Lattice QCD for pedestrians
(Lattice QCD for physicists)

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Overview (1): history of “elementary particle physics”
Overview (2): Standard Model with strong interactions

Matter:

\[
\begin{array}{c|c|c|c}
\nu_e & \nu_\mu & \nu_\tau \\
e & \mu & \tau \\
u & c & t \\
d & s & b \\
\end{array}
\]

Forces:

\[
U(1) \times SU(2) \times SU(3)
\]

Relevant parameters for strong interaction: \(\alpha_{QCD}, m_d, u, s, c, \ldots\) with basic law

\[
S_{QCD} = \frac{1}{2} \text{tr}(G_{\mu\nu}G^{\mu\nu}) + \sum_{q=d, u, s, c} \bar{q}(D + m_q)q
\]

Phenomenology of QCD with 1 + \(N_f\) parameters

- non-Abelian gauge symmetry \(\implies\) non-linear
- asymptotic freedom \(\implies\) perturbation theory at high energy
- confinement \(\implies\) hadrons \(\neq\) fundamental degrees of freedom
- spontaneous breaking of chiral symmetry \(\implies\) \(M_\pi \ll 4\pi F_\pi\)
Overview (3): QCD at high energies

Asymptotic freedom
[t'Hoof 1972, Gross-Wilczek/Politzer 1973]

\[
\frac{\beta(\alpha)}{\alpha} = \frac{\mu \partial \alpha}{\alpha \partial \mu} = \beta_1 \alpha^1 + \beta_2 \alpha^2 + \ldots
\]

\[
\beta_1 = \left( -\frac{11 N_c + 2 N_f}{6 \pi} \right)
\]

with \( N_c = 3 \) gives
\( \beta_1 < 0 \) for \( N_f < 33/2 \)

- virtual gluons anti-screen, i.e. they make a static color source appear stronger at large distance.

- virtual quarks weaken this effect.
In quenched QCD the $\bar{Q}Q$ potential keeps growing, $V(r) = \alpha/r + \text{const} + \sigma r$.

In full QCD it is energetically more favorable to pop a light $\bar{q}q$ pair out of the vacuum, $V(r) \leq \text{const}$. Analysis with explicit $\bar{Q}q\bar{q}Q$ state: Bali et al., PRD 71, 114513 (2005).
Q0: What is the physical meaning of the “wrong sign” of the proton binding energy if current quark masses are used?

Q1: Do we understand strong dynamics sufficiently well as to postdict the mass of the proton?

Q2: If so, can we turn the calculation around and determine $m_{ud} = (m_u + m_d)/2$ from first principles?
Overview (6): separating EW from QCD dynamics

Consider \( D^- \rightarrow K^0 e^- \bar{\nu}_e \), mediated through flavor changing weak decay \( \bar{c} \rightarrow \bar{s} W^- \)

Experiment: \( \Gamma \propto |V_{cs} f_{D}^{K} (q^2_*)|^2 \) and \( \Gamma \propto |V_{cs} f_{Ds}|^2 \) in semileptonic/leptonic decay

How do we separate QCD “contamination” from EW “vertex” and extract \( V_{cs} \)?

Would QCD result be precise enough to track BSM physics through inconsistencies?
Talk outline

(1) Lattice Basics
   – how to put scalars/gluons/quarks on the lattice

(2) Lattice Spectroscopy
   – sea versus valence quarks and (partial) quenching
   – spectra of stable versus unstable hadrons

(3) Lattice Techniques
   – weak and strong coupling expansion
   – numerical aspects, parallel architectures

(4) Lattice Phenomenology
   – quark masses: $m_d, m_u, m_s, m_c$
   – decay constants, form factors and CKM-physics
   – kaon mixing: $B_K, B_{BSM}, K \rightarrow 2\pi$ amplitude

(5) Lattice Outreach
   – baryon sigma terms, nuclear physics, ...
   – QCD thermodynamics at $\mu = 0$ and $\mu > 0$
   – large $N_c$, large $N_f$, different fermion representations
Lattice Basics

- path-integral and euclidean spacetime
- spin models and Metropolis algorithm
- how to put scalars on the lattice
- how to put gluons on the lattice
- how to put fermions the lattice
- Wilson versus Susskind/staggered fermions
Lattice basics (1): path-integral and euclidean spacetime

QFT: \[ e^{iS_M} = e^{i \int L_M \, d^4x_M} \quad x_M = (x^0, \mathbf{x}) = (x^0, x^{1/3}) \quad x^4 \equiv ix^0 \]

\[ L_M = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V[\phi] \quad V[\phi(x)] = \frac{m^2}{2} \phi^2(x) + \frac{\lambda}{4!} \phi^4(x) \]

\[ \partial_\mu \phi \partial^\mu \phi = (\partial_0 \phi)^2 - (\partial_{1/3} \phi)^2 = \left( \frac{\partial \phi}{\partial x^0} \right)^2 - \left( \frac{\partial \phi}{\partial x^{1/3}} \right)^2 \]

\[ L_E \equiv -L_S = \left( \frac{\partial \phi}{\partial x^{1/3}} \right)^2 + \left( \frac{\partial \phi}{\partial x^4} \right)^2 + \frac{m^2}{2} \phi^2(x) + \frac{\lambda}{4!} \phi^4(x) > 0 \quad \text{(for } \lambda > 0) \]

\[ i \int L_M \, dx^0 \, dx^1 \, dx^2 \, dx^3 = \int L_M \, dx^1 \, dx^2 \, dx^3 \, dx^4 = -\int L_E \, dx^1 \, dx^2 \, dx^3 \, dx^4 \]

\[ \implies \text{euclidean standard is } e^{-S_E} \text{ with } S_E = \int L_E \, d^4x_E > 0 \]

Lorentz symmetry
\[(x^0)^2 - x^2 \text{ invariant} \quad (+ - - -) \text{ signature} \quad \longleftrightarrow \quad O(4) \text{ symmetry} \]
\[ x^2 + (x^4)^2 \text{ invariant} \quad (+ + + +) \text{ signature} \]

[box \( L^3 \times T \) (lattice spacing \( a=1 \)) contains \( N = L^3T \) continuous dofs]
Lattice basics (2): spin models and Metropolis algorithm

Ising model (in $d=2$ dimensions):

- $N = N_1 N_2$ sites
- $s_i = \pm 1 \quad \forall i \in \{1, \ldots, N\}$

Toroidal boundary conditions

Spin configuration $s = (s_1, \ldots, s_N)$

Energy/Hamiltonian $H(s) = -J \sum_{\langle ij \rangle} s_i s_j - h \sum_k s_k$

- $J > 0$, parallel preferred ("ferromagnetic")
- $J < 0$, antipar. preferred ("antiferromag.")

Bounded from below, $H(s) > \text{const}$, as in EQFT

Partition function, free energy: $Z = \sum_s e^{-\beta H} \equiv e^{-\beta F}$

Inverse temperature $\beta = 1/(kT)$ is external parameter

Overall $2^N$ contributions ("proliferation of states")

Task: $\langle O \rangle = \frac{\sum_s O(s) e^{-\beta H(s)}}{\sum_s e^{-\beta H(s)}}$

Goal: generate sequence of spin configurations in which specific configuration $s$ shows up with probability $p(s) = \frac{1}{Z} e^{-\beta H(s)}$, $Z \equiv \sum_{s'} e^{-\beta H(s')}$

("Boltzmann distribution", solution MRRTT'53)
Lattice basics (3): how to put scalars on the lattice

\[ S_E = a^4 \sum_{x, \mu} \left\{ \frac{1}{2} (\nabla_\mu \phi)(x)(\nabla_\mu \phi)(x) + V[\phi(.)] \right\} \]  
[drop “E” henceforth]

\[(\nabla_\mu \phi)(x) \equiv \frac{1}{a} [\phi(x+a\hat{\mu}) - \phi(x)] \quad \text{("forward derivative")}
\]

\[(\nabla^*_\mu \phi)(x) \equiv \frac{1}{a} [\phi(x) - \phi(x-a\hat{\mu})] \quad \text{("backward derivative")}
\]

\[= a^4 \sum_x \left\{ -\frac{1}{2} \phi(x) \Delta \phi(x) + V[\phi(.)] \right\} \] > 0 (for \( \lambda > 0 \))

\[(\Delta \phi)(x) = \left\{ \begin{array}{c} (\nabla_\mu \nabla^*_\mu \phi)(x) \\ (\nabla^*_\mu \nabla_\mu \phi)(x) \end{array} \right\} = \sum_\mu \frac{\phi(x+a\hat{\mu}) - 2\phi(x) + \phi(x-a\hat{\mu})}{a^2}
\]

EQFT/simulation exploits formal analogy to statistical mechanics:

\[ Z = \int d\phi(x_1) ... d\phi(x_N) e^{-S[\phi]} \equiv \int D\phi \ e^{-S[\phi]} \]

\[ \langle \phi(x_1) ... \phi(x_n) \rangle \] means \( \langle 0|T\{\phi(x_1)\}...|0 \rangle \), i.e. time–ordered product of \( n=2,3,\ldots \) fields

finite ratio of two high–dimensional integrals, each of the \( N=L^3T \) fields runs from \( -\infty \) to \( +\infty \)
Attempts to put gauge fields $A_\mu(x)$ on the lattice break gauge invariance by $O(a)$ effects. Only path-ordered exponentials $\exp(ig \int A(s)ds)$ are measurable (Aharonov-Bohm).

Wilson: identify $U_\mu(x) \mapsto e^{ig \int_{x+\hat{\mu}} x A_\mu(\bar{x}) d\bar{x}}$ and consider $U_\mu(x) \in SU(3)$ fundamental dof.

- $U_\mu(x)$ is parallel transporter from $x+\hat{\mu}$ to $x$
- cov. derivative $(D_\mu \phi)(x) = U_\mu(x)\phi(x+\hat{\mu}) - \phi(x)$
- $U_\mu(x)$ transforms into $g(x)U_\mu(x+\hat{\mu})g^\dagger(x+\hat{\mu})$
- traced closed loops of links are gauge-invariant

Wilson: simplest gauge action involves $1 \times 1$ loop ("plaquette")

\[ \text{Tr}(P_{\mu\nu}) = N_c - \frac{a^4 g^2}{2} \text{Tr}(F_{\mu\nu}F_{\mu\nu}) \]

$\beta \equiv \frac{2N_c}{g^2}$ plays role of $J$ in Ising model

$\beta \ll 1 \iff g^2 \gg 1$ “strong coupling”

$\beta \gg 1 \iff g^2 \ll 1$ “weak coupling”

\[ S = a^4 \sum_{x,\mu,\nu} \frac{1}{2} \text{Tr}(F_{\mu\nu}(x)F_{\mu\nu}(x)) \]

\[ = \frac{1}{g^2} \sum_{x,\mu,\nu} \left\{ N_c - \text{Tr}(P_{\mu\nu}(x)) \right\} \]

\[ = \frac{2N_c}{g^2} \sum_{x,\mu<\nu(!)} \left\{ 1 - \frac{1}{N_c} \text{ReTr}(P_{\mu\nu}(x)) \right\} \]
Bosons were handy, because they required second-order operator:

\[ S_B = \frac{a^4}{2} \sum_x \left\{ \phi^\dagger(x)(-\Delta \phi)(x) + m^2 \phi^\dagger(x)\phi(x) \right\} \]

Fourier transform \( \hat{p}^2 + m^2 \) for \( m=0 \) with \( \hat{p} \equiv \frac{2}{a} \sin\left(\frac{ap}{2}\right) \)
has only 1 zero in BZ which is \( \left(\left[ -\frac{\pi}{a}, \frac{\pi}{a} \right] \right)^4 \)

Fermions give troubles, since they require first-order operator (“naive fermions”):

\[ S_F = a^4 \sum_x \left\{ \bar{\psi}(x)\gamma_\mu \left( \frac{\nabla_\mu + \nabla^*_\mu}{2} \psi(x) + m\bar{\psi}(x)\psi(x) \right) \right\} \]

Fourier transform \( i\gamma_\mu \bar{p}_\mu + m \) for \( m=0 \) with \( \bar{p} \equiv \frac{1}{a} \sin(ap) \)
has 16 zeros (one doubling per dim) in BZ
\( \rightarrow \) lift 15 of these to \( O\left(\frac{1}{a}\right) \) \ [Wilson]

Simulation as in pure YM, but with Grassmann-valued fermions integrated out:

\[ \langle O \rangle = \frac{\int DU \: O[U] \: \det^{N_f}(D[U]) \: e^{-S_G[U]}}{\int DU \: \det^{N_f}(D[U]) \: e^{-S_G[U]}} \quad \text{with} \quad DU \equiv \prod_{\mu=1}^{4} \prod_{x} \underbrace{dU_\mu(x)}_{\text{Haar measure on SU(3)}} \]
Lattice basics (6): Wilson versus Susskind fermions

Susskind/staggered fermions yield 4 species: 

\[ S_S = \sum_{x,y} \bar{\chi}(x) D_S(x, y) \chi(y) \]

with

\[ D_S(x, y) = \frac{1}{2} \sum_{\mu} \eta_{\mu}(x) \left\{ U_{\mu}(x) \delta_{x+\hat{\mu},y} - U_{\mu}^\dagger(x-\hat{\mu}) \delta_{x-\hat{\mu},y} \right\} \]

Wilson fermions [slower] yield 1 species: 

\[ S_W = \sum_{x,y} \bar{\psi}(x) D_W(x, y) \psi(y) \]

with

\[ D_W(x, y) = \frac{1}{2} \sum_{\mu} \left\{ (\gamma_{\mu} - I) U_{\mu}(x) \delta_{x+\hat{\mu},y} - (\gamma_{\mu} + I) U_{\mu}^\dagger(x-\hat{\mu}) \delta_{x-\hat{\mu},y} + 2 \delta_{x,y} \right\} \]

Overlap construction, traditionally with \( X = D_W - \rho \), makes things even slower:

\[ D_N(x, y) = \frac{\rho}{a} \left( 1 + X(X^\dagger X)^{-1/2} \right) = \frac{\rho}{a} \left( 1 + (XX^\dagger)^{-1/2} X \right) \]

- main advantage of staggered fermions is their expedience [plus flavored symm.]
- main advantage of Wilson-like fermions is 1-to-1 [latt-cont] flavor identification
Lattice basics (7): rationale for “smearing+clover”

info: staggered $D_S$ has (for $m=0$) EV spectrum on imaginary axis

info: overlap $D_N$ has (for $m=0$) EV spectrum on unit circle around $(1,0) \in \mathbb{C}$

→ link smearing in $D_W$ alone does not help on “horizontal jitter”

→ Symanzik improvement $c_{SW} \approx 1$ alone does not help much on “mass shift”

→ smearing and $c_{SW} \approx 1$ cure “mass shift” and “horizontal jitter” in physical branch
Lattice Spectroscopy

- scale hierarchies in LQCD
- sea quarks versus valence quarks
- terminology: QCD / QQCD / PQQCD
- hadron interpolating fields
- spectroscopy of stable particles
- spectroscopy of scattering states
### Lattice spectroscopy (1): scale hierarchies

**typical spacing:** \[0.05 \text{ fm} \leq a \leq 0.20 \text{ fm}\]

\[1 \text{ GeV} \leq a^{-1} \leq 4 \text{ GeV}\]

**typical boxsize:** \[2 \text{ fm} \leq L \leq 6 \text{ fm}\]

**require (UV):** \[a m_q \ll 1\]

**require (IR):** \[M_\pi L \geq 4\]

For each \(\beta\) (a posteriori lattice spacing \(a\)) tune \(1/\kappa_{ud,s,c,...}\) such that \(\{M_\pi^2, 2M_K^2 - M_\pi^2, M_{\eta_c}^2, ...\}/M_\Omega^2\) assume correct values (“sacrificed observables”).
Lattice spectroscopy (2): sea versus valence quarks

Hadronic correlator in $N_f \geq 2$ QCD: \[ C(t) = \int d^4x \ C(t, x) e^{ipx} \] with

\[ C(x) = \langle O(x) O(0) \rangle = \frac{1}{Z} \int DU D\bar{q}Dq \ O(x) O(0)^\dagger \ e^{-S_G-S_F} \]

where $O(x) = \bar{d}(x) \Gamma u(x)$ and $\Gamma = \gamma_5, \gamma_4 \gamma_5$ for $\pi^\pm$ and $S_G = \beta \sum (1 - \frac{1}{3} \text{Re} \text{Tr} \ U_{\mu\nu}(x)), \ S_F = \sum \bar{q}(D+m)q$

\[ \langle \bar{d}(x) \Gamma_1 u(x) \ \bar{u}(0) \Gamma_2 d(0) \rangle = \frac{1}{Z} \int DU \ \text{det}(D+m)^{N_f} \ e^{-S_G} \]
\[ \times \ \text{Tr}\left\{ \Gamma_1(D+m)^{-1}_{x0} \ Gamma_2(D+m)^{-1}_{0x} \right\} \]
\[ \times \ \gamma_5[(D+m)^{-1} x0]^\dagger \ gamma_5 \]

- Choose $m_u = m_d$ to save CPU time, since isospin SU(2) is a good symmetry.
- In principle $m_{\text{valence}} = m_{\text{sea}}$, but often additional valence quark masses to broaden data base. Note that “partially quenched QCD” is an extension of “full QCD”.
- $$(D+m)^{-1}_{x0}$$ for all $x$ amounts to 12 columns (with spinor and color) of the inverse.
Lattice spectroscopy (3): QCD/QQCD/PQQCD terminology

\( N_f = 0 \) “QQCD”  
no dynamical “sea” quarks, only “valence” quarks

\( N_f = 2 \)  
2 dynamical flavors with common mass \( m_{ud} \)

\( N_f = 2+1 \)  
3 dynamical flavors with masses \( m_{ud}, m_s \)

\( N_f = 2+1+1 \)  
4 dynamical flavors with masses \( m_{ud}, m_s, m_c \)

\( N_f = 1+1+1+1 \)  
4 dynamical flavors with masses \( m_d, m_u, m_s, m_c \)

Note: in none of the above cases is \( m_q = m_q^{\text{phys}} \) understood (are to be reached a posteriori through interpolations/extrapolations)

Note: “partially quenched” may mean absent in sea (e.g. \( c \) in \( N_f = 2+1 \)) or present in sea with different mass (i.e. \( m_c^{\text{sea}} \neq m_c^{\text{val}} \))

Note: quenching introduces serious artefacts in theory (non-unitarity, as \( \eta' \) has double-pole rather than shifted single-pole), but numerically effects seemed to be small [these days QQCD is gone]
• asymptotic rise \( V(r) \propto \sigma r \) in QQCD ("string tension" \( \sigma \) well-defined for \( N_f = 0 \))
• string breaks in (full) QCD, can be seen with better technique (cf. overview)
• short distance part is \( V(r) \propto \frac{\alpha}{r} \) or \( V(r) \propto \frac{\alpha(r)}{r} \); this can be used to get \( \alpha_V(r) \)
Lattice spectroscopy (5): meson/baryon interpolating fields

Flavor quantum number is to be kept track of explicitly:

\[ O_{\pi^+}(x) = \bar{d}(x)\gamma_5u(x), \quad O_{\pi^0}(x) = \frac{1}{\sqrt{2}}[\bar{u}(x)\gamma_5u(x) - \bar{d}(x)\gamma_5d(x)], \quad O_{\pi^-}(x) = \bar{u}(x)\gamma_5d(x) \]

\[
\langle O_{\pi^+}(x) O_{\pi^+}(y) \rangle = \langle \bar{d}(x)\gamma_5u(x) \bar{u}(y) | \gamma_5d(y) \rangle \\
= \langle \text{Tr}\{\gamma_5 D_{m_d}^{-1}(y, x) \gamma_5 D_{m_u}^{-1}(x, y)\} \rangle \\
= \langle \text{Tr}\{[D_{m_d}^{-1}(x, y)]^\dagger D_{m_u}^{-1}(x, y)\} \rangle \\
\langle O_{\pi^0}(x) O_{\pi^0}(y) \rangle = 6 \text{ terms, 2 connected and 4 disconnected, latter cancel for } m_u = m_d \\
\langle O_N(x) O_{\bar{N}}(y) \rangle = \langle (\text{contractions}) u(x)u(x)d(x) \bar{u}(y)\bar{u}(y)d(y) \rangle \]
Lattice spectroscopy (6): pseudoscalar meson correlators

Excellent data quality even on our lightest ensemble ($M_\pi \simeq 190$ MeV and $L \simeq 4.0$ fm):

$c_0 = G\tilde{G}/M_0, G\tilde{F}, F\tilde{G}, F\tilde{F}M_0$ (left) and $c_0 = \tilde{G}\tilde{G}/M_0, \tilde{G}\tilde{F}, \tilde{F}\tilde{G}, \tilde{F}\tilde{F}M_0$ (right)  

combined 1-state fit of 8 correlators with 5 parameters yields $M_\pi, F_\pi, m_{\text{PCAC}}$
Lattice spectroscopy (7): spectroscopy of stable states

stable states: meaning is *under strong interactions* (example: $\pi, N, \ldots$)

$$\langle A(x)B(y) \rangle = \sum_{n \geq 0} \frac{1}{2E_n} \langle 0|A(x,0)e^{-E_n x_4}|n\rangle \langle n|e^{+E_n y_4}B(y,0)|0\rangle$$

$$= \sum_{n \geq 0} \frac{1}{2E_n} \langle 0|A(x,0)|n\rangle \langle n|B(y,0)|0\rangle e^{-E_n(x_4-y_4)}$$

Consider local effective mass $M_{\text{eff}}(t) = \frac{1}{2} \log\left(\frac{C(t-1)}{C(t+1)}\right)$ and determine plateau value:
Lattice spectroscopy (8): spectroscopy of unstable/mixing states

unstable states: meaning is *under strong interactions* (example: $\rho, \Delta, \ldots$)

2-particle ($\pi\pi$, $\pi K$, $KK$, $\pi N$, $NN$) states:

Scattering length and phase-shift can be determined in Euclidean space from tower of states in finite volume [Lüscher 1991].

Example: $L$-dependence of states with $\pi\pi$ or $\rho$ quantum numbers is different for small (dashed blue) versus large (full red) $g_{\pi\pi\rho}$.

Original framework by Lüscher refined in many respects [Rummukainen and Gottlieb, Rusetsky et al] and successfully applied to a variety of systems.

Method in practice rather demanding, since limited number of $L$ values available, and extraction of high-lying states remains a challenge.

Results on $\pi\pi$, $\pi K$, $KK$, $\pi D$, $\pi N$, $NN$, ... from various groups, e.g. Beane/Savage et al [NPLQCD], Dudek et al [HSC], Lang et al, Mohler et al, Aoki et al [HAL-QCD], ...
Lattice Techniques

- strong coupling expansion
- weak coupling expansion
- iterative solvers
- CPUs in parallel mode
- GPUs in farming mode
- postprocess: $a \to 0$, $V \to \infty$, $m_q \to m_q^{\text{phys}}$
Lattice techniques (1): strong-coupling perturbation theory

Strong coupling PT: expansion in $\beta = 6/g_0^2$; expansion about “disorder”, i.e. about rough configurations. Rather large $O(20)$ orders can be reached by massive amount of computer algebra.

$$W_{1\times1}(r, t) = \left(\frac{\beta}{2N_c^2}\right)^{rt} (1 + O(\beta))$$

→ confinement proven to leading order in SCPT

Weak coupling PT: expansion in $g_0^2 = 6/\beta$; expansion about “order”, i.e. about smooth configurations. Already 2-loop computations extremely tedious due to broken Lorentz invariance.

→ most successful are “mixed schemes” in which $W_{2\times2}, W_{3\times3}, W_{4\times4}$ are analytically linked to $W_{1\times1}$ and the latter is measured in simulation.
Lattice techniques (2): weak-coupling perturbation theory

$Z$-factors ("renormalization") needed/useful for lattice-to-continuum matching; distinguish operators with/without anomalous dimension, beware of mixing:

$$\langle .|O_i^\text{cont}(\mu)|. \rangle = \sum_j Z_{ij}(a\mu) \langle .|O_j^{\text{latt}}(a)|. \rangle$$

$$Z_{ij}(a\mu) = \delta_{ij} - \frac{g_0^2}{16\pi^2}\left(\Delta_{ij}^{\text{latt}} - \Delta_{ij}^\text{cont}\right) = \delta_{ij} - \frac{g_0^2}{16\pi^2}C_F z_{ij}$$

$$Z_S(a\mu) = 1 - \frac{g_0^2}{4\pi^2}\left[\frac{z_S}{3} - \log(a^2\mu^2)\right] \quad Z_V = 1 - \frac{g_0^2}{12\pi^2}z_V$$

$$Z_P(a\mu) = 1 - \frac{g_0^2}{4\pi^2}\left[\frac{z_P}{3} - \log(a^2\mu^2)\right] \quad Z_A = 1 - \frac{g_0^2}{12\pi^2}z_A$$

Generically $[z_P - z_S]/2 = z_V - z_A$, and for a chiral action either side vanishes.

Typically $n$-loop LPT yields results with leading cut-off effects $O(\alpha^n a)$; usual hope/belief is that with non-perturbative improvement Symanzik scaling window is larger.
Lattice techniques (3): sparse iterative solvers

\begin{equation}
D_{st}(x, y) = \frac{1}{2} \sum \eta_\mu(x) \left\{ U_\mu(x) \delta_{x+\hat{\mu}, y} - U_\mu^\dagger(x-\hat{\mu}) \delta_{x-\hat{\mu}, y} \right\} + m \delta_{x, y}
\end{equation}

\begin{equation}
D_W(x, y) = \frac{1}{2} \sum_\mu \left\{ (\gamma_\mu - I) U_\mu(x) \delta_{x+\hat{\mu}, y} - (\gamma_\mu + I) U_\mu^\dagger(x-\hat{\mu}) \delta_{x-\hat{\mu}, y} \right\} + (4 + m_0) \delta_{x, y}
\end{equation}

- \( D \) is \( 12N \times 12N \) complex sparse matrix, for \( N = 64^3 \times 128 \) this is \( 402 \times 10^6 \times 402 \times 10^6 \)
- each line/column contains only \( 1 + 3 \cdot 2 \cdot 8 = 49 \) non-zero entries
- inverse is full \([\text{non-sparse}]\), example above would require \( 2.4 \times 10^6 \) TB of memory
- CG solver yields \( D^{-1}\eta \simeq c_0 \eta + c_1 D\eta + \ldots + c_n D^n \eta \) with \( n^2 \propto \text{cond}(D^\dagger D) = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \)
Lattice techniques (4): new CPU packing strategies

SMP versus SIMD:

JUQUEEN [IBM BG/Q] 06/2012 - 10/2012
processor type 64-bit PowerPC A2 1.6 GHz (205 Gflops each)
compute node 16-way SMP processor (water cooled)
racks, nodes, cores 8, 8'192, 131'072 28, 28'672, 458'752
memory 16 GB per node, aggregate 131 TB aggregate 448 TB
performance (double) 1678/1380 Teraflops peak/Linpack 5873/4830 Teraflops
power consumption <100 kW/rack, aggregate 0.8 MW aggregate 2.8 MW
network topology 5D torus among compute nodes (incl. global barriers)
network bandwidth 40 Gigabyte/s
network latency 2.5 \( \mu \)sec (light travels 750 meters)
Lattice techniques (5): new GPU programming models

GPUs originally designed for tasks in computer graphics (e.g. rendering).

GPUs nowadays frequently used for OpenMP-parallelizable scientific computations.

Hardware connection via PCI bus (overhead from data transfer before/after computation).

```c
void transform_10000by10000grid(float in[10000][10000], float *out[10000][10000]){
    for(int x=0; x<10000; x++){
        for(int y=0; y<10000; y++){
            *out[x][y] = do_something(in[x][y]); // local operation !!!
        }
    }
}
```

Popular programming languages: CUDA, OpenCL, ...

Issues of single (32bit) versus double (64bit) precision ...

Excellent price/performance ratio paid for by human work ...
Lattice breaks Lorentz symmetry (softly, i.e. recovered in observables under $a \to 0$, $V \to \infty$) but maintains gauge-invariance. Lattice spacing $a$ and quark masses $m_{ud,s,\ldots}^{\mathrm{scheme}}$ are quantities that emerge from the parameters $\beta$ and $1/\kappa_{ud,s,\ldots}$ of the simulations; hence a suitable number of observables must be “sacrificed” to set the lattice spacing and to adjust the quark masses.

$a \to 0$ Symanzik effective theory of cut-off effects has simple consequence: plot data versus correct power of $a$ (e.g. $\alpha a$, depends on action used) and extrapolate linearly.

$V \to \infty$ Chiral perturbation theory predicts that every quantity has asymptotic finite-volume effects which scale exponentially in $M_\pi L$; in relative shift $[f_B(L) - f_B(\infty)]/f_B(\infty) = \text{const} \ e^{-M_\pi L}$ often “const” from ChPT.

$m_q \to m_q^{\mathrm{phys}}$ Traditionally extrapolation $M_\pi^2 \to (134.8 \text{ MeV})^2$ via ChPT, modern simulations often bracket $m_{ud}^{\mathrm{phys}}$ by those in the simulation (in such case linear interpolation seems sufficient).

Almost all lattice computations concern quantities (masses, decay constants, form factors) for which no backrotation to Minkowski spacetime is required.
Lattice Phenomenology

- light quark masses from spectroscopy
- decay-constants and form-factors for CKM physics
- light flavor \((d,u,s)\) physics: \(f_\pi, f_K, \ldots\)
- heavy flavor \((c,b)\) physics: \(f_D, f_{D_s}, f_B, f_{B_s}, \ldots\)
- indirect CP violation: \(B_K, B_{BSM}, B_D, B_B, \ldots\)
- \(K \to 2\pi\) amplitudes and \(\Delta I = 1/2, \epsilon'/\epsilon\)
Quark masses (1): anatomy of $N_f = 2 + 1$ computation

1. Choose observables to be “sacrificed”, e.g. $M_\pi, M_K, M_\Omega$ in $N_f = 2+1$ QCD, and get “polished” experimental values, e.g. $M_\pi = 134.8(3)$ MeV, $M_K = 494.2(5)$ MeV in a world without isospin splitting and without electromagnetism [arXiv:1011.4408].

2. For a given bare coupling $\beta$ (yields $a$) tune bare masses $\frac{1}{\kappa_{ud,s}}$ such that the ratios $M_\pi/M_\Omega, M_K/M_\Omega$ assume their physical values (in practice: inter-/extrapolation).

$M_{\pi,K,\Omega} \longleftrightarrow m_q^{\text{bare}}$

3. Read off $\frac{1}{\kappa_{ud,s}}$ or determine bare $am_{ud,s}$ via AWI and convert them (perturbatively or non-perturbatively) to the scheme of your choice (e.g. $\overline{\text{MS}}$ at $\mu = 3$ GeV).

$m_q^{\text{bare}} \longleftrightarrow m_q^{\text{SF/RI}} \longleftrightarrow m_q^{\overline{\text{MS}}}$

4. Repeat steps 2 and 3 for at least 3 different lattice spacings and extrapolate the (finite-volume corrected) result to the continuum via Symanzik scaling.

Depending on details, step 3 can be rather demanding [RI/MOM, SF renormalization]. Below, guided tour using plots from BMW-collaboration [arXiv:1011.2403, 1011.2711].
Quark masses (2): Final result for ratio $m_s/m_{ud}$

In QCD ratios like $m_s/m_{ud}$ are renormalization group invariant (RGI), hence step 3 in this list is skipped (detail: we invoke $\alpha a$ and $a^2$ scaling).

Final result $m_s/m_{ud} = 27.53(20)(08)$ amounts to 0.78% precision.
Quark masses (3): $N_f = 3$ RI-running extrapolation for $Z_S$

Evolution $Z_S^{RI}(\mu)/Z_S^{RI}(4 \text{ GeV})$ has no visible cut-off effects among three finest lattices:

\[ \mu^2 \left[ \text{GeV}^2 \right] \\
0 \quad 10 \quad 20 \quad 30 \quad 40 \]

\[ \frac{Z_S^{RI}(\mu)}{Z_S^{RI}(4 \text{ GeV})} \]

\[ \beta = 3.8 \]
\[ \beta = 3.7 \]
\[ \beta = 3.61 \]
\[ \beta = 3.5 \]
\[ \beta = 3.31 \]

→ separate continuum limit with $R_S^{RI}(\mu, 4 \text{ GeV}) = \lim_{\beta \to \infty} Z_S^{RI}(4 \text{ GeV})/Z_S^{RI,\beta}(\mu)$
Quark masses (4): $N_f = 3$ RI-scheme-running ratio for $Z_S$

On the finest lattice we make contact within errors to 4-loop PT for $\mu \geq 4 \text{ GeV}$:
Quark masses (5): $N_f = 3$ RI and $\overline{\text{MS}}$ perturbative series for $Z_S$

- RI series (left) converges less convincingly than $\overline{\text{MS}}$ series (right)
- difference “4-loop” to “4-loop/ana” indicates size of 5-loop effects
- ratio suggests that higher-loop effects in RI are $< 1\%$ at $\mu = 4$ GeV
- ratio suggests that higher-loop effects in $\overline{\text{MS}}$ are negligible down to $\mu = 2$ GeV
Quark masses (6): Final results for $m_s$ and $m_{ud}$

Good scaling of $m_{ud,s}^{RI}(4$ GeV) out to the coarsest lattice ($a \sim 0.116$ fm):

Conversion with analytical 4-loop formula at 4 GeV and downwards running in $\overline{MS}$:

<table>
<thead>
<tr>
<th>$m_{ud}$</th>
<th>$m_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{ud}^{RI}(4$ GeV)</td>
<td>3.503(48)(49)</td>
</tr>
<tr>
<td>$m_s^{RI}$</td>
<td>4.624(63)(64)</td>
</tr>
<tr>
<td>$m_s^{\overline{MS}(2$ GeV})</td>
<td>3.469(47)(48)</td>
</tr>
</tbody>
</table>

RGI/$\overline{MS}$ results (table 1.9% prec.) need to be augmented by a $\sim 1\%$ conversion error.
Quark masses (7): splitting $m_{ud}$ with information from $\eta \rightarrow 3\pi$

The process $\eta \rightarrow 3\pi$ is highly sensitive to QCD isospin breaking (from $m_u \neq m_d$) but rather insensitive to QED isospin breaking (from $q_u \neq q_d$), and this is captured in $Q$.

Rewrite the Leutwyler ellipse in the form

$$\frac{1}{Q^2} = 4 \left( \frac{m_{ud}}{m_s} \right)^2 \frac{m_d - m_u}{m_d + m_u}$$

and use the conservative estimate $Q = 22.3(8)$ of [Leutwyler, Chiral Dynamics 09] together with our result $m_s/m_{ud} = 27.53(20)(08)$ to get the asymmetry parameter

$$\frac{m_d - m_u}{m_d + m_u} = 0.381(05)(27) \quad \leftrightarrow \quad m_u/m_d = 0.448(06)(29)$$

from which we then obtain individual $m_u, m_d$ values (note: $m_u = 0$ strongly disfavored)

<table>
<thead>
<tr>
<th></th>
<th>$m_u$</th>
<th>$m_d$</th>
<th>$m_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RI (4 GeV)</td>
<td>$2.17(04)(10)$</td>
<td>$4.84(07)(12)$</td>
<td>$96.4(1.1)(1.5)$</td>
</tr>
<tr>
<td>RGI</td>
<td>$2.86(05)(13)$</td>
<td>$6.39(09)(15)$</td>
<td>$127.3(1.5)(1.9)$</td>
</tr>
<tr>
<td>MS (2 GeV)</td>
<td>$2.15(03)(10)$</td>
<td>$4.79(07)(12)$</td>
<td>$95.5(1.1)(1.5)$</td>
</tr>
</tbody>
</table>
Lattice phenomenology (1): CKM physics ...

\[ \begin{align*}
D & \rightarrow Kl\nu & K & \rightarrow \pi l\nu & B & \rightarrow \pi l\nu \\
D & \rightarrow K^*l\nu & & & B & \rightarrow \rho l\nu \\
\end{align*} \]

\[ V_{CKM} \approx \begin{pmatrix}
1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\
-\lambda & 1 - \lambda^2/2 & A\lambda^2 \\
A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1
\end{pmatrix} \]

\[ \begin{align*}
B^0_d - \bar{B}^0_d & \quad \text{Mixing (} B_B \text{)} \\
B^0_s - \bar{B}^0_s & \quad \text{Mixing (} B_{B_s} \text{)} \\
B & \rightarrow K^* \gamma & \text{Rare Decays} \\
B & \rightarrow D l\nu \\
& \rightarrow D^* l\nu
\end{align*} \]
Lattice phenomenology (2): ... via external currents

Pion decay

\[ \pi^- \left\{ \begin{array}{c} d \\ \bar{u} \end{array} \right\} \rightarrow \mu^- + \bar{\nu}_\mu \]

Kaoon decay

\[ K^- \left\{ \begin{array}{c} s \\ \bar{u} \end{array} \right\} \rightarrow \mu^- + \bar{\nu}_\mu \]

\[ J^\text{CC}_\mu = (\bar{u}, \bar{c}, \bar{t}) \gamma_\mu \frac{1}{2}[1 - \gamma_5] V_{\text{CKM}} \left( \begin{array}{c} d \\ s \end{array} \right) \]

\[ \langle 0 | (\bar{u} \gamma_\mu \gamma_5 d)(x) | \pi^- (p) \rangle = i f_\pi p_\mu e^{ipx} \]

\[ \langle 0 | (\bar{u} \gamma_\mu \gamma_5 s)(x) | K^- (p) \rangle = i f_K p_\mu e^{ipx} \]

\[ \left( \begin{array}{c} d' \\ s' \\ b' \end{array} \right) = \left( \begin{array}{ccc} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{array} \right) \left( \begin{array}{c} d \\ s \\ b \end{array} \right) \]

\[ \Rightarrow \text{strong dynamics restricted to} \]

\[ \text{matrix elements } \langle 0 | A_\mu | \pi \rangle, \langle 0 | A_\mu | K \rangle \]

\[ \text{and form factors } \langle \pi | V_\mu | K \rangle \text{ etc.} \]
\( f_K/f_\pi \) calculation (1): Marciano’s observation

- \( |V_{ud}| \) is known, from super-allowed nuclear \( \beta \)-decays, with 0.03\% precision [HT].

- \( |V_{us}| \) is much less precisely known, but can be linked to \( |V_{ud}| \) via a relation involving \( f_K/f_\pi \), with everything else known rather accurately:

\[
\frac{\Gamma(K \to l\bar{\nu}_l)}{\Gamma(\pi \to l\bar{\nu}_l)} = \frac{|V_{us}|^2 f_K^2}{|V_{ud}|^2 f_\pi^2} \frac{M_K (1 - m_l^2/M_K^2)^2}{M_\pi (1 - m_l^2/M_\pi^2)^2} \left\{ 1 + \frac{\alpha}{\pi} (C_K - C_\pi) \right\}
\]

- CKM unitarity \( |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 \) (with \( |V_{ub}| \) being negligibly small) is genuine to the SM; any deviation is a \textit{unambiguous} signal of BSM physics.

\( \Rightarrow \) calculate \( f_K/f_\pi \) in \( N_f = 2+1 \) QCD (with quark masses extrapolated to the physical point) on the lattice; the precision attained gives the precision of \( |V_{us}| \).
$f_K/f_\pi$ calculation (2): adjusting quark masses

$N_f = 2+1$ lattice QCD: set $m_{ud}$, $m_s$ by adjusting $M_\pi$, $M_K$ to their physical values

→ extract $f_K/f_\pi$ on unitary ensembles and extrapolate to the physical mass point

→ $f_K/f_\pi = 1$ at $m_{ud}=m_s$ means that $f_K/f_\pi - 1$ is calculated with $\sim 5\%$ accuracy
\( \frac{f_K}{f_\pi} \) calculation (3): chiral extrapolation

- chiral \( SU(3) \) formula:

\[
\frac{F_K}{F_\pi} = 1 + \frac{1}{32 \pi^2 F_0^2} \left\{ \frac{5}{4} M_\pi^2 \log\left( \frac{M_\pi^2}{\mu^2} \right) - \frac{1}{2} M_K^2 \log\left( \frac{M_K^2}{\mu^2} \right) \right. \\
\left. - \left[ M_K^2 - \frac{1}{4} M_\pi^2 \right] \log\left( \frac{4M_K^2 - M_\pi^2}{3\mu^2} \right) \right\} + \frac{4}{F_0^2} [M_K^2 - M_\pi^2] L_5
\]

- chiral \( SU(2) \)-plus-strange formula [RBC/UKQCD 08], simplified form:

\[
\left. \begin{array}{c}
\frac{F_K}{F_\pi} = \left. \frac{F_K}{F_\pi} \right|_{m_{ud}=0} \\
\end{array} \right\} \left( 1 + \frac{5}{8} \frac{M_\pi^2}{(4\pi F)^2} \log\left( \frac{M_\pi^2}{\Lambda^2} \right) \right)
\]

- polynomial expansion \( F_\pi/F_K = d_0 + d_1 (M_\pi - M_\pi^{\text{ref}}) + d_2 (M_\pi - M_\pi^{\text{ref}})^2 \), e.g. around \( M_\pi^{\text{ref}} = 300 \text{ MeV} \), at fixed physical \( m_s \), with \( \Delta_{\pi,K} \equiv (M_{\pi,K}^2 - M_\pi^{\text{ref}}^2)/M_\Omega^2 \) suggests:

\[
\frac{F_K}{F_\pi} = c_0 + c_1 \Delta_{\pi} + c_2 \Delta_{\pi}^2 + c_3 \Delta_K
\]

\( \rightarrow \) use all of them and count spread towards systematic uncertainty
**$f_K/f_\pi$ calculation (4): infinite volume extrapolation**

- finite volume effects on $F_K, F_\pi$ are known at the 2-loop level [CDH 05]

\[
\frac{F_\pi(L)}{F_\pi} = 1 + \sum_{n=1}^{\infty} \frac{m(n)}{\sqrt{n}} \frac{1}{M_\pi L} \frac{1}{F_\pi} \frac{M_\pi^2}{(4\pi F_\pi)^2} \left[ I_{F_\pi}^{(2)} + \frac{M_\pi^2}{(4\pi F_\pi)^2} I_{F_\pi}^{(4)} + \ldots \right]
\]

\[
\frac{F_K(L)}{F_K} = 1 + \sum_{n=1}^{\infty} \frac{m(n)}{\sqrt{n}} \frac{1}{M_\pi L} \frac{F_\pi}{F_K} \frac{M_\pi^2}{(4\pi F_\pi)^2} \left[ I_{F_K}^{(2)} + \frac{M_\pi^2}{(4\pi F_\pi)^2} I_{F_K}^{(4)} + \ldots \right]
\]

with $I_{F_\pi}^{(2)} = -4K_1(\sqrt{n} M_\pi L)$ and $I_{F_K}^{(2)} = -\frac{3}{2}K_1(\sqrt{n} M_\pi L)$, where $K_1(.)$ is a Bessel function of the second kind, and lengthy expressions for $I_{F_\pi}^{(4)}, I_{F_K}^{(4)}$

- finite volume effects cancel partly in the ratio, as evident from the 1-loop formula

\[
\frac{F_K(L)}{F_\pi(L)} = \frac{F_K}{F_\pi} \left\{ 1 + \sum_{n=1}^{\infty} \frac{m(n)}{\sqrt{n}} \frac{1}{M_\pi L} \frac{M_\pi^2}{(4\pi F_\pi)^2} \left[ \frac{F_\pi}{F_K} I_{F_K}^{(2)} - I_{F_\pi}^{(2)} \right] \right\}
\]

- BMW uses $\frac{F_K(L)}{F_\pi(L)}/\frac{F_K}{F_\pi}$ at 1-loop and 2-loop level, and $F_\pi(L)/F_\pi$ at 2-loop level
\( \frac{f_K}{f_\pi} \) calculation (5): combined fits

\[ M_\pi^2 [\text{MeV}^2] \]

\( \frac{F_K}{F_\pi} \)

\( a \sim 0.125 \text{ fm} \)
\( a \sim 0.085 \text{ fm} \)
\( a \sim 0.065 \text{ fm} \)
\( a = 0 \) (extr.)

→ plot shows data \( (M_\pi^2, 2M_K^2 - M_\pi^2) \) − fit \( (M_\pi^2, 2M_K^2 - M_\pi^2) \) + fit \( (M_\pi^2, [2M_K^2 - M_\pi^2]_{\text{phys}}) \)

→ \( \frac{f_K}{f_\pi} \) scales rather nicely [note \( a^2/\text{fm}^2 = 0.0042, 0.0072, 0.0156 \)]

\[ \Rightarrow \frac{f_K}{f_\pi} = 1.192(7)(6) \text{ at physical } m_{ud} \text{ and } m_s, \text{ in continuum, in infinite volume} \]
Calculation (6): update on $|V_{us}|$ and CKM unitarity

- Latest nuclear structure calculations [Hardy Towner'09] give
  $$|V_{ud}| = 0.97425(22).$$

- Plug experimental information $\Gamma(K \rightarrow \mu\bar{\nu})/\Gamma(\pi \rightarrow \mu\bar{\nu}) = 1.3363(37)$ [PDG'08] and $C_K - C_\pi = -3.0 \pm 1.5$ [Marciano] into Marciano's equation; this yields
  $$\frac{|V_{us}|}{|V_{ud}|} \frac{f_K}{f_\pi} = 0.27599(59).$$

- Upon combining the previous one/two points and our value for $f_K/f_\pi$ we obtain
  $$\frac{|V_{us}|}{|V_{ud}|} = 0.2315(19) \quad \text{and} \quad |V_{us}| = 0.2256(17).$$

- Upon including $|V_{ub}| = 3.39(36)10^{-3}$ [PDG'08] we end up with [BMW, 1001.4692]
  $$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1.0001(9).$$
$f_K/f_\pi$ calculation (7): FLAG summary

- lattice result for $f^+(0), N_f = 2+1$
- lattice result for $F_K/F_\pi, N_f = 2+1$
- lattice results for $N_f = 2$ combined
- unitarity
- our estimate (lattice + unitarity)
- nuclear $\beta$ decay
Lattice Outreach

- baryon sigma terms and dark matter
- nuclear physics from first principles
- QCD thermodynamics at $\mu = 0$
- QCD thermodynamics at $\mu > 0$
- hadronic contributions to muon $g-2$
- isospin splitting and electromagnetism
- large $N_c$, larger $N_f$, different representations
Lattice outreach (1): WIMPS via nucleon sigma terms

Traditionally large uncertainty from matrix elements

\[
\sigma_{ud} = m_{ud} \langle N| \bar{u}u + \bar{d}d|N \rangle
\]

\[
\sigma_s = m_s \langle N| \bar{s}s|N \rangle
\]

(RGI, dimension of mass)

Universe: 73% dark energy
23% dark matter
4% baryons

Dark matter stays dark, unless WIMP-Nucleon scattering can be probed down to tiny cross-sections.
Lattice outreach (2): sigma terms via Feynman-Hellmann

Lattice can compute $\sigma_{ud}$ and $\sigma_s$ directly or via Feynman-Hellmann theorem:

$$\sigma_{ud} = m_{ud} \frac{\partial M_N}{\partial m_{ud}} = M^2_\pi \frac{\partial M_N}{\partial M^2_\pi}$$

and

$$\sigma_s = m_s \frac{\partial M_N}{\partial m_s} = (2M^2_K - M^2_\pi) \frac{\partial M_N}{\partial (2M^2_K - M^2_\pi)}$$

we find

$$\sigma_{ud} = 39(4)(^{+18}_{-7}) \text{ MeV} \quad \text{and} \quad \sigma_s = 34(14)(^{+28}_{-24}) \text{ MeV}$$

in consequence

$$m_{ud}\langle N|\bar{u}u + \bar{d}d|N \rangle \simeq m_s\langle N|\bar{s}s|N \rangle$$
Lattice outreach (3): nuclear physics from first principles

Nuclear “valley of stability” from first principles ambitious due to contractions:

\[
\begin{align*}
\text{N energy} & : 3! \times 3! = 36 \text{ contractions} \\
\text{\textsuperscript{4}He energy} & : 6! \times 6! = 518'400 \text{ contractions} \\
\text{\textsuperscript{12}C energy} & : 18! \times 18! = 4 \times 10^{31} \text{ contractions}
\end{align*}
\]
Lattice outreach (4): QCD thermodynamics at $\mu = 0$

Established: QCD with physical $m_{ud}, m_s$ at zero chemical potential (as relevant in early universe) shows crossover.

Different definitions of “transition temperature” $T_c$ yield different values [$P, \langle \bar{\psi}\psi \rangle, ...$], but for one definition everyone should agree in the continuum.

Long standing discrepancy between Wuppertal-Budapest (left) and HotQCD (right) now resolved.

S. Dürr, BUW/JSC

PhD course Heidelberg/Neckarzimmern, 14 Feb 2013
At non-zero baryon density (equivalent: chemical potential $\mu \neq 0$) the fermion determinant becomes complex, which creates a major difficulty to the concept of importance sampling.

A clear establishment of a second-order endpoint would be a major leap forward.

In QCD many approaches to solve the sign problem have been tried:
- absorb phase in observable [ancient]
- two-parameter reweighting from $\mu = 0$ [Fodor Katz]
- work at imaginary $\mu$ and continue [Philipsen deForcrand]
- compute Taylor coefficients at $\mu = 0$

In QCD-inspired models many tricks/reformulations become possible.
Lattice outreach (6): hadronic contributions to muon $g-2$

Hadronic contributions to vacuum polarization provide one of the major sources of systematic uncertainty in the computation of $a_\mu = (g_\mu - 2)/2$. Can the lattice help?

$$a_{\ell}^{\text{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 \ f(Q^2) \ \bar{\Pi}(Q^2)$$

with known $f$ and $\bar{\Pi}(Q^2) = \Pi(Q^2) - \Pi(0)$ and $\Pi_{\mu\nu}(q) = (q^2 g_{\mu\nu} - q_\mu q_\nu) \Pi(q^2)$ can be computed as the Fourier transformed 2-point function of the electromagnetic current.

Recent computations include:
Lattice outreach (7): isospin splittings and electromagnetism

In standard $N_f = 2 + 1$ lattice studies two sources of isospin breaking are ignored (up-down mass difference, electromagnetic). Since they are both small, it would appear reasonable to include both of them a posteriori, by reweighting the configurations.

PACS-CS has long experience with reweighting in the quark mass; they used reweighting in $m_{ud}$ to shift $M_\pi$ from 156 MeV to 135 MeV.

In arXiv:1205.2961 they extend this approach to account for QED effects and the up-down quark mass difference. They find $M_{K^0} > M_{K^\pm}$.

Pioneering publication for QCD+QED on the lattice is Duncan et al, Phys. Rev. Lett. 76 (1996) 3894-3897 [hep-lat/9602005].

Lattice outreach (8): $N_f = 1+1+1+1$ plus QED simulations

- 2002-20??:
  
  $N_f = 2+1$ QCD requires 3 polished input values [e.g. $M_\pi$, $M_K$, $M_\Omega$ in theory with $m_u, m_d \rightarrow (m_u+m_d)/2$ and $e \rightarrow 0$]
  
  $\rightarrow$ analysis suggests $M_\pi = 134.8(3) \text{MeV}$, $M_K = 494.2(5) \text{MeV}$ [see FLAG report]

- 2010-????:
  
  $N_f = 2+1+1$ QCD requires 4 polished input values [ditto and $M_{D_s}$ in theory with $m_u, m_d \rightarrow (m_u+m_d)/2$ and $e \rightarrow 0$]
  
  $\rightarrow$ charm unquenched, but no conceptual change on isospin issue

- 2014-????:
  
  $N_f = 1+1+1+1$ QCD requires 5 input variables [e.g. $M_{\pi^\pm}$, $M_{K^\pm}$, $M_{K^0}$, $M_{D_s}$, $M_\Omega$]
  
  $\rightarrow$ requires disconnected contribution to flavor-singlet quantities
  
  $\rightarrow$ analysis of $\pi^0$-$\eta$-$\eta'$-$\gamma$ mixing mandatory to extract physical masses
  
  $\rightarrow$ QED and QCD renormalization intertwined ($m_s/m_d$ is RGI, $m_u/m_d$ is not)
  
  $\rightarrow$ final word on $m_u \not\equiv 0$ [in QCD+QED] will be possible
Lattice outreach (9): Large $N_c$, larger $N_f$, higher representations

QCD with $N_c \to \infty$ and fixed $\lambda = g^2 N_c$ gets much simpler [weakly coupled hadrons, OZI exact, chiral loops $\sim 1/N$, axial anomaly $\sim 1/N$]; lattice is almost unnecessary ;-)
Summary

- Lattice solves QCD from first principles: euclidean QFT, analytical and numerical methods

- Remnants of lattice formulation to be removed:
  - continuum extrapolation: $a \to 0$
  - infinite volume extrapolation: $L \to \infty$
  - chiral inter/extrapolation: $m_q \to m_q^{\text{phys}}$

Hadron spectroscopy with one stable particle on in and out side is simple

Hadron spectroscopy with multiparticle states on in or our side is challenging

Wealth of applications in flavor physics, nuclear physics, (perhaps) BSM physics

Formulation useful for addressing conceptual issues in euclidean QFT
Epilogue: lattice literature


• Ch. Gattringer and Ch. Lang, Quantum Chromodynamics on the Lattice, An Introductory Presentation, Springer, 2009.


