8) Quantisation of the Dirac field

$$\mathcal{L}(x) = \bar\psi \, (i \, \gamma \cdot \partial - m) \, \psi(x)$$

canonical momentum: $\Pi_\alpha(x) = \dfrac{\partial \mathcal{L}}{\partial \dot\psi_\alpha} = (\bar\psi \, i \gamma^0)_\alpha$

$$\Pi(x) = i \, \psi^\dagger(x)$$

note: $\bar\Pi = \dfrac{\partial \mathcal{L}}{\partial \dot{\bar\psi}} = 0$

Hamiltonian density: (Schrödinger picture, suppress x^0)

$$\mathcal{H}_D(\vec x) = \Pi(\vec x)\dot\psi(\vec x) - \mathcal{L}(\vec x) = \bar\psi \, (i \, \vec\gamma \cdot \vec\partial + m) \, \psi$$

Hamiltonian: $H = \int d^3x \; \mathcal{H}_D = \int d^3x \; \psi^\dagger \left(i \gamma^0 \vec\gamma \cdot \vec\partial + \gamma^0 m \right) \psi$

Insert $\psi(x) = \int \text{"}dp\text{"} \sum\limits_{s = \pm 1/2} \Big(e^{ipx} \, v_s(p) \, \beta^*_s(\vec p)$

$$+ e^{-ipx} \, u_s(p) \, \alpha_s(\vec p) \Big)$$

<u>Use</u>: $\gamma^0 \left(i \, \vec\gamma \cdot \vec\partial + m \right) u_s(p) \, e^{-ipx}$

$= \gamma^0 \left(\quad + \vec\gamma \cdot \vec p + m \right) u_s(p) \, e^{-ipx}$

$= \gamma^0 \big(\gamma^0 p^0 - \underbrace{(\gamma^\mu p_\mu - m)\big) \, u_s(p)}_{=0} \, e^{-ipx}$

$= p^0 \; u_s(p) \, e^{-ipx}$

and $\gamma^0 \left(i \vec{\gamma} \vec{\partial} + m \right) v_s(p) e^{ipx} = -p^0 v_s(p)$

Result:

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \sum_{s=\pm 1/2} \left(\alpha_s^*(\vec{p}) \alpha_s(\vec{p}) - \beta_s(\vec{p}) \beta_s^*(\vec{p}) \right)$$

Note: So far no $\alpha(\vec{p})$, $\beta(\vec{p})$ etc. had to be passed by each other!

- Promote to operators:

$$\alpha_s(\vec{p}) \longrightarrow a_s(\vec{p})$$
$$\alpha_s^*(\vec{p}) \longrightarrow a^+_s(\vec{p})$$
$$\beta_s(\vec{p}) \longrightarrow b_s(\vec{p})$$
$$\beta_s^*(\vec{p}) \longrightarrow b_s^+(\vec{p})$$

Problem: If impose commutation relations
$\rightsquigarrow$ H unbounded because of $- b_s(p) b_s^+(p)$

Only way out: anti-commutation relations:

$$\left\{ a_r(\vec{p}), a_s^+(\vec{p}') \right\} = \delta_{rs} (2\pi)^3 \cdot 2E_p \; \delta^3(\vec{p} - \vec{p}')$$
$$\left\{ b_r(\vec{p}), b_s^+(\vec{p}') \right\} = \delta_{rs} (2\pi)^3 \; 2E_p \; \delta^3(\vec{p} - \vec{p}')$$
$$\left\{ a, a \right\} = 0 = \left\{ a^+, a^+ \right\} = \left\{ b^+, b^+ \right\} = \{ b, b \}$$

$$\Rightarrow H = \int \frac{d^3p}{(2\pi)^3} \frac{\hbar}{2E_p} \frac{\hbar}{2} \sum_s \left(a_s^+(\vec{p})\, a_s(\vec{p}) + b_s^+(\vec{p})\, b_s(\vec{p}) \right)$$

$$\underbrace{- E_0}_{\text{vacuum energy}} \longrightarrow \text{discarded}$$

Interpretation:

- H is now bounded below $\Longrightarrow$ $|0\rangle$ normalised vacuum state $\langle 0|0\rangle$
- Same logic as for scalar field reveals:

$$a_s(\vec{p})|0\rangle = 0 = b_s(\vec{p})|0\rangle$$

- $a_s^+(\vec{p})$, $b_s^+(\vec{p})$: creation operators

$$a_s^+(\vec{p})|0\rangle =: |e^-(p,s)\rangle$$

$$b_s^+(\vec{p})|0\rangle =: |e^+(p,s)\rangle$$

where $p = \begin{pmatrix} E_p \\ \vec{p} \end{pmatrix}$

$s = \pm 1/2 \longleftrightarrow$ spin

$e^-(p,s)$, $e^+(p,s)$ are different particle species of the same mass, but described by the same complex field $\psi(x)$.
But they differ in one conserved quantum number called charge.

Charge is a manifestation of a global symmetry of the theory.

Insert: **Noether's theorem:**

Consider $S = \int d^4x \, \mathcal{L}(\phi, \partial\phi)$ which is invariant under a transformation

$$\phi \longrightarrow \phi' = (1 + \delta_\alpha + \dots)\,\phi$$

where α is a continuous parameter that is independent of position.

Then this symmetry leads to a conserved current $J^\mu(x)$

$$\partial_\mu J^\mu(x) = 0 \quad \text{and a conserved charge}$$

$$Q(t) = \int d^3x \; J^0(t, \vec{x}) \quad : \quad \frac{\partial}{\partial t} Q = 0$$

$Q(t)$ is the generator of the symmetry.

Proof: By assumption $\delta_\alpha S = 0$ for α indep. of x.

Promote $\alpha \mapsto \alpha(x)$

$$\Rightarrow \delta_\alpha S = \int d^4x \; J^\mu \partial_\mu \alpha(x) = -\int d^4x \; \partial_\mu J^\mu \alpha(x)$$

Since $\delta_\alpha S = 0$ for α indep. of $x \Rightarrow \partial_\mu J^\mu = 0$

$$\frac{\partial}{\partial t} Q = \int d^3x \; \underbrace{\partial_0 J^0 + \partial_i J^i}_{=0} - \partial_i J^i = -\int d^3x \; \partial_i J^i = 0.$$

Back to Dirac: $\mathcal{L} = \bar{\psi}(i\gamma\cdot\partial - m)\psi$

Symmetry: $\psi(x) \longrightarrow e^{i\alpha}\psi(x)$ $\quad \alpha$ constant

$$= (1 + i\alpha + \dots)\,\psi(x)$$

Promote $\alpha \rightarrow \alpha(x)$:

$$\delta_\alpha S = -\int d^4x \, (\bar\psi \gamma^\mu \psi)\, \partial_\mu \alpha(x)$$

$\Rightarrow$ $J^\mu = -e\,\bar\psi \gamma^\mu \psi$ is conserved ✓

(e : constant)

$$Q = -e\int d^3x\, \bar\psi \gamma^0 \psi = -e \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left(a_s^+(\vec p)\, a_s(\vec p) - b_s^+(\vec p)\, b_s(\vec p) \right)$$

| Note the | sign !

$\Rightarrow$ $Q\; a_s^+(\vec p)|0\rangle = -e \;\; a_s^+(\vec p)|0\rangle$

$Q\; b_s^+(\vec p)|0\rangle = +e \;\; b_s^+(\vec p)|0\rangle$

$\Longleftrightarrow$ $|e^-(\vec p, s)\rangle \longleftrightarrow$ charge $-e$

$|e^+(\vec p, s)\rangle \longleftrightarrow$ charge $+e$

<u>Note</u>: At this stage "charge" just refers to the eigenvalue under Q. Later we will see that this eigenvalue describes coupling to the el.-magn. field.

$\rightarrow$ Dirac theory predicts 2 species of matter

e^- $\longleftrightarrow$ "matter" "electron"

e^+ $\longleftrightarrow$ anti-matter "positron"

Note: Q indeed generates the transformation $\psi \to e^{i\alpha}\psi$ in the sense

$$e^{-i\alpha Q}\, \psi\, e^{+i\alpha Q} = e^{i\alpha}\psi \qquad \text{(see tutorial)}$$

- Back to Fock space:

The anti-commutation relations enforce Fermi-Dirac statistics:

$$|e^-(\vec{p}_1, s_1)\, e^-(\vec{p}_2, s_2)\rangle = a^\dagger_{s_1}(\vec{p}_1)\, a^\dagger_{s_2}(\vec{p}_2)\, |0\rangle$$

$$= -a^\dagger_{s_2}(\vec{p}_2)\, a^\dagger_{s_1}(\vec{p}_1)\, |0\rangle$$

$$= -\, |e^-(\vec{p}_2, s_2)\, e^-(\vec{p}_1, s_1)\rangle$$

In particular: $|e^-(\vec{p}_1, s_1)\, e^-(\vec{p}_1, s_1)\rangle$ is forbidden

(Pauli-exclusion-principle)

More generally the spin-statistics-theorem correlates:

$s \in \mathbb{Z} + \frac{1}{2}$ $\longleftrightarrow$ anti-commutation relations $\longleftrightarrow$ Fermi-Dirac statistics

$s \in \mathbb{Z}$ $\longleftrightarrow$ commutation relations $\longleftrightarrow$ Bose-Einstein statistics

- Orthogonality:

$$\langle e^-(\vec{p}, s) | e^-(\vec{p}\,', s')\rangle = \langle 0 | \{a_s(\vec{p}), a^\dagger_{s'}(\vec{p}\,')\} | 0\rangle$$

$$= (2\pi)^3\, 2E_p\, \delta^{(3)}(\vec{p} - \vec{p}\,')\, \delta_{ss'}$$

Time ordered product & propagator

For perturbation theory we need

$$T\left(\psi_\alpha(x)\,\bar{\psi}_\beta(y)\right) = \begin{cases} \psi_\alpha(x)\,\bar{\psi}_\beta(y) & x^0 > y^0 \\ -\bar{\psi}_\beta(y)\,\psi_\alpha(x) & y^0 > x^0 \end{cases}$$

or

$$T\left(\psi(x)\,\bar{\psi}(y)\right) = \begin{cases} \psi(x)\,\bar{\psi}(y) & x^0 > y^0 \\ -\bar{\psi}(y)\,\psi(x) & y^0 > x^0 \end{cases}$$

The Feynman propagator is defined as

$$i\,S_{F\,\alpha\beta}(x-y) = \langle 0 | T\,\psi(x)_\alpha\,\bar{\psi}(y)_\beta | 0 \rangle$$

One can prove that

$$i\,S_{F\,\alpha\beta}(x-y) = i\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip\cdot(x-y)}}{\gamma\cdot p - m + i\varepsilon}$$

$$\equiv i\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip(x-y)}}{p^2-m^2+i\varepsilon}\left(\gamma^\mu p_\mu + m\mathbb{1}\right)_{\alpha\beta}$$

and $\quad (i\gamma\cdot\partial - m)\,S_F(x-y) = \delta(x-y)$