Solutions

Motivated by $(\partial^2 + m^2)\psi = 0$ we make the ansatz

$$\psi_\alpha(x) = u_\alpha(p)\, e^{-ipx} \qquad p^\mu = \begin{pmatrix} E \\ \vec{p} \end{pmatrix} \qquad E = \sqrt{\vec{p}^{\,2} + m^2}$$

or $\psi_\alpha(x) = v_\alpha(p)\, e^{ipx}$

Back into $(i\gamma^\mu \partial_\mu - m)\psi_\alpha = 0$

$$\Rightarrow (\gamma^\mu p_\mu - m)\, u(p) = 0$$

$$(\gamma^\mu p_\mu + m)\, v(p) = 0$$

Since $\gamma \cdot p - m = \begin{pmatrix} E - m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(E+m) \end{pmatrix}$ we make

the ansatz: $u(p) = \begin{pmatrix} \chi \\ \phi \end{pmatrix}$ $\qquad \chi, \phi$: 2-component spinors

$$\Rightarrow (E-m)\chi - \vec{\sigma}\cdot\vec{p}\,\phi = 0$$

$$\vec{\sigma}\cdot\vec{p}\,\chi - (E+m)\phi = 0 \quad \rightarrow \quad \phi = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi$$

$$\Rightarrow u_s(p) = \sqrt{E+m} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \end{pmatrix} \qquad \chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(s = \pm 1/2) \qquad \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Similarly: $v_s(p) = +\sqrt{E+m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \\ \chi_s \end{pmatrix}$

Note: $\sum_{s=\pm 1/2} u_s(p)\bar{u}_s(p) = \gamma \cdot p + m \equiv \not{p} + m$

$$\sum_{s=\pm 1/2} v_s(p)\bar{v}_s(p) = \gamma \cdot p - m$$

General solution:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{s=\pm 1/2} \left\{ e^{ipx} v_s(p) \beta^*_s(p) + e^{-ipx} u_s(p) \alpha_s(p) \right\}$$

$\alpha_s(p), \beta_s(p)$: independent coefficient functions

Spin

Consider rotation around axis with normalised direction $\vec{n}$ by angle Θ

$$\Lambda = \exp\left(-\frac{i}{4} M^{ij} \omega_{ij}\right) \qquad \omega_{ij} = \Theta n_k \varepsilon_{ijk}$$

The spinor transforms by

$$S(\Lambda) = \exp\left(-\frac{i}{4} \sigma^{ij} \omega_{ij}\right) = \exp\left(-\frac{i}{4} \Theta n_k \sigma^{ij} \varepsilon_{ijk}\right)$$

Use: $\frac{1}{2} \varepsilon_{ijk} \sigma^{ij} = \frac{1}{2} \varepsilon_{ijk} \frac{i}{2} [\gamma^i, \gamma^j] = \frac{i}{2} \varepsilon_{ijk} \gamma^i \gamma^j$

$$= \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \equiv \Sigma^k$$

Σ^k is called spin operator

$$\Rightarrow S(\Lambda) = \exp\left(-\frac{i}{2} \Theta n_k \Sigma_k\right)$$

$$= \begin{pmatrix} e^{-i/2 \Theta n_k \sigma_k} & 0 \\ 0 & e^{-i/2 \Theta n_k \sigma_k} \end{pmatrix}$$

$$\rightarrow S(\Lambda)\, u(p) = \sqrt{E+m} \begin{pmatrix} \chi_R \\ \frac{\vec{\sigma} \cdot R\vec{p}}{E+m} \chi_R \end{pmatrix}$$

where $\chi_R = e^{-\frac{i}{2}\Theta\, n_k \sigma_k} \chi$

$$\vec{\sigma} \cdot R\vec{p} = e^{-\frac{i}{2}\Theta\, n_k \sigma_k}\, \vec{\sigma} \cdot \vec{p}\; e^{\frac{i}{2}\Theta\, n_k \sigma_k}$$

$\rightarrow$ $\chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ transform like spin $1/2$ objects

Chirality

- Define $\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$

$$(\gamma^5)^\dagger = \gamma^5 \qquad \{\gamma^5, \gamma^\mu\} = 0 \qquad (\gamma^5)^2 = \mathbb{1}$$

- Consider the massless Dirac equation

$$i\gamma^\mu \partial_\mu \psi = 0 \quad \Rightarrow \quad i\gamma^\mu \partial_\mu (\gamma_5 \psi) = 0$$

- Since $(\gamma_5)^2 = \mathbb{1}$ we can define projection operators:

$$P_L = \frac{1}{2}(\mathbb{1} - \gamma_5) \qquad P_R = \frac{1}{2}(\mathbb{1} + \gamma_5)$$

$$P_L^2 = P_L\,, \quad P_R^2 = P_R\,, \quad P_L P_R = 0 = P_R P_L\,, \quad P_L + P_R = \mathbb{1}$$

- A spinor is called chiral / left-chiral if

$$P_L \psi = \psi$$

and anti-chiral / right-chiral if

$$P_R \psi = \psi$$

$$\Rightarrow \left.\begin{matrix} \psi_L = P_L \psi \\ \psi_R = P_R \psi \end{matrix}\right\} \text{ is the } \left\{\begin{matrix} \text{left-chiral part} \\ \text{anti-chiral part} \end{matrix}\right\} \text{ of a given } \psi$$

- For $m = 0$ if ψ satisfies Dirac eqn., then also ψ_L and ψ_R
- Consider u_L : $P_L u_L = u_L$, again for massless case

$$\Leftrightarrow \frac{1}{2}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}\begin{pmatrix} \chi_L \\ \frac{\vec{\sigma}\circ\vec{p}}{E}\chi_L \end{pmatrix} = \begin{pmatrix} \chi_L \\ \frac{\vec{\sigma}\circ\vec{p}}{E}\chi_L \end{pmatrix}$$

$$\Leftrightarrow \boxed{\frac{1}{2}(\vec{\sigma}\circ\hat{\vec{p}})\,\chi_L = -\frac{1}{2}\chi_L} \qquad \hat{\vec{p}} = \frac{\vec{p}}{E} \text{ for a massless particle}$$

and u_R : $P_R u_R = u_R$

$$\Leftrightarrow \boxed{\frac{1}{2}(\vec{\sigma}\circ\hat{\vec{p}})\,\chi_R = +\frac{1}{2}\chi_R}$$

Def.: The helicity of a particle is the component of the spin along the direction of motion.
It is measured by $\frac{1}{2}\vec{\Sigma}\circ\hat{\vec{p}}$ $\qquad \vec{\Sigma}$: spin operator

$$\Rightarrow \boxed{\begin{matrix} \psi_L & \Leftrightarrow & \text{helicity } -\frac{1}{2} \\ \psi_R & \Leftrightarrow & \text{helicity } +\frac{1}{2} \end{matrix}}$$

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- Given a general ψ one can decompose

$$\psi(x) = \psi_L(x) + \psi_R(x)$$

Since $\psi_L(x) = \frac{1}{2}(1-\gamma_5)\psi(x)$

$$\overline{\psi_L}(x) = \psi_L^\dagger \gamma^0 = \psi^\dagger \left(\tfrac{1}{2}(1-\gamma_5^\dagger)\right)\gamma^0$$

$$= \psi^\dagger \gamma^0 \cdot \tfrac{1}{2}(1+\gamma_5) = \overline{\psi}\, \tfrac{1}{2}(1+\gamma_5)$$

$$\Rightarrow \overline{\psi_L}\psi_L = 0 = \overline{\psi_R}\psi_R$$

Similarly: $\overline{\psi_R}\gamma^\mu \psi_L = 0 = \overline{\psi_L}\gamma^\mu \psi_R = 0$

$$\Rightarrow \mathcal{L} = \overline{\psi}(x)\left(i\gamma^\mu \partial_\mu - m\right)\psi(x)$$

$$= \overline{\psi_R}(x)\left(i\gamma\cdot\partial\right)\psi_R(x) + \overline{\psi_L}(i\gamma\cdot\partial)\psi_L$$

$$- m\left[\overline{\psi_R}\psi_L + \overline{\psi_L}\psi_R\right]$$

$\Rightarrow$ i) $m = 0 \rightarrow$ theory decomposes into chiral / anti-chiral part

ii) $m \neq 0 \rightarrow$ chiral symmetry is broken

The fact that chirality is a sensible concept only for $m = 0$ is consistent with the fact that for a massive particle, helicity depends on the frame!