II. Pre-requisites

1. Relativistic kinematics
2. Wave description of free particles
3. Scattering process and transition amplitudes
4. Cross section and phase space

1. Relativistic kinematics

1.1 Notations

- 4-vector
  - contra-variant form $x^{\mu} = \left(x^{0}, \vec{x}\right) = (t, \vec{x}) \quad p^{\mu} = \left(p^{0}, \vec{p}\right) = (E, \vec{p})$
  - covariant form $x_{\mu} = \left(x^{0}, -\vec{x}\right) = (t, -\vec{x}) \quad p_{\mu} = \left(p^{0}, -\vec{p}\right) = (E, -\vec{p})$
- Metric tensor

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad x_{\mu} = g_{\mu\nu} x^{\nu} \quad x^{\mu} = g^{\mu\nu} x_{\nu}$$

- Derivative operator

$$\partial^{\mu} = \frac{\partial}{\partial x_{\mu}} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

- Scalar product

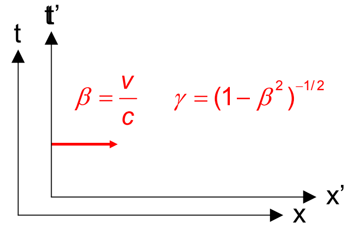
$$ab = a_{\mu} b^{\mu} = g_{\mu\nu} a^{\nu} b^{\mu} = (a^{0} b^{0} - \vec{a} \cdot \vec{b})$$

1.2 Lorentz invariants

Lorentz transformation:

moving particle with $p = (E, \vec{p})$

$$p' = \begin{cases} \begin{pmatrix} E' \\ \vec{p}'_l \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E \\ \vec{p}_l \end{pmatrix} \\ \vec{p}'_t = \vec{p}_t \end{cases}$$



$\beta = \frac{v}{c}$ $\gamma = (1-\beta^2)^{-1/2}$

w/r to rest frame: $\beta = \frac{|\vec{p}|}{E}$ $\gamma = \frac{E}{m}$

Scalar products are invariant under Lorentz transformations: $a'b' = ab$

Example 1: invariant mass

$$p^2 = p_\mu p^\mu = E^2 - \vec{p}^2 = m^2$$

Example 2: center-of-mass energy of 2 particle collision

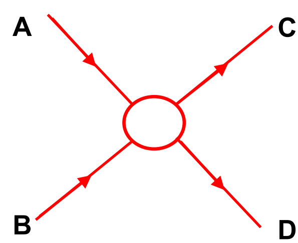
$$s = (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$$

Lorentz scalars can only by functions of other Lorentz invariants (scalars).

Examples of Lorentz invariants: $\frac{1}{E}\frac{d\sigma}{d^3p}$ and $E \cdot \Gamma$ Lorentz invariant cross section / decay width

1.3 Mandelstam variables

$$A + B \rightarrow C + D$$



(unpolarized particles)

What are the Lorentz scalars the cross section can depend on ?

$p_i p_k$ with $p_{i,k} = p_A, p_B, p_C, p_D$

$p_i^2 = m_i^2$ — 4 constraints

4-mom. conservation: 4 constraints

2 indep. products

Use usually 2 out of the 3 **Mandelstam variables**

$$s = (p_A + p_B)^2$$
$$t = (p_A - p_c)^2$$
$$u = (p_A - p_D)^2$$

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$$

2. Wave description of free particles

2.1 Schrödinger Equation for non-relativistic free particles

$$i\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\nabla^2\psi$$

Solution for energy $E = \frac{p^2}{2m}$

$$\psi(\vec{r},t) = \frac{1}{\sqrt{V}}\exp[i(\vec{p}\vec{x} - Et)]$$

Continuity equation:

$$\frac{\partial\rho}{\partial t} + \nabla\vec{j} = 0$$

$$\rho = |\psi|^2$$

$$\vec{j} = \frac{1}{2im}\left(\psi^*(\nabla\psi) - (\nabla\psi^*)\psi\right)$$

Schrödinger Eq uses classical E-p relation $E^2=p^2/2m$ and the replacement $E \to i\frac{\partial}{\partial t}$ and $\vec{p} = -\vec{\nabla}$

2.2 Klein-Gordon Equation

Starts from relativistic energy relation $E^2=p^2+m^2$:

Describes relativistic Spin 0 particles

$$\frac{\partial^2}{\partial t^2}\phi - \nabla^2\phi + m^2\phi = 0$$

Solutions for energy values:

$$E_\pm = \pm\sqrt{p^2 + m^2} \gtrless 0$$

$$\phi(\vec{r},t) = N\exp[i(\vec{p}\vec{x} - E_\pm t)]$$

negative E values cannot be ignored as otherwise solutions are incomplete

Continuity equation: $\frac{\partial\rho}{\partial t} + \nabla\vec{j} = 0$

$$\frac{\partial}{\partial t}\left(i\phi^*\frac{\partial}{\partial t}\phi - i\phi\frac{\partial}{\partial t}\phi^*\right) + \nabla\left(-i\phi^*\vec{\nabla}\phi - i\phi\vec{\nabla}\phi^*\right) = 0$$

For the solution: $\phi(\vec{r},t) = N \exp\left[i(\vec{p}\vec{x} - E_{\pm}t)\right]$

$$\vec{j} = \left(-i\phi^{*}\vec{\nabla}\phi - i\phi\vec{\nabla}\phi^{*}\right) \qquad \vec{j} = 2\vec{p}|N|^2$$

$$\rho = \left(i\phi^{*}\frac{\partial}{\partial t}\phi - i\phi\frac{\partial}{\partial t}\phi^{*}\right) \qquad \rho = 2E|N|^2$$

What are negative probabilities for the E < 0 solutions ?

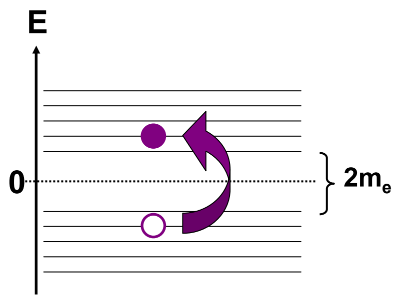
Normalization schemes:

$N = 1/\sqrt{2E} \Rightarrow$ 1 particle per unit volume

$N = 1 \Rightarrow$ 2E particles per unit volume

2.3 Anti-particles

Dirac interpretation for fermions: **Vacuum = sea of occupied neg. E levels**



For fermions the negative energy levels are w/o influence as long as they are fully occupied

Missing e^- w/ negative energy corresponds to to a positron w/ E>0

<u>e^+e^- annihilation:</u>

Free energy level in the sea. e^- drops into the hole and releases energy by photon emission: $E_\gamma > 2m_e$

<u>Photon conversion for $E_\gamma > 2m_e$</u>

Excitation of e^- from neg. energy level to pos. level: $\gamma \rightarrow e^+e^-$

Model predicts anti-particles (Discovery of positron by Anderson in 1933)

Discovery of positron

The Positive Electron

Carl D. Anderson, *California Institute of Technology, Pasadena, California*

(Received February 28, 1933)

Out of a group of 1300 photographs of cosmic-ray tracks in a vertical Wilson chamber 15 tracks were of positive particles which could not have a mass as great as that of the proton. From an examination of the energy-loss and ionization produced it is concluded that the charge is less than twice, and is probably exactly equal to, that of the proton. If these particles carry unit positive charge the curvatures and ionizations produced require the mass to b less than twenty times the electron mass. These particle will be called positrons. Because they occur in group associated with other tracks it is concluded that they mu be secondary particles ejected from atomic nuclei.

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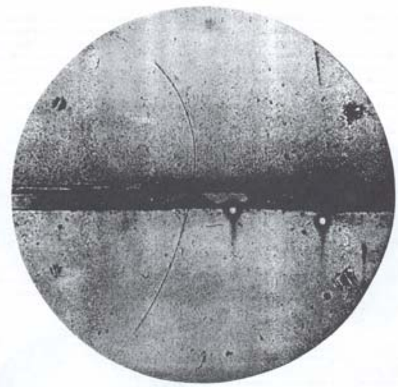


Fig. 1. A 63 million volt positron ($H\rho = 2.1\times10^5$ gauss-cm) passing through a 6 mm lead plate and emerging as a 23 million volt positron ($H\rho = 7.5\times10^4$ gauss-cm). The length of this latter path is at least ten times greater than the possible length of a proton path of this curvature.

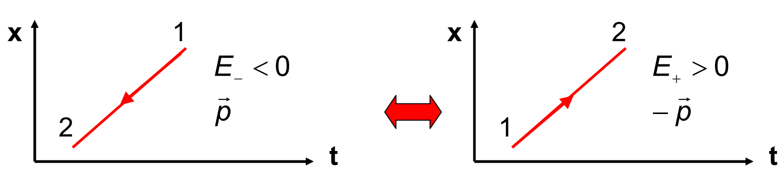
Feynman Stückelberg interpretation

Solutions with neg. energy propagate backwards in time:

$$E_+ = E: \quad \phi_+ = \frac{1}{\sqrt{2E}}\exp(i\vec{p}\vec{x} - iEt)$$

$$E_- = -E: \quad \phi_+ = \frac{1}{\sqrt{2E}}\exp(i\vec{p}\vec{x} + iEt)$$

Solutions describe anti-particles propagating forward in time:



Neg. probability density

$$\left.\begin{aligned} \rho &= 2E|N|^2 \\ \vec{j} &= 2\vec{p}|N|^2 \end{aligned}\right\} \times q \;\Rightarrow\; \begin{aligned} J^0 &= q\cdot 2E|N|^2 \\ \vec{J} &= q\cdot 2\vec{p}|N|^2 \end{aligned}$$

Charge density / currents

Example

Particle T^- with $q = -e$ and energy $E_- = -E < 0$

$$J^0(T^-) = (-e)\cdot 2(-E)|N|^2 = (+e)\cdot 2(+E)|N|^2 = J^0(T^+)$$

$$\vec{J}(T^-) = (-e)\cdot 2\vec{p}|N|^2 = (+e)\cdot 2(-\vec{p})|N|^2 = \underbrace{\vec{J}(T^+)}$$

$$T^+ \text{ with } E(T^+) > 0,\ \vec{p}_{T^+} = -\vec{p}_{T^-}$$

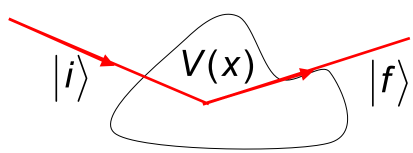
Description of creation and annihilation:

- Emission of anti-particle $\bar{T}$ with $p^\mu = (E, \mathbf{p}) \Leftrightarrow$ absorption of particle T with $p^\mu = (-E, -\mathbf{p})$
- Absorption of anti-particle $\bar{T}$ with $p^\mu = (E, \mathbf{p}) \Leftrightarrow$ emission of T with $p^\mu = (-E, -\mathbf{p})$

3. Scattering process and transition amplitudes

3.1 Fermi's "golden rule"

Scattering at potential $V(x) = V(t, \vec{x})$



Transition from $|i\rangle$ to $|f\rangle$ is described in 1st order perturbation theory by the amplitude

$$T_{fi} = -i\int \phi_f^*(x)V(x)\phi_i(x)d^4x$$

For a static potential $V = V(\vec{x})$

$$T_{fi} = -i\underbrace{\int \phi_f^*(\vec{x})V(\vec{x})\phi_i(\vec{x})d^3x}_{M_{fi}} \cdot \underbrace{\int \exp(i(E_f - E_i)t)dt}_{2\pi\delta(E_f - E_i)}$$

$$T_{fi} = -i \quad M_{fi} \quad \cdot \quad 2\pi\delta(E_f - E_i)$$

Transition rate

Fermi's "golden rule"

$$W = \frac{\int |T_{fi}|^2}{\Delta t} = 2\pi \int |M_{fi}|^2 \delta(E_f - E_i)\rho_f(E_f)dE_f$$

$$= 2\pi |M_{fi}|^2 \rho_f(E_i)$$

Final state density

3.2 Invariant amplitude (matrix element)

Independent of the interaction the amplitude $\boldsymbol{T_{fi}}$ can be written in the general form:

$$T_{fi} = -i \cdot (2\pi)^4 \underbrace{N_i N_f}_{\text{normalization}} \underbrace{\delta^4(p_i - p_f)}_{\text{4-mom. conservation}} \cdot M_{fi}$$

$\boldsymbol{M_{fi}}$ = Lorentz invariant transition amplitude (matrix element)

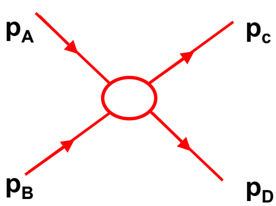
- Describes dynamics of interaction
- Determined by Feynman rules

Generalized “transition rate density” per final state

$$W_{fi} = \frac{|T_{fi}|^2}{\Delta t \cdot V} = (2\pi)^4 \frac{1}{V^2} \delta(p_i - p_f) \cdot |M_{fi}|^2$$

4. Cross section and phase space

$$A + B \rightarrow C + D$$



$$W_{fi} = \frac{(2\pi)^4}{V^4} \delta^4(p_A + p_B - p_C - p_D) \cdot |M_{fi}|^2$$

Differential cross section:

$$d\sigma = \frac{W_{fi}}{\Phi_i} d\rho_f$$

$\boldsymbol{d\rho_f}$ final state density
$\boldsymbol{\Phi_i}$ incident particle flux of A and B

4.1 Final state density $d\rho_f$

Number of possible states in box with volume V for particles with momentum $\in [\vec{p}, \vec{p} + d\vec{p}]$

$$d\rho_f = \frac{Vd^3p}{h^3} = \frac{Vd^3p}{\hbar^3 (2\pi)^3} = \frac{Vd^3p}{(2\pi)^3}$$

Normalization such that there are *2E* particles per unit volume V

$$\Rightarrow \text{state density } d\rho_f = \frac{Vd^3p}{2E(2\pi)^3}$$

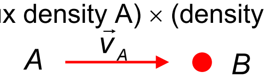
$$d\rho_f(A + B \to C + D) = \frac{Vd^3p_C}{2E_C(2\pi)^3} \frac{Vd^3p_D}{2E_D(2\pi)^3}$$

4.2 Incident particle flux Φ_i

Special case:

choose system where particle B is at rest

Φ_i = (flux density A) × (density B)



$$\Phi_i = |\vec{v}_A| \frac{2E_A}{V} \cdot \frac{2E_B}{V}$$

4.3 Lorentz invariant phase space factor

$$d\sigma = \frac{(2\pi)^4}{V^4} \delta^4(p_A + p_B - p_C - p_D) \cdot |M_{fi}|^2 \frac{V^2}{|\vec{v}_A| 2E_A 2E_B} \cdot \frac{Vd^3p_c}{2E_C(2\pi)^3} \cdot \frac{Vd^3p_D}{2E_D(2\pi)^3}$$

$$= \frac{|M_{fi}|^2}{|\vec{v}_A| 2E_A 2E_B} \cdot (2\pi)^4 \delta^4(p_A + p_B - p_C - p_D) \cdot \frac{Vd^3p_c}{2E_C(2\pi)^3} \cdot \frac{Vd^3p_D}{2E_D(2\pi)^3}$$

Particle flux ***F*** Lorentz invariant phase space factor ***dL***

Remark: volume V drops out !

Particle flux ***F***

$A \xrightarrow{\vec{v}_A} \bullet\; B$ $\qquad F = |\vec{v}_A| \cdot 2E_A \cdot 2E_B \qquad$ with $\vec{v}_A = \dfrac{\vec{p}_A}{E_A}$

$A \xrightarrow{\vec{v}_A} B \xrightarrow{\vec{v}_B}$ $\qquad F = |\vec{v}_A - \vec{v}_B| \cdot 2E_A \cdot 2E_B = 4\left((p_A p_B)^2 - m_A^2 m_B^2\right)^{1/2}$

$A \xrightarrow{\vec{p}_A} \xleftarrow{\vec{p}_B} B$ $\qquad F = 4|\vec{p}_i| \cdot (E_A + E_B) = 4|\vec{p}_i|\sqrt{s}$

$\vec{p}_A = -\vec{p}_B = \vec{p}_i$

Phase space factor ***dL*** for two-particles final-state

CM System: $\vec{p}_A = -\vec{p}_B \quad \vec{p}_C = -\vec{p}_D$

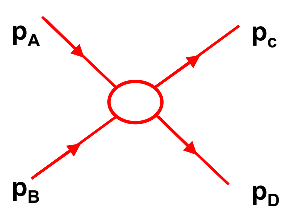
$$dL \xrightarrow{\int} \int dL = \frac{1}{4\pi^2}\int \delta^3(\vec{p}_C + \vec{p}_D)\delta(E_A + E_B - E_C - E_D)\frac{d^3p_C}{2E_C}\frac{d^3p_D}{2E_D}$$

$$\int dL = \frac{1}{4\pi^2}\int \frac{|\vec{p}_f|}{4\sqrt{s}}\, d\Omega$$

$\vec{p}_f = \vec{p}_C = -\vec{p}_D$
$s = (E_A + E_B)^2$

4.4 Differential cross section …putting everything together

$$A + B \rightarrow C + D$$

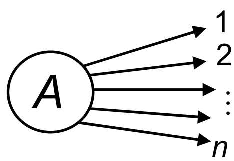


$$d\sigma = \frac{|M_{fi}|^2}{F} dL = \frac{1}{64\pi^2} \cdot \frac{1}{s} \cdot \frac{|\vec{p}_f|}{|\vec{p}_i|} \cdot |M_{fi}|^2 d\Omega$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{s} \cdot \frac{|\vec{p}_f|}{|\vec{p}_i|} \cdot |M_{fi}|^2$$

5. Decay width, lifetime and Dalitz plots

5.1 Decay width



$$\tau = \frac{1}{\Gamma} \qquad \Gamma = \sum \Gamma_i$$

Diff. decay width:

$$d\Gamma_i(A \to 1 + 2 + \ldots + n) = \frac{W_{fi}}{n_A} d\rho_f$$

$$d\Gamma_i = \frac{|M_{fi}|^2}{2E_A} \cdot (2\pi)^4 \delta^4(p_A - p_1 - p_2 - \ldots - p_n) \cdot \frac{d^3p_1}{2E_1(2\pi)^3} \cdot \frac{d^3p_2}{2E_2(2\pi)^3} \cdot \ldots \cdot \frac{d^3p_n}{2E_n(2\pi)^3}$$

Two-body decay:

$$A \to 1 + 2$$

$$d\Gamma_i(A \to 1+2) = \frac{|M_{fi}|^2}{2E_A} \cdot dL = \frac{|M_{fi}|^2}{2E_A} \frac{1}{4\pi^2} \frac{|\vec{p}_f|}{4\sqrt{s}} d\Omega$$

wie oben

$$\sqrt{s} = E_A = m_A$$

$$d\Gamma_i(A \to 1+2) = \frac{|\vec{p}_f|}{32\pi^2 m_A^2} \int |M_{fi}|^2 d\Omega$$

Three-body decay:

$$A \to 1 + 2 + 3$$

scalar

$$\int dL = \frac{1}{32\pi^3} \int dE_1 dE_2$$

flat in E_1 and E_2

$$d\Gamma_i(E_1, E_2) = \frac{1}{64\pi^3} \frac{1}{2m_A} |M_{fi}|^2 dE_1 dE_2$$

<u>Remark:</u> Instead of variables E_1 and E_2 one can use variables m_{12}^2 and m_{23}^2 = invariant mass of pairs (i,j)

$$m_{ij}^2 = (p_i + p_j)^2$$

$$dE_1 dE_1 = C \cdot dm_{12}^2 dm_{23}^2$$

$$d\Gamma_i(m_{12}, m_{23}) = \frac{1}{256\pi^3} \frac{1}{m_A^3} |M_{fi}|^2 dm_{12} dm_{23}$$

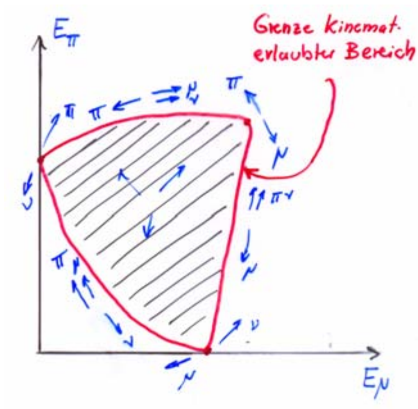
If phase space is flat in E_i then it is also flat in m_{ij}

Experimental method to explore behavior of $\boldsymbol{M_{fi}}$: Dalitz Analysis

5.2 Dalitz Plots

<u>Method:</u>

Put every measured decay into a 2-dim,. (E_1,E_2) distribution. A flat distribution over the allowed region corresponds to a “flat matrix element”. Structures in the dirstribution point to a varying matrix elem.



Ex.: $K^0 \rightarrow \pi^{\pm} \mu^{\mp} \nu$

Dalitz Plot: $\pi^+ \overline{K}^0 p$ final state at 3 GeV

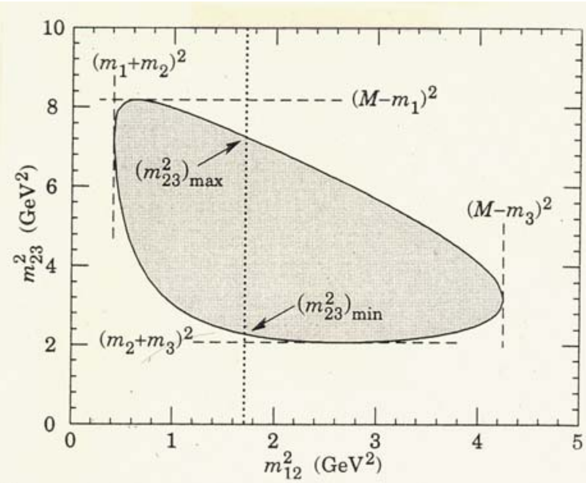


Figure 34.3: Dalitz plot for a three-body final state. In this example, the state is $\pi^+\overline{K}^0 p$ at 3 GeV. Four-momentum conservation restricts events to the shaded region.

Dalitz Plot: Example

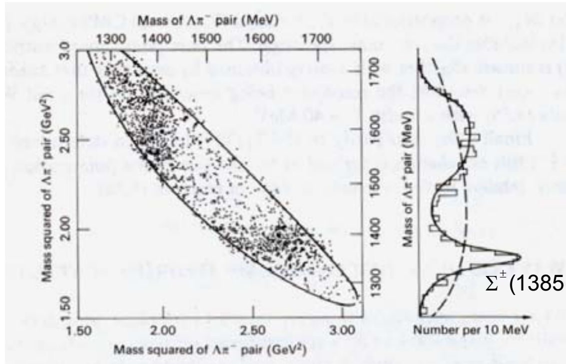
$$K^- + p \to \pi^+ \pi^- \Lambda$$
$$\hookrightarrow p\pi$$

$\Rightarrow$ 2 bands at $\Lambda\pi^\pm$ inv. masses of ~1385 MeV

<u>Explanation:</u>

$\Lambda\pi^\pm$ form an intermediate resonance state:

$$\Sigma^\pm(1385) \to \Lambda\pi^\pm$$



4.8 Dalitz plot of the $\Lambda\pi^+\pi^-$ events from reaction (4.36), as measured by Shafer , for 1.22-GeV/c incident momentum. The effective $\Lambda\pi^+$ mass spectrum is shown at ished curve is that expected for a phase-space distribution (ordinate equal to the inte within the Dalitz-plot boundary), while the full curve corresponds to a Breit-W nce expression fitted to the $\Lambda\pi^+$ and $\Lambda\pi^-$ systems.

Experiment: Kaon beam on a liquid H_2 bubble chamber