7 Spin - 1/2 - Fields : Classical Dirac theory

- Spin-1/2 objects are described by spinor fields, defined by the following transformation under Lorentz transformations:

  Suppose you find a set of matrices $(\gamma^\mu)_{\alpha\beta}$

  $$\mu : 0, 1, 2, 3 \qquad \alpha, \beta = 1, \dots, n$$

  such that: $\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \mathbb{1}_{(n \times n)}$

  (Clifford or Dirac algebra)

  Then: $\sigma^{\mu\nu} := \frac{i}{2} [\gamma^\mu, \gamma^\nu]$

A spinor $\psi_\alpha(x)$ is a field on which $(\sigma^{\mu\nu})_{\alpha\beta}$ acts as follows:

as $\quad x^\mu \longmapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \Lambda = \exp\left(-\frac{i}{4} M^{\mu\nu} \omega_{\mu\nu}\right)$

$$\psi_\alpha(x) \longmapsto \boxed{\psi'_\alpha(x') = S_{\alpha\beta}\, \psi_\beta(\Lambda^{-1} x')}$$

where $\quad S_{\alpha\beta} = \exp\left(-\frac{i}{4} \sigma^{\mu\nu} \omega_{\mu\nu}\right)_{\alpha\beta}$

Note: The index "α" of ψ_α is a spinor index.

- The simplest set of γ-matrices $(\gamma^\mu)_{\alpha\beta}$ satisfying the Clifford algebra are 4×4 matrices:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where σ^i are the 2-dim. Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma^i \sigma^j = \delta^{ij} + i \varepsilon^{ijk} \sigma_k$$

$\Rightarrow$ Dirac spinor ψ_α $\quad \alpha = 1,2,3,4$ $\quad$ spinor indices

Note: Don't confuse $\mu, \nu = 0,1,2,3$ and
$\alpha, \beta = 1,2,3,4$

- That ψ_α indeed corresponds to a spin-$\frac{1}{2}$ object follows from its behavior under spatial rotations:

$\omega_{\mu\nu} \rightarrow \omega_{ij}$

$$\sigma^{ij} = \frac{i}{2}[\gamma^i, \gamma^j] = \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \varepsilon^{ijk} \cdot \Sigma^k$$

- We want to find a Lorentz invariant equation for $\psi_\alpha(x)$ s.t. it describes a particle/field of mass m.

The simplest is the

Dirac equation: $\boxed{(i\gamma^\mu \partial_\mu - m)\psi(x) = 0}$

$\rightsquigarrow$ Lorentz invariance follows with the help of

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu$$

$\rightsquigarrow$ mass m:

$$(-i\gamma^\mu \partial_\mu - m)(i\gamma^\nu \partial_\nu - m)\psi(x)$$
$$= \left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu + m^2\right)\psi(x) = (\partial^2 + m^2)\psi(x) \checkmark$$

- To define an action we need a method to create scalars from $\psi_\alpha(x)$.

Define $\bar{\psi}(x) = \psi^\dagger \gamma^0$ where $\psi^\dagger = (\psi^*)^T$

It satisfies: $\bar{\psi}(x) \underset{x' = \Lambda x}{\longmapsto} \bar{\psi}(\Lambda^{-1} x') S^{-1}(\Lambda)$

(Proof: See tutorial)

$\rightsquigarrow$ we can form the following quantities

$S(x) = \bar{\psi}(x)\, \psi(x)$ scalar

$j_\mu(x) = \bar{\psi}(x)\, \gamma_\mu\, \psi(x)$ vector

$S_{\mu\nu}(x) = \bar{\psi}(x)\, \sigma_{\mu\nu}\, \psi(x)$ tensor

(pseudo scalar/vector later)

- The Dirac action is $S = \int d^4x\, \mathcal{L}$ with

$$\boxed{\mathcal{L} = \bar{\psi}(x) \left(i \gamma^\mu \partial_\mu - m \right) \psi(x)}$$

E.o.m. $\dfrac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\mu \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0$

$\rightsquigarrow \left(i \gamma^\mu \partial_\mu - m \right) \psi(x) = 0$ ✓

Solutions

Motivated by $(\partial^2 + m^2)\psi = 0$ we make the ansatz

$$\psi_\alpha(x) = u_\alpha(p)\, e^{-ipx}$$

or $\psi_\alpha(x) = v_\alpha(p)\, e^{ipx}$ $\qquad p^\mu = \begin{pmatrix} E \\ \vec{p} \end{pmatrix} \qquad E = \sqrt{\vec{p}^{\,2} + m^2}$

Back into $(i\gamma^\mu \partial_\mu - m)\psi_\alpha = 0$

$$\Rightarrow \quad (\gamma^\mu p_\mu - m)\, u(p) = 0$$

$$(\gamma^\mu p_\mu + m)\, v(p) = 0$$

Since $\gamma \circ p - m = \begin{pmatrix} E - m & -\vec{\sigma}\circ\vec{p} \\ \vec{\sigma}\circ\vec{p} & -(E+m) \end{pmatrix}$ we make

the ansatz: $\quad u(p) = \begin{pmatrix} \chi \\ \phi \end{pmatrix} \qquad \chi, \phi$: 2-component spinors

$$\Rightarrow (E-m)\chi - \vec{\sigma}\circ\vec{p}\,\phi = 0$$

$$\vec{\sigma}\circ\vec{p}\,\chi - (E+m)\phi = 0 \quad \longrightarrow \quad \phi = \frac{\vec{\sigma}\circ\vec{p}}{E+m}\chi$$

$$\Rightarrow \quad u_s(p) = \sqrt{E+m}\begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\circ\vec{p}}{E+m}\chi_s \end{pmatrix} \qquad \chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Similarly: $\quad v_s(p) = +\sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\circ\vec{p}}{E+m}\chi_s \\ \chi_s \end{pmatrix}$

Note: $\displaystyle\sum_{s=\pm 1/2} u_s(p)\bar{u}_s(p) = \gamma\circ p + m = \not{p} + m$

$$\sum_{s=\pm 1/2} v_s(p)\bar{v}_s(p) = \gamma\circ p - m$$

General solution:

$$\psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \sum_{s=\pm 1/2} \left\{ e^{ipx} \, v_s(p) \beta^*_s(p) + e^{-ipx} \, u_s(p) \, \alpha_s(p) \right\}$$

$\alpha_s(p), \beta_s(p)$: independent coefficient functions