt{T}}\,, \qquad d_2^M = d_1 - \sigma\sqrt{T}. \tag{77}$$

This is the Merton model credit spread.

8.3 A Merton-type structural model within statistical ExV approach

Let us here further define the firm's mean return as μ. By doing the same steps as in Sec. 8.1, but with the ExV SOPA instead of the B&S model, we find

$$CS_{\text{ExV}} = -\frac{1}{T}\log\left(\mathcal{N}(c_2) + \frac{V_0}{De^{-(\mu+\sigma^2/2)T}}\mathcal{N}(-c_1)\right) , \tag{78}$$

where

$$c_1^M = \frac{\log(V_0/D) + \left(\mu + \sigma^2\right)T}{\sigma\sqrt{T}} , \qquad c_2^M = c_1 - \sigma\sqrt{T} . \tag{79}$$

This concludes the analysis of the Merton-type structural model within ExV approach.

8.4 A Merton-type structural model within statistical AExV approach

Here, let us define also the firm's ARCH(1) parameter α_1. Once more, we shall do the same steps as Sec. 8.1. However, since AExV is not analytically solvable, we must start from Eq. (75), with the option price calculated via AExV approach:

$$B = -(\text{price of put option with strike } D)_{AExV} + De^{-r_f T} = -\mathcal{P}^{AExV}_{\mu,\sigma,r_f,\alpha_1}(T, S_0, D) + De^{-r_f T} . \tag{80}$$

By writing $B = De^{-(r_f+CS)T}$, we find

$$CS_{\text{AExV}} = -\frac{1}{T}\log\left[\frac{e^{r_f T}}{D}\left(- \mathcal{P}^{AExV}_{\mu,\sigma,r_f,\alpha_1}(T, S_0, D) + De^{-r_f T}\right)\right] = -\frac{1}{T}\log\left(1 - \frac{WPP^A}{D}\right) . \tag{81}$$

This concludes the analysis of the Merton-type structural model within AExV approach.

8.5 Comparison and conclusions

The three different 'versions' of the Merton-type Model we explored in the previous sections are compared in Fig. 13. The curve related to AExV has been interpolated with *spline* functions. The error of each $G^A_{T=1,\ldots,10}$ function has been kept below 1%.

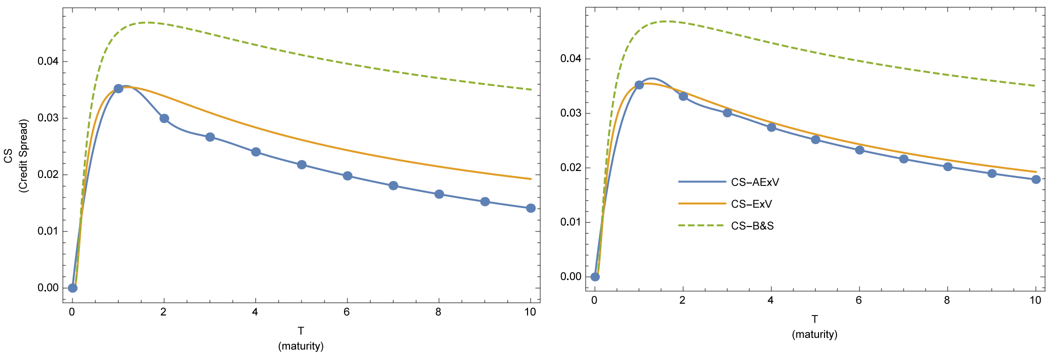

Figure 13: Credit Spread as obtained from AExV, ExV and B&S Merton approaches. Parameters are $V_0 = 1$, $D = 0.8$, $\sigma = 0.3$, $r_f = \mu = 0$. The ARCH parameter α_1 is set as 0.5 (left panel) and 0.1 (right panel).

All approaches present the hump shape that is typical of the CS obtained from the Merton approach. As known, the hump curve suggests that the creditworthiness of a firm has a minimum for a certain maturity. Such a minimum evidently corresponds to the maturity value for which CS attains its maximum. Saying it more simply, a firm that receives a loan with maturity T has statistically the highest probability to default on that loan if T coincides with the maturity that yields the maximum Merton's CS. On the other hand, the convergence of CS to zero for long maturities indicates that it is statistically less probable to default on loans with long maturities, which is related to the growth of the firm market value, as seen in Sec. 7.2.1. Finally, it is also improbable to default on loans with short maturities (T$\simeq$ 0), since the random fluctuations would in that case not be enough to bring the firm's market value below the debt at inception D, provided that $V_0 > D$. This is related to the fact that the Merton model does not consider discrete jumps in the firm market value, since the dynamics of this latter is based on a step that is infinitesimal in time.

The ExV and AExV curves lie below the B&S curve, and converge to zero faster than B&S for large maturities. This suggests that ExV and AExV approaches give more creditworthiness to firms than B&S approach does, for any loan maturity T. A further remark is that AExV coincides with ExV for T=1, because $G_1^A = G_1$, as already remarked at pg. 37.

We would like to stress that one could analogously build another (new) Merton-type structural model by using AR(1) option pricing method that we described in Sec. 4, or any other process whose joined PDF is known, along the same lines we presented here. One could also calculate the CS confidence interval related to a certain confidence level α by using the VaR SOPA for the option price.

9 Options sold by real banks: the Credit Spread in the option price

As mentioned at pg 10, we (the bank) have limited resources, so we might in principle default. If we default, we will not give money to the customer who bought an option from us that turns out to be in-the-money at maturity. Until now we did not consider this, and therefore the option prices we investigated only apply to an ideal bank that never defaults. In order to apply to a real bank, or any real option issuer, we should discount the option price by the issuer's CS, also called 'yield spread' (Gatev and Strahan 2006). We shall describe how to do this in the present section.

If options are sold in secondary markets, the CS could depend on the margin that the clearing house requires the bank to deposit and to maintain, so to trade the option. We shall completely neglect such a dependence here. In other words, we shall consider options that are sold Over-The-Counter (OTC).

The procedure we are going to describe can be applied to B&S option pricing method, as well as any of the expected values approaches we presented so far, namely ExV, AExV, and RExV. Nevertheless, in order to be clearer and more concise, we shall only consider ExV. Moreover, as we shall explain later on in Sec. 10, the same study can be conducted with any model for which the joint PDF of returns for different days is known.

9.1 Defining the Credit Spread

Let us take a call option. We shall use the CS as defined in Section 8 within ExV approach, i.e. Eq. (78). Now, the CS defined in Eq. (78), depends on the debt D to be paid at maturity T. So the first question to be answered is: "How much is the debt D of the bank to be paid at maturity T to the customer who bought the option?" Such a debt is unfortunately not defined yet at time $t = 0$, since it will be just the payoff at maturity T, i.e. $D = \mathrm{Max}[0, S_0 e^{rT} - K]$. The debt to the customer is thus only known at maturity T. Nevertheless, we can calculate the *expected* debt to be paid at maturity T, denoted by $\langle D\rangle$, which is nothing but the ExV call option price with no risk free discounting:

$$\langle D\rangle = S_0 e^{T(\mu+\sigma^2/2)}\, \mathcal{N}(c_1) - K\mathcal{N}(c_2)\,, \tag{82}$$

where, as in Eq. (12), the arguments are

$$c_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(\mu + \sigma^2\right)T}{\sigma\sqrt{T}}\,, \qquad c_2 = c_1 - \sigma\sqrt{T}\,. \tag{83}$$

Therefore the CS of the bank that sold the option to the customer, within the ExV Merton-like model for the CS, can be taken as

$$CS^{call}_{\mathrm{ExV}}(V_0) = -\frac{1}{T}\log\left(\mathcal{N}(c_2') + \frac{V_0}{\langle D\rangle\, e^{-(\mu+\sigma^2/2)T}}\mathcal{N}(-c_1')\right)\,, \tag{84}$$

where

$$c_1^{M'} = \frac{\log\left(V_0/\langle D\rangle\right) + \left(\mu + \sigma^2\right)T}{\sigma\sqrt{T}}\,, \qquad c_2^{M'} = c_1 - \sigma\sqrt{T}\,. \tag{85}$$

and where $\langle D\rangle$ is defined in Eq. (82), while V_0 is the market value of the bank at initial time.

9.2 Bank market value is given externally

Now we are ready to present SOPAs with Credit Spread in the price. In order to do that, we will just replace the discounting factor $e^{-r_f T}$ with the factor $e^{-\left(r_f+CS\right)T}$, in all discounting processes that we did so far, where $CS = CS^{call}_{\text{ExV}}(V_0)$ as defined in Eq. (84). The ExV call option price with CS included would be

$$C^{CS_{\text{ExV}},\,\text{ExV}}_{\mu,\sigma,r_f}(T, S_0, K, V_0) = \left[S_0 e^{T(\mu+\sigma^2/2)}\, \mathcal{N}(c_1) - K\mathcal{N}(c_2)\right] e^{-\left(r_f + CS^{call}_{\text{ExV}}(V_0)\right)T}\ , \tag{86}$$

where $c_{1,2}$ are defined in Eq. (12).

In this section, we have considered that the option price will *not* increase the market bank value V_0. The market value V_0 is in fact given as external parameter. This is true if the bank does not use the money raised by selling the option to increase its market value (e.g. to buy assets). In the next section, we shall on the other hand consider the case when the bank does keep the money raised by selling the option to immediately increase its market value.

9.3 Bank market value rises due to the option purchase

Let us consider the case when the bank market value (V_0) increases as soon as the option has been sold, due to the money raised by selling the option. Similarly to the previous section, we replace the discounting factor $e^{-r_f T}$ with $e^{-\left(r_f+CS\right)T}$, in all discounting processes that we did so far. However, since this time we consider that the bank market value increases by the option price, the credit spread will be $CS = CS^{call}_{\text{ExV}}(V_0 + C)$, where C is the option price and V_0 is the bank market value right before selling the option.

The ExV call option price with CS included, which we shall call $C^{CS^+_{\text{ExV}},\,\text{ExV}}$, would be then the numerical solution of the following equation:

$$C^{CS^+_{\text{ExV}},\,\text{ExV}} = \left[S_0 e^{T(\mu+\sigma^2/2)}\, \mathcal{N}(c_1) - K\mathcal{N}(c_2)\right] e^{-\left(r_f + CS^{call}_{\text{ExV}}(V_0 + C^{CS^+_{\text{ExV}},\,\text{ExV}})\right)T}\ , \tag{87}$$

where $c_{1,2}$ are defined in Eq. (12), $CS^{call}_{\text{ExV}}(y)$ is defined in Eq. (84). We omitted all parameters and variable dependencies of the call option price $C^{CS^+_{\text{ExV}},\,\text{ExV}}$ for the sake of clarity.

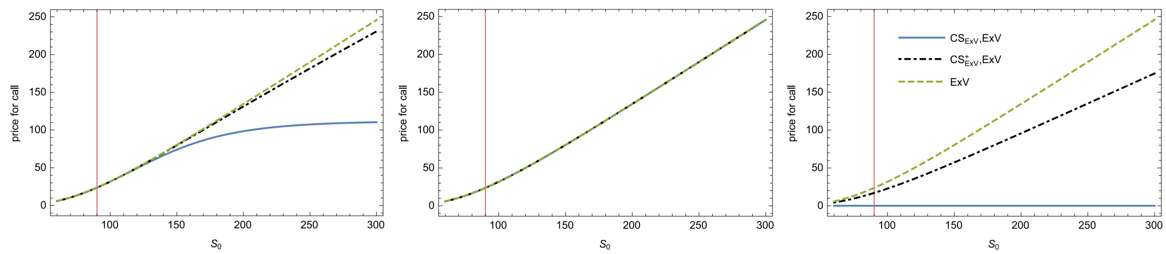

Figure 14: Comparison between $C^{CS^+_{\mathrm{ExV}},\,\mathrm{ExV}}_{\mu,\sigma,r_f}$, $C^{CS_{\mathrm{ExV}},\,\mathrm{ExV}}_{\mu,\sigma,r_f}$, $C^{\mathrm{ExV}}_{\mu,\sigma,r_f}$. Parameters are: $r_f = \mu = 0$, $\sigma = 0.15$, $T = 10$, $K = 90$. Left panel: $V_0 = 100$; middle panel: $V_0 = 1000$; right panel: $V_0 = 0$. The red vertical bar denotes the strike price.

9.4 Credit Spread in the option price: a comparison among approaches

A comparison between $C^{CS_{\mathrm{ExV}},\,\mathrm{ExV}}_{\mu,\sigma,r_f}$, $C^{\mathrm{ExV}}_{\mu,\sigma,r_f}$ and $C^{CS^+_{\mathrm{ExV}},\,\mathrm{ExV}}_{\mu,\sigma,r_f}$ is made in Fig. 14. Let us start analyzing the left panel. It is noticeable that the CS starts playing a role when the option price is close to the bank market value V_0, as it is reasonable to expect. The option price $C^{CS_{\mathrm{ExV}},\,\mathrm{ExV}}_{\mu,\sigma,r_f}$ has its asymptote at approximately V_0. This means that the client is not supposed to pay an option price higher than the bank market price (i.e. the bank reliability). This is indeed reasonable. On the other hand, one can see that the deviation between $C^{CS^+_{\mathrm{ExV}},\,\mathrm{ExV}}_{\mu,\sigma,r_f}$ and $C^{\mathrm{ExV}}_{\mu,\sigma,r_f}$ approaches is not very dramatic. This means that those banks that invest the money raised by selling the option on their capital value significantly reduce their default risk.

In normal cases, the bank market price is far higher than the option price. Therefore, the inclusion of the Credit Spread in the option price does not normally make any difference, as shown in the middle panel of Fig. 14.

In Fig. 14, right panel, we study the interesting case of a bank with no market value. This is, for example, the ideal case of a newly born bank created right at the moment of the option purchase by a person who is good willing to pay back the future debts. That person will wisely use (invest) the earnings from the option trade so to increase the bank market value immediately, but has no capital at the moment. As seen from the plot, the calculation made with $C^{CS_{\mathrm{ExV}},\,\mathrm{ExV}}$ gives no reliability to such a bank: The option price the customer must pay is zero. On the other hand the calculation made with $C^{CS^+_{\mathrm{ExV}},\,\mathrm{ExV}}$ does give the bank some reliability: The option price the customer must pay is somewhat lower than that one for a bank with sound market value, but definitely not zero. Thus this interestingly shows that in principle anyone with an empty wallet and good asset allocation skills could fairly trade options at a lower price.

10 Possible further studies

10.1 Dependency

An immediate follow up of the present study would be the inclusion of correlation or, more generally, of dependency among assets. This addition will strengthen the idea that dependencies within assets should be priced in, when trading options on several assets simultaneously. The inclusion of dependency can be easily done within the formalism we used in Section 2. We shall here briefly outline how this problem can be tackled.

To study dependency between assets, one must firstly model the joint PDF of returns of asset A and B, for the day i, with a mathematical function. Let us call such a function $f(r^A_{\mathrm{d}i}, r^B_{\mathrm{d}i})$. One must be aware that such a function might depend also on some of $(r^A_{\mathrm{d}i-1}, r^A_{\mathrm{d}i-2}, ..., r^B_{\mathrm{d}i-1}, r^B_{\mathrm{d}i-2}, ...)$. This was the case for ARCH and AR type assets we investigated in Sections 5 and 4.

A typical model for $f(r^A_{\mathrm{d}i}, r^B_{\mathrm{d}i})$ would be a binormal distribution function (which is directly related to the Gaussian copula density (Nelsen 2006)):

$$f(r^A_{\mathrm{d}i}, r^B_{\mathrm{d}i}) = \frac{\exp\left(-\frac{-\frac{\rho_{AB}\left(r^B_{\mathrm{d}i}-\mu_B\right)\left(r^A_{\mathrm{d}i}-\mu_A\right)}{\sigma_A\sigma_B}+\frac{\left(r^A_{\mathrm{d}i}-\mu_A\right)^2}{2\sigma_A^2}+\frac{\left(r^B_{\mathrm{d}i}-\mu_B\right)^2}{2\sigma_B^2}}{1-\rho_{AB}^2}\right)}{2\pi\sqrt{1-\rho_{AB}^2}\sigma_A\sigma_B}, \tag{88}$$

where ρ_{AB} is the correlation coefficient, while $\mu_{A,B}$, $\sigma_{A,B}$ are mean and standard deviation of the two assets. Such a joint PDF of returns well describes correlation between assets, while disregards the dependence among different days, as well as non-linear dependencies.

After the model has been chosen, one may build the convolution between day 1 and day 2 as follows:

$$G_2(s_A, s_B) = f(r^A_{\text{d}2}, r^B_{\text{d}2}) * f(r^A_{\text{d}1}, r^B_{\text{d}1}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \text{d}r^A_{\text{d}1}\text{d}r^B_{\text{d}1}\; f(r^A_{\text{d}1}, r^B_{\text{d}1})\, f(s_A - r^A_{\text{d}1}, s_B - r^B_{\text{d}1})\,. \quad (89)$$

Similarly, one can build the convolution for T days, and therefore the function $G_T(s_A, s_B)$, as

$$G_T(s_A, s_B) = f(r^A_{\text{d}T}, r^B_{\text{d}T}) * ... * f(r^A_{\text{d}1}, r^B_{\text{d}1}) \quad (90)$$

This function would be the equivalent of Eq. (8) for two correlated (or dependent) assets.

One may be then interested in pricing two options sold for the two correlated assets A and B, say a call option sold for both assets with the same maturity T. To do this, one could proceed within the ExV approach, viz. one can write the probability weighted call payoff for the two options (WCP^{AB}), which is the expected calls payoffs, as

$$WCP^{AB} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(\text{Max}[0, S^A_0 e^{r^A_T} - K^A] + \text{Max}[0, S^B_0 e^{r^B_T} - K^B]\right) G_T(r^A_T, r^B_T)\, \text{d}r^A_T \text{d}r^B_T\,. \quad (91)$$

The option prices would then be set equal to the discounted expected payoffs:

$$\mathcal{CC}^{\text{ExV}}_{\mu_1,\mu_2,\sigma_1,\sigma_2,r_f}(T, S^A_0, S^B_0, K^A, K^B) \equiv e^{-r_f T}\; WCP^{AB}, \quad (92)$$

where $\mathcal{CC}$ indicates that a call option for both assets A and B has been sold.

Similar studies can be carried out for all combinations of options (i.e. for $\mathcal{CP}$, $\mathcal{PC}$, $\mathcal{PP}$), for assets with different maturities, and within approaches VaR and ESF.

The results could be compared with the price obtained by selling the two options separately, and also with known studies on option pricing methods that account for correlation between assets and that are based upon financial mathematics and the B&S model (Haug 2007; Zhang 1997; Wilmott 2007).

10.2 Extension to any model for return distribution

In Sections 2, 4 and 5, we studied three models of return distribution, namely the Normally distributed returns, as well as AR and ARCH distributed returns. Such studies can be easily extended to any return distribution for which the joint density probability function of returns for different days is known. More specifically, given the joint density probability function of returns of day 1 and 2, $g(r_{\mathrm{d1}}, r_{\mathrm{d2}})$, one can always build the PDF of the sum of the continuous returns as

$$G_2^{\mathrm{model}}(s) = \int_{-\infty}^{+\infty} g(r_{\mathrm{d1}}, s - r_{\mathrm{d1}})\, \mathrm{d}r_{\mathrm{d1}} \ , \tag{93}$$

Thus, the ExV, VaR and ESF analyses - together with the relative option prices, put/call parity relation, credit spread analysis and Merton model - could be conducted **for any given** $g(r_{\mathrm{d1}}, r_{\mathrm{d2}})$, by following the steps described in detail in the sections above. The solution would be at least numerically available. To be more explicit, in our analysis we had $g(r_{\mathrm{d1}}, r_{\mathrm{d2}}) = g_N(r_{\mathrm{d1}})\, g_N(r_{\mathrm{d2}})$ for assets whose returns are normally distributed, where $g_N(x)$ is defined in Section 2. On the other hand, we had $g(r_{\mathrm{d1}}, r_{\mathrm{d2}}) = g_A^{r_{\mathrm{d0}}}(r_{\mathrm{d1}})\, g_A^{r_{\mathrm{d1}}}(r_{\mathrm{d2}})$ for ARCH(1) assets, while we had $g(r_{\mathrm{d1}}, r_{\mathrm{d2}}) = g_R^{r_{\mathrm{d0}}}(r_{\mathrm{d1}})\, g_R^{r_{\mathrm{d1}}}(r_{\mathrm{d2}})$ for AR(1) assets, where g_A and g_R functions are defined in Secs. 5 and 4, respectively.

10.3 Other possible studies

Other interesting studies that are possible with the formalism here presented might be a) testing SOPAs and FOPAs on illiquid markets, since FOPAs are supposed to fail due to the lack of liquidity; b) including the dependence on previous returns in AR and ARCH type assets; c) analyzing the main Greeks (Δ, ν, Θ, ρ, Γ) as well as the modified duration of the SOPAs option prices, and then comparing them to the B&S's Greeks. Such differences also imply different hedging strategies, when, for example, Delta hedging or Vega hedging is pursued; d) analyzing non-normal return distributions with constant volatility, such as Student-t distribution, and compare the results to what found in Sec. 2.

11 Conclusions

In this paper we studied Statistical Option Pricing Approaches (SOPAs). Instead of starting with a basic infinitesimal step that is characteristic of Financial-mathematics Option Pricing Approaches (FOPAs), we worked with statistical distributions and their quantiles. This way of proceeding is more alike actuarial mathematics. Thus, one could loosely say that we presented the way an insurance institution would price options. Said it differently, we demonstrated how a bank can price options without hedging requirements, but rather using a risk pool. We found that some developed SOPAs allow to analytically incorporate risk measures within the option price, such as Value-at-Risk or Expected Shortfall. Pricing option within those SOPAs would therefore be an advantage for any option issuer, such as a financial institution, since it would allow more control on the profit and loss to be expected out of the option payoff, when compared to analytical option pricing methods based on financial mathematics. SOPAs go in the direction of regulators, since they allow to analytically estimate the maximum loss within a certain confidence level (i.e., the Value-at-Risk) out of the option trade, which is a required estimate to be made within Basel II regulations. Moreover, Basel regulation will enforce Expected Shortfall estimations within the next future (Banking Supervision 2016; Banking Supervision 2012), which fact is also analytically accounted for within SOPAs. Therefore, we conclude that the presented SOPAs could become a valid alternative to standard analytical option pricing methods based on financial mathematics, such as the Black and Scholes (B&S) method.

Perhaps even more importantly, we showed that, beside assets whose returns are normally distributed, SOPAs allow for analytically pricing options on assets whose returns are not normally distributed or with variable mean/standard deviation, such as AR or ARCH processes. Truly, any asset distribution, for which the joint return distribution is known, can be used in SOPAs, as we outlined in Sec. 10. While for some asset distributions analytical solutions are found for the option price (e.g., AR), for others a numerical evaluation of integrals must be performed (e.g., ARCH). This possibility is undoubtedly a great advantage to option issuers since it overcomes limitations related to the normality assumption, which is inherent in some approaches like the B&S approach.

Therefore, within SOPA, effects like conditional heteroskedasticity or non-markovianity of returns are already priced in, analytically.

Another typical limitation of FOPA is related to the requirement of market liquidity and market completeness (Shah, Brorsen, and Anderson 2009; Ludkovski and Shen 2013). In fact FOPAs, like B&S, are based on infinitesimal steps, which presupposes that market positions can be adjusted on a time scale that is very small in comparison with the time during which the asset value changes considerably, with negligible costs. This evidently supposes high liquidity and market completeness. On the other hand, SOPAs do not make use of such requirements. In fact, SOPAs are totally independent of liquidity, as they merely rely on the probability that the asset value is below, above or equal to a certain value after T trading time steps. Similarly, there is no need to adjust nor to re-price the portfolio at each time in SOPAs, so market completeness is not explicitly used. The trading time step is not fixed a priori either. Within this paper, we considered it to be one day, since we preferred to work with daily continuous returns. Nevertheless, the time trading step could be analogously chosen to be anything else, such as 10 minutes, 2 days, or 1 month.

Another evident advantage of SOPAs is the possibility to include complex asset dependencies, as we outlined in the Sec. 10.

We may spend a few words on the effective performances that we found by testing SOPA on simulated and empirical data, as compared to B&S price. B&S option pricing method, fed with historical volatility (for empirical data) and with effective volatility (for simulated data), has been found to fall below the option average payoff. ExV option price has been found to match the expected payoff, both in simulated and real events. VaR option pricing method performed as supposed, by guaranteeing a positive return in the percentage of trades that was set by the confidence level, both in simulated and empirical data. Similarly, ESF option pricing method performed as expected, by being close to the realized statistic. Finally, AExV approach performance also turned out to be good, especially in simulated data, as it matched the average payoff. With empirical data, AExV was closer than B&S to the average payoff, but was still somewhat far from it.

With the SOPAs described, new put/call parity relations and new structural models along the line of the Merton model have been built. Curiously, we found that, when SOPAs are used, the derived structural models give more reliability to firms. In other words, firms are expected to default less often, which is the same as saying that their credit spread is reduced.

Toward the end of the paper, we described how to insert the credit spread of the issuer into the option price. We found reasonable results that are dependent on the market value of the issuer. Interestingly, we found that an option issuer with no market value could still honestly trade options by using the money raised by selling the option, and by pricing the option slightly cheaper than what an issuer with a solid market value would do.

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