F47 — Cyclotron frequency in a Penning trap

Blaum group

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1 Introduction

The mass \( m \) is closely related to mass-energy \( E \) via the famous relationship

\[
E = mc^2
\]

This relationship can be used to convert mass measurements to energy measurements. For example, the mass difference between the mother nucleus and daughter nucleus in a radioactive decay is equivalent to the decay energy. Measuring such energies is important in many areas of physics:

Astrophysics: One of the “11 Science Questions for the New Century” that were formulated by the NRC in 2002 was: “How were the elements from iron to uranium made?” Mass spectrometry provides key parameters (e.g. neutron separation energies) for building models that explain how nuclei are formed in supernovas and stars.

Nuclear physics: The measurement of nuclear masses helps to improve mass models. The masses of radioactive nuclei are of particular interest. One goal is to find the “island of stability”, which is predicted by many different models of nuclear masses.

Tests of QED: Measuring the binding energy of the 1s-electron in hydrogen-like uranium (\( \text{U}^{91+} \)) allows to test QED in strong fields.\(^1\) The ionization energy of \( \text{U}^{91+} \) from the ground state is approximately 200 keV and far out of the reach of laser spectroscopy. High precision mass measurements allow to measure this energy.

Neutrino physics: The determination of the neutrino mass is an active research field in particle physics and astrophysics. A precise measurement would yield an important parameter in theories beyond the Standard Model of physics. The neutrino mass is also needed for estimating the energy that is required for neutrino production during the big bang. Penning traps can help to measure this value by determining \( \beta \)-decay energies. The difference between the total energy of a \( \beta \)-decay and the maximum energy that the \( \beta \)-electrons can have (provided by a separate measurement) is equal to the mass energy of the \( \bar{\nu}_e \) neutrino.

Penning-traps are useful tools in mass spectrometry, but they can also be used to measure other properties of charged particles, such as magnetic moments or electric dipole moments.

Table 1.1: Applications of mass spectrometry and relative mass uncertainty that is needed.

<table>
<thead>
<tr>
<th>Field</th>
<th>Rel. mass uncert.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chemistry: identification of molecules</td>
<td>( 10^{-5} - 10^{-6} )</td>
</tr>
<tr>
<td>Nuclear physics: shells, sub-shells, pairing</td>
<td>( 10^{-6} )</td>
</tr>
<tr>
<td>Nuclear fine structure: deformation, halos</td>
<td>( 10^{-7} - 10^{-8} )</td>
</tr>
<tr>
<td>Astrophysics: r-process, rp-process, waiting points</td>
<td>( 10^{-7} )</td>
</tr>
<tr>
<td>Nuclear mass models and formulas: IMME</td>
<td>( 10^{-7} - 10^{-8} )</td>
</tr>
<tr>
<td>Weak Interaction studies: CVC hypothesis, CKM unitarity</td>
<td>( 10^{-8} )</td>
</tr>
<tr>
<td>Atomic physics: binding energies, QED</td>
<td>( 10^{-9} - 10^{-11} )</td>
</tr>
<tr>
<td>Metrology: fundamental constants, CPT</td>
<td>(&lt; 10^{-10} )</td>
</tr>
</tbody>
</table>

\(^1\)The electric field near the uranium nucleus is one of the strongest available electrostatic fields (cf. Schwinger limit).
2 Theory

2.1 Cyclotron motion

When a particle with mass \( m \) and charge \( q \) moves through a magnetic field \( \vec{B} \), it experiences the Lorentzian force

\[
\vec{F}_L = q \vec{v} \times \vec{B}.
\]

In the special case \( \vec{v} \perp \vec{B} \), the particle’s trajectory will be a circle (Fig. 2.1 left). In the more general case, \( \vec{v} \) has a component parallel to the magnetic field, and the trajectory is a spiral. By setting \( \vec{F}_L \) equal to the centripetal force, you can quickly show that the frequency of the circular motion is independent of the velocity:

\[
\omega_c = \frac{q}{m} B.
\]

This frequency is called the **cyclotron frequency**. The basis of Penning-trap mass spectrometry is to determine the cyclotron frequencies of two different particles in the same magnetic field \( \vec{B} \). When the ratio of these frequencies is calculated, the magnetic field cancels out and the charge ratio reduces to an integer fraction. The frequency measurement can therefore be used to calculate a mass ratio:

\[
\frac{\omega_{c1}}{\omega_{c2}} = \frac{q_1}{q_2} \frac{m_2}{m_1}.
\]

By using one particle with a well-known or defined mass, such as a \(^{12}\)C\(^+\) ion, the mass of the other particle can be determined.

State-of-the-art superconducting magnets can be carefully tuned to have a magnetic field that changes only 1 part in \( 10^8 \) over a 1 cm\(^3\) region in the center. When a charged particle is inserted into such a field, it orbits around the field lines (in other words, it is trapped radially), but it is free to drift axially, along the field lines, out of the homogeneous part of the magnetic field. This considerably reduces the useful time in which the particle can be studied or in which its cyclotron frequency can be determined.

The idea of a Penning trap is to have a weak, electric field that axially pushes the particles towards a defined point. But before describing this field, let us consider the motion of particles in strong \( \vec{B} \)-fields with simultaneous, weak \( \vec{E} \)-fields.

2.2 Guiding center motion

If in addition to the magnetic field \( \vec{B} \) there is a weak electric field \( \vec{E} \), the trajectory of charged particles can be separated into a fast cyclotron motion, caused by \( \vec{B} \), and a slow drift of the center of the cyclotron motion, caused by \( \vec{E} \). The center of the cyclotron motion is also called the guiding center.

For example: If \( \vec{E} \parallel \vec{B} \), the radial motion is undisturbed, and the particle is accelerated axially, along the field lines. The trajectory is a spiral that has more and more space between its loops (Fig. 2.1 center).

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1Parts of this chapter are based on the upcoming doctoral thesis by Martin Höcker.
2Weak meaning that the electric forces are much smaller than the magnetic forces.
Figure 2.1: Motion of a charged particle in a magnetic field (left), in magnetic field parallel to an electric field (center), and a magnetic field orthogonal to an electric field (right).

When \( \vec{E} \perp \vec{B} \), the particle accelerates and decelerates as it moves on its cyclotron orbit. The radius of the orbit grows as the particle moves in the direction of \( \vec{E} \) and shrinks again during the other half of the orbit. This causes the center of the cyclotron motion to drift sideways, orthogonal to both the electric and the magnetic field (Fig. 2.1 right). This drift is called the \( E \)-cross-\( B \) drift.

The general case can be described by a superposition of \( \vec{E}_\parallel \) and \( \vec{E}_\perp \).

We now have the tools to describe the motion of charged particles in a Penning trap.

2.3 Penning trap motion

This section gives a qualitative description of the motion. For a more rigorous quantitative treatment, see Sec. 2.5

By convention, the magnetic field points towards the \( z \)-direction, so that \( \vec{B} = B \hat{e}_z \). The electric field that leads to something called Penning trap can then be written as

\[
\vec{E} = c \begin{pmatrix} x/2 \\ y/2 \\ -z \end{pmatrix},
\]

(2.3)

with some arbitrary constant \( c \) (Fig. 2.2). The important part is happening in the \( E_z \)-component: Assuming the signs of \( c \) and charge \( q \) are the same, the electric field pushes the particles towards \( z = 0 \). Since this force is proportional to the \( z \)-displacement, like in a spring, the \( z \)-motion is a harmonic oscillation (Fig. 2.3).

Figure 2.2: Cut-view of the electric field inside a Penning trap.
The field in the $xy$-plane (the “radial” plane) is a necessary trade-off, as dictated by Gauss’s law: The field lines that are coming in towards the trap center in the $z$-direction, need to come out somehow. In an ideal Penning trap, the radial components of the $\vec{E}$ point evenly away from the trap center.

The radial components of $\vec{E}$ pull the particles away from the trap center. As they accelerate outwards, their cyclotron orbit increases. When the cyclotron motion takes the particles back towards the trap center, their cyclotron orbit decreases. In total the center of the cyclotron motion (the guiding center) performs a slow $E$-cross-$B$-drift around the center of the trap [Fig. 2.3]. In Penning trap terms, this slow drift is called the magnetron motion.

The drift of the guiding center slightly reduces the frequency of the fast cyclotron motion, see [Sec. 2.5].

It is important to point out that the magnetron motion is unstable. The particles sit on a potential hill in the radial plane. If there are any damping processes, such as collisions with background gas, the particles roll down the potential hill and are lost. In cryogenic traps that are cooled to 4 K, particles can easily be stored for years and longer, but in the room-temperature Penning-trap of this experiment, the storage times are on the order of 100 ms.

### 2.4 Penning trap electrodes

The electric field $\vec{E}$ of a Penning trap corresponds to the potential

$$\Phi = \frac{c}{2} \left( -\frac{\rho^2}{2} + z^2 \right) , \quad (2.4)$$

where $\rho^2 = x^2 + y^2$. To create this potential inside the Penning trap, there are three cylindrically symmetric electrodes that follow the equipotential lines of $\Phi$: Two electrodes follow the contours

$$z(\rho) = \pm \sqrt{\frac{z_0^2}{2} + \frac{\rho^2}{2}} , \quad (2.5)$$

It can be argued that the term Penning “trap” is a misnomer, but “Penning arbitrarily long storage device” does not have the same ring to it.

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Figure 2.3: Motion of a positively charged particle in a Penning trap. The motion consists of the slow magnetron drift around the trap center (blue), the axial motion (red), and the cyclotron motion (yellow). The superposition of all three motions is shown in purple.
with $z_0$ as the closest point to the trap center. These electrodes are called the endcaps (Fig. 2.4). Both endcaps are held at the same potential. The third electrode, called the ring electrode, follows the contour:

$$\rho(z) = \pm \sqrt{\rho_0^2 + 2z^2}, \quad (2.6)$$

with $\rho_0$ being the distance between trap center and ring electrode. In the trap of the FP-experiment, $\rho_0 = \sqrt{2}z_0$.

Let $V_0$ be the voltage between the ring electrode and the endcaps. Then the potential in the trap can be written as:

$$\Phi(\rho, z) = \frac{V_0 z_0}{z_0^2 + 2\rho_0^2} \left(\frac{1}{2}\rho^2 + z^2\right) = \frac{V_0}{2z_0^2} \left(-\frac{1}{2}\rho^2 + z^2\right) \quad (2.7)$$

2.5 Penning trap frequencies

To quantify our qualitative understanding of the Penning trap trajectories, we have to solve the equations of motion

$$\ddot{\mathbf{r}} = qm \left( \dot{\mathbf{r}} \times \mathbf{B} + \mathbf{E} \right). \quad (2.8)$$

With $\mathbf{B} = B\mathbf{e}_z$ and Equation 2.7 for the potential $\Phi(\rho, z)$ that defines the electric field $\mathbf{E} = -\nabla \Phi$, we can rewrite this as

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = qmB \begin{pmatrix} -\dot{x} \\ \dot{y} \\ 0 \end{pmatrix} + \frac{qV_0}{2mz_0^2} \begin{pmatrix} x \\ y \\ -2z \end{pmatrix} \quad (2.9)$$

The $z$-component of this differential equation is independent of the radial components, and it has the same structure as the equation of an undamped, harmonic oscillator. It is solved by

$$z(t) = z_0 e^{i\omega_z t} \quad \text{with} \quad \omega_z = \sqrt{\frac{qV_0}{mz_0^2}}. \quad (2.10)$$

The frequency $\omega_z$ is called the axial frequency. Using the above definition for $\omega_z$ and Equation 2.1 for $\omega_c$, the radial equations of motion can be written as

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \omega_c \begin{pmatrix} \dot{y} \\ -\dot{x} \end{pmatrix} + \frac{1}{2} \omega_c^2 \begin{pmatrix} x \\ y \end{pmatrix} \quad . \quad (2.11)$$
These equations can be solved by introducing the function \( u(t) = x(t) + iy(t) \). With the help of this function, the radial equations of motion can be reformulated as

\[
\ddot{u} = -i\omega_c \dot{u} + \frac{1}{2} \omega_z^2 u . \tag{2.12}
\]

Guessing a solution of the form \( u(t) = u_0 e^{-i\omega t} \) leads to two independent solutions with the frequencies

\[
\omega_{\pm} = \frac{1}{2} \left( \omega_c \pm \sqrt{\omega_c^2 - 2\omega_z^2} \right) . \tag{2.13}
\]

The frequency \( \omega_+ \) is the faster frequency. This frequency is often called the reduced cyclotron frequency, because for typical trap parameters, it is only slightly smaller than the free space cyclotron frequency \( \omega_c \). The other frequency, \( \omega_- \), is called the magnetron frequency.

In order for the trajectory to be stable, the frequencies must be real-valued. This is only the case if \( \omega_c^2 > 2\omega_z^2 \). This requirement is equivalent to

\[
B > \sqrt{\frac{2mV_0}{qz_0^2}} . \tag{2.14}
\]

In other words, if the voltage between the trap electrodes is too big, then the electric forces become too large and the ions are no longer contained radially by the magnetic field. This is called the stability limit.

In typical Penning traps, \( \omega_- \ll \omega_z \ll \omega_+ < \omega_c \). The magnetron frequency \( \omega_- \) may become larger than \( \omega_z \), but this is only true if the trap is operated dangerously close to the stability limit.

The fact that the motion of particles in a Penning trap can be broken down into three independent, harmonic modes is very important from a practical standpoint: It ensures that the frequencies can be measured separately and that the measurements do not depend on difficult-to-control initial conditions.

The free space cyclotron frequency \( \omega_c \) can easily be calculated from the trap eigenfrequencies \( \omega_+ \), \( \omega_- \), and \( \omega_z \). There are two ways to do it:

\[
\omega_c = \omega_+ + \omega_- \tag{2.15}
\]

\[
\omega_z^2 = \omega_+^2 + \omega_-^2 + \omega_z^2 \tag{2.16}
\]

These relationships can easily be verified using the results of Equation 2.13 and Equation 2.10. In ideal Penning traps, they give identical results.

One further relationship is quite useful when calculating one frequency from two measured frequencies:

\[
\omega_+ \omega_- = \frac{1}{2} \omega_z^2 . \tag{2.17}
\]

It follows directly from Equation 2.13.

### 2.6 Real Penning traps

In a real Penning trap, the convenient separation of the particle motion into three independent, harmonic modes is not strictly true. Imperfections lead to frequency offsets, to energy dependent frequency shifts (each frequency depends on the energies of all modes) and to coupling (energy exchange) between the modes. Relevant imperfections are:
• Particle interactions. If there is more than one charged particle in the Penning trap, particle interactions lead to chaotic frequency shifts and couplings. High precision Penning traps are designed to work with single ions, or with at most two ions in the trap. The interactions in the electron cloud of the FP-experiment are the biggest limitation of this device.

• Misalignment between $\vec{B}$ and $\vec{E}$. This can happen when the trap electrodes are tilted with respect to the magnet. In Penning traps used for mass spectrometry, typical frequency shifts are on a $10^{-7}$-level, but it can be shown that this shift cancels out when using Equation 2.16 to calculate $\omega_c$.

• Ellipticity of the $\vec{E}$ field, where the radial components of $\vec{E}$ are stronger in one direction than the other. This affects measurements on a $10^{-10}$-level.

• Cylindrically symmetric $\vec{E}$-field imperfections. When, for example, $E_z$ is not perfectly proportional to $z$ but additionally also depends on $z^3$, then the trapped particles’ $z$-oscillation has the characteristics of a “stiffening” (or “weakening”, depending on the sign) spring, and the axial frequency changes as a function of amplitude. Relevant on a $10^{-8}$-level.

• Cylindrically symmetric $\vec{B}$-field imperfections. They lead to similar effects as the $\vec{E}$-field imperfections, but are typically relevant on a $10^{-10}$-level.

• Relativistic shifts. These are especially important for light particles. For light ions, the reduced cyclotron frequencies are typically shifted on a $10^{-10}$-level.

• Image charges in the trap electrodes. Image charges always pull the charged particles to the closest electrode, away from the trap center. This modifies the equations of motions and is relevant on a $10^{-10}$-level.

Most of these frequency shifts are energy dependent. Current research in Penning trap mass spectrometry pushes for lower motional amplitudes (lower ion energy), smaller trap imperfections, and better characterization of the remaining shifts.
3 Detection and Manipulation

3.1 Detecting electrons

When an electron sits motionless inside the trap, electrostatic induction causes image charges to form on the trap electrodes. When the electron moves, the image charges move as well. The moving image charges are called image currents. The radial motion of trapped electrons leads to an image current that mostly flows in the ring electrode. Since the ring electrode is rotationally symmetric, this current can flow freely and does not lead to a measurable voltage drop across the ring electrode. On the other hand, the image current of the axial motion mostly flows between the endcaps, and depending on the connection between the endcaps, causes a voltage that can be measured.

The image current is low ($I \sim 10^{-12} \text{A}$), and according to Ohm’s law,

$$V = ZI,$$

we need a high impedance ($|Z| \sim 10^6 \Omega$) for a measurable voltage drop ($V \sim 10^{-6} \text{V}$). To achieve this we could connect a $1 \text{M} \Omega$ resistor between the endcaps and then measure the voltage drop across the resistor. However, the traps have a self-capacitance $C_{\text{eff}}$ of several pF, which is electrically parallel to the resistor we would connect. Therefore connecting a resistor would actually decrease $Z$, since it will be connected in parallel.

At typical axial frequencies of a few tens of MHz, the overall impedance is dominated by $C_{\text{eff}}$, and is only a few kΩ. (Try to verify this)

We can solve the problem by connecting an inductor $L$ across the endcaps [Fig. 3.1]. The self-capacitance of the trap, the inductor, and the resistive losses $R$ of the inductor form an RLC-circuit. The impedance of this RLC circuit is given by

$$Z = \frac{i\omega C_{\text{eff}} (i\omega L + R_{\text{coil}})}{i\omega C_{\text{eff}} + (i\omega L + R_{\text{coil}})}.$$

You can show that the impedance is maximal, and real, at a frequency of $\omega_{\text{RLC}} \approx \frac{1}{\sqrt{LC}}$ [Fig. 3.2].

The $z$-oscillation of the trapped particles acts as a current source parallel to the trap capacitance $C_{\text{eff}}$. If the particles oscillate with $\omega_z$ near $\omega_{\text{RLC}}$, their image current causes a

![Figure 3.1: Electrical model of the endcap connection.](image)
measurable voltage drop across the coil. As a side effect, the voltage drop also back-acts on the moving charges and damps their motion, until they are in thermal equilibrium with the RLC-circuit. When new electrons are loaded into the experiment, they start out hot and therefore appear as a peak on top of the frequency spectrum of the detection circuit. However, they quickly cool down and the peak is lost before it can be measured.

Therefore, we use a different method in order to detect them (see Fig. 3.3).

We use a device that sends sinusoidal signals of different frequencies through the circuit and measures the amplitudes of the returned waves after they went around in the circuit. Signals with frequencies substantially different from the circuit’s resonance frequency will mostly decay as they go around the circuit and will return with a greatly reduced amplitude. However, signals with frequencies close to the circuit’s resonance frequency will largely remain unaffected, and if we plot the voltage as a function of the frequency we would expect to get the graph shown in Fig. 3.2.

Instead, we get an additional dip in the center as shown in Fig. 3.3. This dip is caused by the interaction of the electrons with the circuit:
Just like how the motion of the trapped electrons create an induced current at the circuit, current running through the circuit applies forces on the trapped electrons. We set the electron’s axial oscillation frequency $\omega_z$ to be the same as the resonance frequency of the circuit $\omega_{RLC}$ (try to think how we can do that). We then send signals of different frequencies through the circuit.

Most of these signals will apply forces which are non-resonant with the electron’s axial motion. Therefore they will not excite the electrons much. (Also, these signals will mostly decay in the circuit anyhow because they are not resonant with the circuit either)

However, if we send a signal with the same frequency as the other two (so that $\omega_{\text{signal}} = \omega_z = \omega_{RLC}$), much of its amplitude will be spent on exciting the motion of the electrons, and the returned signal will be low in amplitude. We will then see a dip in the spectrum for this frequency. This is how we can see that we have electrons in the trap - By measuring a dip in the spectrum.

By the way, if the axial frequency is not close enough to the circuit’s resonance frequency, the impedance will have a non-negligible imaginary component and as a result instead of a dip we will get a peak and then a dip (a dispersion curve). (see Fig. 3.3)

**An alternative explanation:**

When the resonance frequency of the electrons, the resonance frequency of the circuit, and the frequency of the signal we send through the circuit are all identical, a significant image current flows through the electrons across the endcaps. This image current provides an additional path for the signal to flow, and it typically has a much lower impedance than the RLC circuit. The electron cloud then acts as a short and the voltage drop on the RLC circuit is reduced, producing a dip.

Note that the device sending the signals throughout the circuit is not connected directly, rather it is connected to the circuit through a capacitor with very small capacitance (1 pF). The capacitor is there to allow a voltage difference between the signal generating device and the circuit, otherwise the signal generating device would continuously force the circuit’s voltage to follow the generated signal.

### 3.2 Driving and Coupling the Modes

An electric field (or any vector field) can be described by:

$$
\vec{E} = E_{x,1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + E_{y,1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + E_{z,1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + E_{x,x} \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} + E_{y,y} \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} + E_{z,z} \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}
$$

$$
+ E_{x,y} \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix} + E_{x,z} \begin{pmatrix} 0 \\ z \\ 0 \end{pmatrix} + E_{z,x} \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} + E_{x,x} \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} + E_{z,y} \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}
$$

$$
+ \text{many non linear terms, such as } E_{x,x}x^2 \begin{pmatrix} x^2 \\ 0 \\ 0 \end{pmatrix}, E_{z,x}x^{19} = \begin{pmatrix} 0 \\ 0 \\ x^{19} \end{pmatrix}, \ldots
$$

Where $E_{\text{whatever}}$ are constants determining how strongly each term contributes. Another way of writing this would be:
\[
\vec{E} = \begin{pmatrix}
E_{x,1} + E_{x,x} + E_{x,x}x^2 + \ldots \\
E_{y,1} + E_{y,y} + E_{y,y}y^2 + \ldots \\
E_{z,1} + E_{z,z} + E_{z,z}z^2 + \ldots
\end{pmatrix} + \begin{pmatrix}
E_{x,y}y + E_{x,z}z \\
E_{y,x}x + E_{y,z}z \\
E_{z,x}x + E_{z,y}y
\end{pmatrix}
\]
(3.3)

For strongly homogenous electric fields, only the first few elements are non-zero. For less homogenous fields, more terms are non-zero. The more in-homogenous a field is, the more terms we need to describe it.

In our experiment we have an antenna - a metal wire that is placed near the trap electrodes. The field that this antenna creates is extremely non-homogenous and so contains many different terms. We can send electrical signals to the antenna and so have it create a complicated electrical field containing many terms. If the signal we send the antenna is oscillating with frequency \(\omega_{\text{antenna}}\), it makes each component of the field oscillate by making its coefficient time-dependent:

\[
E_{\text{whatever}} \rightarrow E_{\text{whatever}} \cdot \cos (\omega_{\text{antenna}} t)
\]
(3.4)

Note that, while the coefficients have different amplitudes and spatial dependencies, their time dependency is the same. (They all oscillate with the same frequency and phase)

The antenna has two uses, driving the modes and coupling them.

### 3.2.1 Driving the Modes

The antenna can be used to drive (energize) the modes. The electric field that the antenna creates contains linear non-coupling terms - for instance \(E_{z,1} \cos (\omega_{\text{antenna}} t) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\).

When we send a signal with frequency \(\omega_{\text{antenna}} = \omega_z\) we will get a forced and damped harmonic oscillator equation for the z direction (see Sec. 5.1). The amplitude of the motion will increase due to the driving force. There are two damping effects which limit the amplitude of the electrons:

1. As the amplitude of the motion increases, the effects of the anharmonic imperfections of the trap become more significant, the effective frequency changes, the force goes out of resonance and the electrons are no longer driven by a driving force.
2. As the electrons move, they create induced currents in the circuit, which decay over time due to resistive losses. These image currents create a voltage difference across the endcaps which is out of the phase with the electron’s motion and so counteracts it.

Note that integer multiples of \(\omega_{\text{antenna}} = n\omega_z\) (where \(n\) is an integer) will also drive the oscillator.

Driving the z motion is useful because it increases the effects of the electrons on the circuit and thus allows us to detect them more easily.
3.2.2 Coupling the Modes

The non coupling terms are named as such because if our field is only composed of such terms and we write down the differential equations of motion ($F = ma \Rightarrow q\vec{E} = m\ddot{r}$) we will get three independent differential equations - the equation for the motion in the x axis will only be dependent on the x coordinate, and so on for y and z. There will be no mixing - no coupling.

(In our specific case there is of course the coupling of the x and y axes caused by the magnetic field, even without introducing additional coupling terms by using the antenna.)

The coupling terms mess things up by making coordinates appear in equations of motion other than their own (for instance, we might have x or higher powers of x appear in the equation for the y axis, and so on). This couples the equations together and changes the modes of motion.

The effects are complicated, but practically the result is that there will be energy transfer between the modes. This energy transfer will be negligible (inefficient) for most antenna frequencies that we will choose and then the coupling terms will have no practical effect.

However, a calculation reveals that specific antenna frequencies will create efficient energy transfer. It shows that if we choose the antenna frequency to be an integer multiple of either the sum or a difference of the frequencies of the two involved modes, efficient energy transfer will occur.

As an example, a field component of the form:

$$\vec{E}_{\text{couple}} \propto \begin{pmatrix} z \\ 0 \\ x \end{pmatrix} \cos(\omega_{\text{antenna}} t + \phi)$$  \hspace{1cm} (3.5)

can be used to couple (create energy transfer between) the axial and the radial modes (either $z$ and $+$ or $z$ and $-$), by selecting either $\omega_{\text{antenna}} = \omega_z \pm \omega_-$ or $\omega_{\text{antenna}} = \omega_z \pm \omega_+$, respectively.

and a field component of the form:

$$\vec{E}_{\text{couple}} \propto \begin{pmatrix} y \\ x \\ 0 \end{pmatrix} \cos(\omega_{\text{antenna}} t + \phi)$$  \hspace{1cm} (3.6)

can be used to couple (create energy transfer between) the $+$ and $-$ modes, by selecting $\omega_{\text{antenna}} = \omega_+ \pm \omega_-$.  

The calculation shows that there are two types of efficient energy transfer - Rabi oscillations and exponential increase.

Rabi oscillations refer to a state where the amplitudes of the two modes change periodically, with a 90 degrees phase difference between them. For example, when coupling the $z$ and the $-$ modes with Rabi oscillations, the amplitude of the $z$ motion will increase while that of the $-$ motion will decrease, until the amplitude of the $z$ motion will start decreasing back, at which point the amplitude of the $-$ motion will start increasing back, and so on. If the amplitude is great enough this can cause the electrons to hit the trap electrodes and escape the trap.

Exponential increase is the state where both amplitudes increase exponentially. For example, when coupling the $z$ and the $-$ modes with exponential increase, the amplitude of each will increase exponentially. This will make the electrons hit the trap electrodes and escape it.

It follows from the calculation that for every pair of efficiently energy transferring frequencies, one of them will cause Rabi oscillations and the other will cause an exponential increase.

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For instance, $\omega_{\text{antenna}} = \omega_z + \omega_- \text{ will cause Rabbi oscillations while } \omega_{\text{antenna}} = \omega_z - \omega_- \text{ will cause exponential increase.}$

Coupling the motions is useful because we can use it to measure the frequencies of the motion ($\omega_{z,\pm}$) by using the following method:

We can observe our electrons detection signal while exciting the antenna with different frequencies. The electrons detection signal will remain roughly constant for most choices of the antenna frequency (because most antenna frequencies will create inefficient energy transfer and no electrons will be lost), however for certain frequencies it will drop significantly (because these antenna frequencies will cause efficient energy transfer which will make at least one of the modes grow so large the electrons will hit the trap).

We can write down these frequencies, and, knowing that these frequencies are actually integer multiples of sums or differences of the frequencies of motion, we can extract the frequencies of motion from them.

To be able to do this we need to know which pair of frequencies of motion a given antenna frequency that we found is a sum or a difference of. (For instance, given that a certain frequency we found makes the electrons detection signal drop, we still don’t know whether its $f_z - f_-$, or maybe $f_z + f_+$ and so on).

We can identify the pair by repeating the measurement with different ring voltages or coil currents. Different frequencies of motion have different dependencies on the ring voltage and the coil current, and so by observing whether and how a given antenna frequency shifted as a result of changing either the ring voltage or the coil current (or both), we can tell which frequencies of motion it is comprised of.

Go back to the theory section, look at the formulas for the different frequencies and try to determine which parameters one needs to modify to cause them to shift, and in which direction.

The calculation that these results are based on is available at the appendix (see chapter 5). You are not expected to know it by heart or to be able to reproduce it, but you should know the basic idea of how it goes.
4 Experiment

This chapter describes the experimental setup, the measurement control and the data analysis. You will learn how to use professional measurement equipment, how to control experimental routines by software and about experimental techniques used in high-precision mass spectrometry.

4.1 Overview of the experimental setup

The Penning trap electrodes are mounted inside a vacuum tube. A cylindrical coil wrapped around the outside of the tube generates the magnetic field. Electrical feedthroughs connect the electrodes inside the vacuum to the measurement electronics on the outside. The timing sequences are controlled by an Arduino single-board microcontroller, and data is read out by a PC.

![Figure 4.1: Overview of the experimental setup.](image)

4.1.1 Mechanical setup

To reach a sufficiently low vacuum pressure \(10^{-8} \text{ mbar or better}\), the vacuum chamber is pumped by a turbopump backed by a scroll pump. The turbopump is connected to a CF63 cross. Connected to the other three ports of the cross are a viewport, a pressure gauge, and the vacuum tube that houses the Penning trap. Through the viewport, you can see the trap, the electron gun and the trap electrodes. The cylindrical coil for generating the magnetic field is mounted on the outside of the vacuum tube. The other end of the vacuum tube is closed off by a flange that has feedthroughs built into it for the electrical connections with the Penning trap.

The Penning trap consists of three hyperbolically shaped electrodes that are insulated from each other. The central electrode, called ring electrode, is made of copper, and the two endcaps are made of wire mesh. Next to the lower endcap is the electron gun. The

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1The trap is mounted horizontally, but by convention, the endcap closer to the viewport is called “bottom endcap”.

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electrons are produced by thermal emission: A tungsten filament is heated by an electrical current running through it. When this filament is set to a negative voltage, electrons are repulsed from the filament through a grounded aperture plate (a plate with a hole in the middle) towards the trap.

The electrons can enter the trap region through the wire mesh of the lower endcap. These primary electrons cannot be trapped, since their energy is too high. Instead, secondary electrons, which are produced by collisions of the primary electrons with the residual gas, are trapped. During a measurement sequence with electrons in the trap, loading of further electrons is suppressed by switching the filament to ground potential.

A small wire between the top endcap and the ring is used as an antenna. When AC signals are sent through the antenna, an electromagnetic field will be produced inside the trap. This field is highly inhomogeneous and so includes components of many orders – dipole, quadrupole and higher. This creates coupling (practically, energy transfer) between the modes of motion. The effectiveness of the coupling is determined by the frequency of the signal we introduce to the antenna. Certain frequencies will create efficient energy transfer between the modes. Assuming that we continuously excite (give energy to) the axial mode, what happens when we efficiently couple it to another mode? What will happen to the trapped electron? Which frequencies do we expect to create efficient coupling? How can we use the antenna to measure these frequencies?

4.1.2 Measurement electronics

Fig. 4.2 provides an overview over the Penning trap and the measurement electronics.

The magnetic field is generated by running a current of several 100 mA through a cylindrical coil. The current is controlled by a programmable power supply that can apply currents of up to \( \approx 1.3 \, \text{A} \) through the coil’s resistance of \( \approx 50 \, \Omega \).

The electrostatic trapping field is generated by applying a positive voltage to the ring electrode while the two endcaps are grounded DC-wise (We still send AC signals through them to excite the axial motion of the electrons, but the average voltage is 0). The ring electrode voltage is generated by a function generator, and then amplified with a constant (but adjustable) gain by the HV200 amplifier. (You’ll need to know the gain for your measurements. How can you measure it?) The function generator supplies the ring voltage with a signal that decreases linearly from a certain (programmable) starting point to 0, in a periodic fashion. In this way we scan (ramp) the ring voltage. We do this in order to detect the electrons (Sec. 3.1). (How does scanning the ring voltage help us to detect the electrons? Hint: Which parameter depends on the ring voltage and what happens when it reaches a certain special value?)

The radio-frequency excitation signal for the \( f_{\text{ant}} \) is generated by a HP8657 signal generator. In order to apply the excitation during a well-defined period, the generator is not directly connected to the antenna, but through a radio-frequency switch instead. The switch is closed for 0 and open for 5 V, as in, if the voltage at the control port of the switch is 0 V, the RF generator signal goes into a 50 \( \Omega \) resistor and nothing special happens, and if its 5 V, the RF generator signal goes to the antenna. Thus, the excitation time and period can be defined by the length of the trigger pulse applied to the control port of the radio-frequency switch.

The current used to heat the filament is provided by one of the 4 channels of the 4-channel Hameg power supply. This channel “floats”, meaning that neither the positive nor negative side of the channel are connected directly to ground. This allows the channel to be floated upwards or downwards (it can be “biased”) by another, non-floating channel of the power supply. This filament bias voltage is applied through a fast HV-switch, which switches between the negative bias voltage and ground.

The coil and the low-noise amplifier of the axial-frequency detection system are enclosed in a metal box that is mounted on the outside of the vacuum system, onto the feedthrough flange. The supply voltages for the amplifier are provided by a 3-channel Hameg power supply.
Figure 4.2: Overview of the measurement electronics.

The detection system, trap and cables form an RLC circuit. It is connected to the DSA 815 network analyzer (spectrum analyzer) which excites it (and so the axial frequency of the electrons) when in Tracking Generator (TG) mode. The resulting voltage signal is amplified by a low-noise amplifier and given back to the DSA815 as input. The trigger signals for the measurement sequence (loading the trap with ions, exciting them with the antenna, and sweeping the ring voltage to detect them) are generated by an Arduino-based trigger generator. An oscilloscope displays the voltages of the measurement sequence.

4.1.3 Measurement control

The experimental setup is controlled by a computer. You will receive a folder containing ready-made functions for controlling the setup, as well as an example file for setting up a basic measurement, which you will need to modify and expand. The scripts will save data into text files which can be viewed using Scilab (see shortcut on the desktop).

The trapped electrons only remain trapped for a few 100ms, and are lost after they’re measured (some are lost by the measurement process, which involves exciting them them to the point where they are lost from the trap, and the rest are then lost due to the scanning of the ring voltage. When its negative all electrons are expelled from the trap). Therefore, in order to perform multiple measurements we need to continuously perform a cycle of loading new electrons, exciting them, detecting them and expelling them from the trap. Each cycle lasts 250ms and is composed of the following steps:

The measurement scheme thus consists out of following steps (see Fig. 4.3):

1. Loading the trap ($t_{\text{load}}$, typ. 25 ms)
2. Waiting for voltages to stabilize ($t_{\text{wait1}}$, typ. 25 ms)
3. Exciting the electrons with $f_{\text{ant}}$ ($t_{\text{exc}}$, typ. 50 ms)
Figure 4.3: Timings of the measurement sequence. All times are controlled by the Arduino trigger generator, except $t_{\text{ramp}}$ and $t_{SWT}$. $t_{\text{ramp}}$ is given by the frequency of the function generator, and $t_{SWT}$ is a setting on the spectrum analyzer.
4. Additional waiting time ($t_{\text{wait2}}$, typ. 25 ms)

5. “Counting”/Detection of the electrons ($t_{\text{det}}$, typ. 100 ms)

6. Expelling all electrons. This happens at the end of $t_{\text{det}}$.

**Loading the trap:** During the loading time the filament is switched to a negative bias voltage, as described in Sec. 4.1.1. The HV-switch used for switching the bias voltage can be controlled by a TTL (5V) signal, which is generated by the trigger generator.

**Excitation of the Electrons:** As described in Sec. 4.1.2, $f_{\text{ant}}$ is not directly applied to the antenna, but is instead connected to the input of an RF switch, which has its output connected to the antenna. The switch is open (allows the signal to pass through the switch to the antenna) when it gets 5V and is closed when it gets 0V. It can be controlled with a TTL signal, which is generated by the trigger generator.

**Counting/Detection of the Electrons:** As explained in Sec. 3.1, the electrons can be detected by scanning over different values of the ring voltage. The electrons’ axial frequency depends on the ring voltage, so by scanning over the ring voltage we scan over the electrons’ axial frequency.

When the electrons’ axial frequency equals the RLC circuit’s resonance frequency, the electrons short out the excitation signal, reducing its amplitude and signaling that we have electrons in the trap.

Just like how the trapped electrons create an induced signal in the circuit, so will an excitation signal running through the circuit apply induced forces on the trapped electrons. The effect of the excitation signal on the electrons will be insignificant unless its frequency is the same as the electron’s resonance frequency, in which case it will give them energy, and lose energy itself due to conservation of energy.

The lower the amplitude, the more electrons we have in the trap. (We will learn how to do this in the next section) The ring voltage is ramped to a negative value, so that all electrons are expelled from the trap, and is then set back to a positive value, to prepare for the next measurement cycle.

### 4.2 Measurement

We will measure the different frequencies (cyclotron, reduced cyclotron, axial and magnetron) as a function of the coil current and the ring voltage and use the resulting graphs to measure different system parameters (go over the equations in the theory section and try to think which parameters can be measured from such graphs). We will next detail the steps for such measurements. Keep in mind that original ideas are not only permitted, but encouraged.

#### 4.2.1 Identifying the Components

Before we start the measurements, get more familiar with the experimental setup by identifying the different devices shown in Fig. 4.2. Follow the cables and see what connects to what and try to understand what each device does. **Please don’t plug/unplug any cables or change the values of the power supplies manually. This can mess up the experiment and even destroy certain components.** (Its okay to change the ring voltage and the coil current remotely using the computer) The devices you can manually operate are the Network Analyzer, the oscilloscope and the HP8657A antenna power source. Feel free to play with them and fully explore their functionalities. Push lots of buttons, its fun.

Make yourself familiar with the minimum/maximum ratings given in Tab. 4.1
Table 4.1: Typical values and maximum-/minimum-ratings.

<table>
<thead>
<tr>
<th>VALUE</th>
<th>DEVICE</th>
<th>TYP</th>
<th>MIN</th>
<th>MAX</th>
<th>UNIT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure</td>
<td>MaxiGauge</td>
<td>$1.2 \cdot 10^{-8}$</td>
<td>$1 \cdot 10^{-7}$</td>
<td>mbar</td>
<td></td>
</tr>
<tr>
<td>Coil Current</td>
<td>3CH Hameg</td>
<td>1</td>
<td>$-1.3$</td>
<td>1.3</td>
<td>A</td>
</tr>
<tr>
<td>Filament Current</td>
<td>4CH Hameg</td>
<td>0.42</td>
<td></td>
<td>0.45</td>
<td>A</td>
</tr>
<tr>
<td>Filament Bias</td>
<td>4CH Hameg</td>
<td>$-10$</td>
<td>$-30$</td>
<td>0</td>
<td>V</td>
</tr>
<tr>
<td>$f_{ant}$ power</td>
<td>HP8657A</td>
<td>5</td>
<td></td>
<td>15</td>
<td>dBm</td>
</tr>
<tr>
<td>$f_{det}$ power</td>
<td>DSA815</td>
<td>$-30$</td>
<td></td>
<td>$-10$</td>
<td>dBm</td>
</tr>
<tr>
<td>Ring Voltage</td>
<td>Agilent 33500B</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>V</td>
</tr>
<tr>
<td>RF Switch DC+</td>
<td>4CH Hameg</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>V</td>
</tr>
<tr>
<td>RF Switch DC−</td>
<td>4CH Hameg</td>
<td>$-5$</td>
<td>$-5$</td>
<td>$-5$</td>
<td>V</td>
</tr>
<tr>
<td>Amplifier Supply</td>
<td>3CH Hameg</td>
<td>3.4</td>
<td>3.4</td>
<td>3.4</td>
<td>V</td>
</tr>
<tr>
<td>Amplifier Gate 1</td>
<td>3CH Hameg</td>
<td>$-0.84$</td>
<td>$-1.5$</td>
<td>0</td>
<td>V</td>
</tr>
<tr>
<td>Amplifier Gate 2</td>
<td>3CH Hameg</td>
<td>$-0.34$</td>
<td>$-1.5$</td>
<td>0</td>
<td>V</td>
</tr>
</tbody>
</table>

A general note (relevant to all steps, not only this one):
When the trap is not in use the coil current should be 0.1A (arbitrary low value to prevent it from heating up significantly).
Therefore, at the end of each day in the lab before going home, if you’re not performing a measurement, set the coil current to 0.1A, and if you are, make sure there is a line of code to change the coil current to 0.1A at the very end of your file, so that the coil can cool down after your script finishes running.

4.2.2 Finding the Resonance Frequency

We need to find the resonance frequency of the $RLC$ circuit. If we knew the capacitance and the inductance we could have calculated it outright (what is the formula for the resonance frequency of an $RLC$ circuit?), however; we don’t. We will use a device called a (scalar) Network Analyzer to determine the frequency – the DSA815. The DSA815 has both input and output connections connected to the $RLC$ circuit. To use it, first preset it to its factory settings. In its default mode of operation, the DSA815 shows a Fourier transform of the received signal. (The horizontal axis is the frequency and the vertical axis is the amplitude of that frequency). What graph did you expect to get and which graph did you really get? Why are the two not identical? Try to play with the settings of the device (frequency range, scale...) to find the resonance. It is expected to be somewhere in the [10;150] MHz range. You will notice that the graph is noisy. To fix this we’ll turn on the Tracking Generator (TG). In this mode, the DSA815 plots the same graph as before, but instead of relying on the input pulse alone in order to measure it, it sends pulses of its own with different frequencies (to the $RLC$ circuit) and measures the amplitudes of the returned pulses. (Why does this make the graph less noisy?)

Turning the TG mode on might not seem to do anything. Try to find a parameter in the TG menu which you can modify to fix this. You will get a graph which is different from the one you would expect to get.

What graph did you expect to get? What are possible causes for the differences? How can we still determine the resonance frequency from this graph? Write down the value you found.
for the resonance frequency of the RLC circuit.

4.2.3 Preparing for an Antenna Frequency Scan

Next we will use another mode of operation for the network analyzer called zero span mode in order to detect the electrons that are in the trap. Unlike in the previous modes, where the network analyzer displays amplitude vs frequency, in zero span mode the network analyzer displays amplitude vs time of a specific frequency (actually a narrow band of frequencies around a specific frequency). It sends pulses to the RLC circuit, as before, only that in zero span mode they all have the same frequency, and the returned amplitude (scaled by some irrelevant proportionality constant) is shown as a function of time. To use it, set the center frequency to the one you found in the previous part and set the DSA815 to zero span mode. (Don’t turn TG mode off)

Next we will synchronize the scan of the DSA815 with the scanning of the ring voltage in the trapping cycles (see Fig. 4.3). This effectively turns the horizontal axis from time to ring voltage. (Why?) (Yes, these trapping cycles are continuously happening, also while you are reading this. Look around and try to verify it by looking at the relevant instrument)

To synchronize the scan, set the trigger mode to external. Next set the Arduino trigger generator to continuous mode by using the following python script:

```python
import devices
devices.set_trigger_num(0)
devices.send_trigger()
```

Now we see the amplitude of the RLC circuit’s resonance frequency as a function of time (or, as a function of the ring voltage) You will initially not see anything and will need to overcome several problems:

1. Modify some of the settings of the DSA815.
   Hint: What are the units of each axis? What are the ranges of values of each axis? Which ranges would we like them to have?

2. Verify that the trap settings are tuned for electron capture.
   Hint: Which parameters related to the trap can you control using the python scripts? What are their current values? What are their typical values?

You should now see a graph, but not the one you expect. Which graph did you expect and which graph did you get?

Change the center frequency of the zero span mode (by no more than 0.5MHz in either direction) while observing the graph and find the optimal frequency for which you get the graph you would expect to get. Try to maximize the MMD (Maximal Minimal Depth – the vertical distance between the highest and the lowest points in the graph) You can use the wheel to perform this smoothly.

Why do we need to change the frequency? Why is using the circuit’s resonance frequency not good enough?

Hint: What changed in the system between this step and the previous one that could have affected the resonance frequency?

Did you get a dip or a peak? Why one and not the other? Why does the dip/peak appear where it does and not elsewhere?

Hint: what is special about its position / what happens in that position?

How can we change the position of the dip/peak? Try to do so by changing one of the experimental setup’s parameters by using the python code. In case where the peak/dip keeps appearing and disappearing, you’re not using the right frequency. (Even a 0.5MHz deviation can cause this problem)
4.2.4 Manual Antenna Frequency Scan

The HP8657 signal generator is capable of sending an AC signal to the antenna. As mentioned in the theoretical section, this makes the antenna produce an electromagnetic field containing higher order components (dipole, quadrupole and higher) which couple the three modes of motion in the trap (in other words, it creates energy transfer between the modes).

We can select different frequencies for the antenna’s electromagnetic field. Most will not cause any noticeable effects, because they will not create any noticeable energy transfer, but specific frequencies will.

Which frequencies are these, and what affect do you think they will have on the measured signal? (Refer to the theoretical section for help)

Test your guess by turning the HP8657 on and manually scanning for different frequencies while observing the network analyzer’s signal. Try to find frequencies which strongly affect the network analyzer’s signal and write down a few of those frequencies. Don’t forget to write down the values of other important parameters like ring voltage and coil current.

A common problem is that the signal of the antenna doesn’t seem to affect the system. If that happens double check the settings of the HP8657.

4.2.5 Rough Scripted Antenna Frequency Scan

The HP8657 can be remotely controlled using the computer. Use the example python file to scan over the antenna excitation frequencies in a range of a few dozen MHz which contain some of the special frequencies you found in the previous part, and measure the MMD of the detected signal for each of them. Observe the resulting graph and try to understand the result. Write down the ring voltage and coil current values you used (remember – The ring voltage you give as input in the python script is amplified before it is sent to the trap. What is the real voltage? What is the amplification factor? (Hint: use the oscilloscope)). Now change the ring voltage or the coil current and repeat the measurement, writing down the new values. What is the difference between the two graphs? Hint: Which features moved and which didn’t? Which of them would be affected by changing the ring voltage, the coil current, or both?

How can we use this information to measure the frequencies we want to measure? You might want to set the RSA to display the data in linear scale. Some data points have very low values, close to 0. When plotting in log scale, values that are close to 0 become close to minus infinity, and so when using log scale you’ll get noisy graphs with lots of false dips.

4.2.6 Fine Scripted Antenna Frequency Scan

Modify the script further to perform two sets of measurements across a larger frequency range of [1;350]MHz, one set with constant ring voltage and changing coil current and one set with changing ring voltage and constant coil current. (Refer to the table for the values you should use, or play with the ring voltage and the coil current [reminder – only by using the computer’s scripted commands, not physically]). These measurements will take a few hours to complete and can be left to run on their own after you leave for the day.

Remember to make sure there’s a line of code at the very end of the script to change the coil current to 0.1A.

4.2.7 Data Analysis

Analyze the data and obtain the different frequencies from it. Plot graphs of the frequencies as functions of either the ring voltage or the coil current and fit them. What parameters can you measure using the graphs?
4.2.8 Optional

These are optional: (Pick at most one, as in either one or none)

1. Make a graph of the MMD as a function of the waiting time and use it to determine the lifetime of the electrons in the trap.

2. Find the phase space for trapping (all possible combinations of ring voltage and coil current that enable trapping)

3. Pick a few dips that you found and try to fit them (after vertically inverting them) to a gaussian, a lorentzian and a sum of the two. Plot the fitting and the residuals (difference between the measured dip and the fitted graphs) to find the best fit and try to explain the results.

4. Be original, try to think of new things you can measure or do.

Hopefully:
Learn a lot and have fun.
Figure 4.4: Typical antenna frequency scan graph. Try to notice and understand the patterns.
5 Appendix - Coupling the Modes

In this chapter we will go through the mathematical derivation for mode coupling in more detail (see Sec. 3.2.2). We will show that:

1. Efficient energy transfer between two modes occurs when the antenna excitation frequency is chosen to be an integer multiple of a sum or a difference of the frequencies of the two coupled modes.

2. For every such pair (sum and difference) of frequencies, one will cause Rabbi oscillations and the other will cause exponential increase.

The calculation is not rigorous but should still be somewhat convincing.

5.1 Forced Harmonic Oscillator

First we will need to prove that for a 1D forced harmonic oscillator, a driving force oscillating at the resonance frequency of the oscillator $F_{	ext{drive}} \propto \cos (\omega_0 t)$ will inspire a motion with the same frequency but with a 90 degrees phase lag: $x \propto t \cos (\omega_0 t - \frac{\pi}{2}) = t \sin (\omega_0 t)$.

The equation for a 1D forced harmonic oscillator is:

$$F_{\text{spring}} + F_{\text{drive}} = m\ddot{x}$$

More explicitly,

$$-\omega_0^2 x + A \cos (\omega_0 t) = \ddot{x}$$, with $\omega_0^2 \equiv \frac{f}{m}$.

Pick $\omega_{\text{drive}} = \omega_0$.

Guess $x = B t \sin (\omega_0 t)$.

Derivations yield:

$$\dot{x} = B \sin (\omega_0 t) + Bt \omega_0 \cos (\omega_0 t)$$

$$\ddot{x} = 2B \omega_0 \cos (\omega_0 t) - Bt \omega_0^2 \sin (\omega_0 t)$$

Substitution in the equation of motion yields:

$$-\omega_0^2 B t \sin (\omega_0 t) + A \cos (\omega_0 t) = 2B \omega_0 \cos (\omega_0 t) - Bt \omega_0^2 \sin (\omega_0 t)$$

$$B = \frac{A}{2\omega_0}$$

And so the solution is $x = \frac{A}{2\omega_0} t \sin (\omega_0 t)$.

Conclusion: For a forced harmonic oscillator, the motion lags 90 degrees behind the force.
5.2 Coupling z and +

Let's examine the effects of a force of the form \( F \propto \begin{pmatrix} z \\ 0 \\ x \end{pmatrix} \cos (\omega_{RF} t) \) with initial conditions of zero amplitude for the x motion - \( x = 0 \), and non-zero amplitude for the z motion, which is oscillating at its resonance frequency - \( z(t) \propto \cos(\omega_z t + \phi_z) \).

We will first pick \( \omega_{RF} = \omega_z - \omega_+ \).

The driving force on the x axis is then:

\[ F_x \propto (\omega_z t + \phi_z) \cos (\omega_{RF} t) \]

Using the following trigonometric identity:

\[
\cos (a) \cos (b) = \frac{1}{2} \cos (a + b) + \frac{1}{2} \cos (a - b)
\]

We get \( F_x \propto \cos ((2\omega_z - \omega_+) t + \phi_z) + \cos (\omega_+ t + \phi_z) \)

The first term is non-resonant and will have no significant effect. Neglecting it, we end up with an equation for a forced harmonic oscillator. The driving force has phase \( \phi_z \), so the movement it will inspire will lag 90 degrees behind it and have a phase of \( \phi_z - \frac{\pi}{2} \):

\[ x \propto \cos (\omega_+ t + \phi_z - \frac{\pi}{2}) \]

The x amplitude will therefore start to grow. The driving force on the z axis will then be:

\[ F_z \propto \cos (\omega_+ t + \phi_z - \frac{\pi}{2}) \cos (\omega_{RF} t) \]

\[ F_z \propto \cos (\omega_+ t + \phi_z - \frac{\pi}{2}) + \cos ((2\omega_z - \omega_+) t + \phi_z - \frac{\pi}{2}) \]

Once again we will neglect the first term because it is non-resonant, and once again we get a forced harmonic oscillator. The driving force has a phase of \( \phi_z - \frac{\pi}{2} \), and so the movement it will inspire will be \( z \propto \cos (\omega_+ t + \phi_z - \pi) \). This motion is at 180 degrees to the original z motion and so the z amplitude will decrease. We see that the x amplitude increases and the z amplitude decreases.

Eventually the z amplitude will be 0 and the x amplitude will be at its maximal value. At this point there's no driving force on the x motion anymore (because the z amplitude is 0), but the x amplitude is still decreasing because the x amplitude is still positive. As a result the z amplitude will become negative, the driving force in the x axis will switch sign and the x amplitude will start to decrease.

So now the x amplitude decreases and the z amplitude increases (in absolute values). This will go on until the z amplitude is at its maximal value and the x amplitude is 0. Following the same logic, the x amplitude will then become negative, the driving force in the z axis will become negative and the process will repeat itself.

Conclusion: \( \omega_{RF} = \omega_z - \omega_+ \) causes Rabi oscillations.

Let's now explore the other option of picking \( \omega_{RF} = \omega_z + \omega_+ \).

The driving force on the x axis: \( \omega_{RF} = \omega_z + \omega_+ \)

\[ F \propto \cos ((2\omega_z + \omega_+) t + \phi_z) + \cos (\omega_+ t - \phi_z) \]

dropping the first term we are left with a forced harmonic oscillator. The driving force has a phase of \(-\phi_z\) and so the inspired movement will have a phase of \(-\phi_z - \frac{\pi}{2} \): \( x \propto \cos (\omega_+ t - \phi_z - \frac{\pi}{2}) \) The motion in the x direction will start to grow.

The driving force on the z axis:

\[ F_z \propto \cos (\omega_z t - \phi_z - \frac{\pi}{2}) \cos (\omega_{RF} t) \]

\[ F_z \propto \cos ((2\omega_z + \omega_+) t - \phi_z - \frac{\pi}{2}) + \cos (\omega_+ t + \phi_z + \frac{\pi}{2}) \]

dropping the first term we are left with a forced harmonic oscillator. The driving force has a phase of \( +\phi_z + \frac{\pi}{2} \) and the inspired movement has a phase of \( \phi_z \): \( z \propto \cos (\omega_+ t + \phi_z) \).
This motion is in phase with the original motion and so the z amplitude will increase. Both the x and the z amplitudes will increase.

Conclusion: $\omega_{RF} = \omega_z + \omega_+ \text{ causes exponential increase.}$

### 5.3 Other Couplings

The same logic holds for the other couplings, with the difference that, the - motion is caused by the $\vec{E} \times \vec{B}$ drift, not by a returning force of the form $\vec{F} = -k\vec{x}$, and a calculation shows that such a force inspires a motion which is at 90 degrees ahead of the force, not behind. The result is that the roles are reversed - $\omega_z + \omega_-$ causes Rabi oscillations while $\omega_z - \omega_-$ causes exponential increase. Similarly $\omega_+ + \omega_-$ causes Rabi oscillations while $\omega_+ - \omega_-$ causes exponential increase.