6) Interacting Fields & Perturbation Theory

6.a) The S-matrix

Suppose $\mathcal{L} = \underbrace{\mathcal{L}_0}_{\text{free Lagrangian}} + \underbrace{\mathcal{L}_I}_{\text{interactions}}$

e.g. $\mathcal{L}_0 = -\frac{1}{2} \phi\left(\partial^2 + m^2\right)\phi$

$$\mathcal{L}_I = -\frac{g}{3!}\phi^3(x) - \frac{\lambda}{4!}\phi^4(x)$$

Define the Hamiltonian $H = H_0 + H_I$

Note: H is time-independent & defined in terms of

$$\phi(x)\Big|_{t=0} = \phi(0,\vec{x})$$

$$H_0 = \int d^3x \left\{ \frac{1}{2}\pi^2(0,\vec{x}) + \frac{1}{2}\left(\nabla\phi(0,\vec{x})\right)^2 + \frac{m^2}{2}\phi^2(0,\vec{x})\right\}$$

$$H_I = -\int d^3x\, \mathcal{L}_I\Big|_{t=0} = d^3x \left\{\frac{g}{3!}\phi^3(0,\vec{x}) + \frac{\lambda}{4!}\phi^4(0,\vec{x})\right\}$$

Glossing over many details, one goes to the interaction picture as follows:

- Define $\overline{H}_I(t) = e^{iH_0 t} H_I e^{-iH_0 t}$

& $\phi_I(t) = e^{iH_0 t}\,\Phi(0,\vec{x})\,e^{-iH_0 t}$

$$\Rightarrow \overline{H}_I(t) = \int d^3x \left\{\frac{g}{3!}\Phi_I^3(t,\vec{x}) + \frac{\lambda}{4!}\phi_I^4(t,\vec{x})\right\}$$

- One finds:

$$\boxed{i\frac{d}{dt}\left|\Phi(t)\right\rangle = \overline{H}_I(t)\left|\phi(t)\right\rangle}$$

for $\left|\Phi(t)\right\rangle \equiv \Phi_I(t)\left|0\right\rangle$ as state in the interaction picture

• Solution: $|\phi(t)\rangle = |\phi_i\rangle + \frac{1}{i}\int_{-\infty}^{t} dt' \, H_I(t') |\phi(t')\rangle$

$|\phi_i\rangle$: asymptotic state for a part at $t_i = -\infty$

• Evolution from $t_i = -\infty$ up to t is governed by

$$|\phi(t)\rangle = U(t)\,|\phi_i\rangle$$

where: $$U(t) = \mathbb{1} + \int_{-\infty}^{t} dt' \, H_I(t')\, U(t')$$

$$\Rightarrow \quad i\frac{d}{dt}U(t) = H_I(t)\, U(t)$$

⇒ Solution in terms of perturbation series:

$$U(t) = \mathbb{1} + \frac{1}{i}\int_{-\infty}^{t} dt' \, H_I(t') +$$
$$+ \frac{1}{i^2}\int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' \, H_I(t')\, H_I(t'')$$
$$+ \dots$$

In terms of the time ordered product

$$T\left(A(t_1)A(t_2)\dots A(t_n)\right) = A(t_{i_1})\,A(t_{i_2})\dots A(t_{i_n})$$

where $t_{i_1} \geq t_{i_2} \geq \dots \geq t_{i_n}$

$$U(t) = \mathbb{1} + \frac{1}{i}\int_{-\infty}^{t} dt' \, H_I(t') +$$
$$+ \left(\frac{1}{i}\right)^2 \frac{1}{2!}\int_{-\infty}^{t} dt' \int_{-\infty}^{t} dt'' \, T\left(H_I(t')\, H_I(t'')\right)$$
$$+ \left(\frac{1}{i}\right)^3 \frac{1}{3!}\int_{-\infty}^{t} dt' \int_{-\infty}^{t} dt'' \int_{-\infty}^{t} dt''' \, T\left(H_I(t')\, H_I(t'')\, H_I(t''')\right)$$

$$\equiv \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{t} dt_1 \dots dt_n \, T\left(H_I(t_1)\dots H_I(t_n)\right)$$

Define the $\underline{\text{S-matrix}}$

$$S = \lim_{t\to\infty} U(t)$$

that describes scattering from $t_i = -\infty$ to $t_f = +\infty$

$$\Rightarrow \quad S = T \exp\left\{ -i \int_{-\infty}^{\infty} dt\, \overline{H}_I(t) \right\}$$

Since $\overline{H}_I(t) = -\int d^3x\, \mathcal{L}_I(t,\vec{x})$ we find:

$$\boxed{\begin{array}{l} |i\rangle \longrightarrow |f\rangle = S|i\rangle \\ S = T\exp\left(i\int d^4x\, \mathcal{L}_I(x)\right) \end{array}}$$

The S-matrix element for $|i\rangle \to |f\rangle$ is defined as

$$\boxed{\langle f|S|i\rangle \equiv \delta_{fi} + i(2\pi)^4\, \delta^{(4)}(P_f - P_i)\, T_{fi}}$$

$\delta_{fi} \longleftrightarrow$ from "1" in exp

T_{fi} is called scattering amplitude

significance:

The unnormalised probability for scattering to a range of final states is:

$$\mathcal{P} = \sum_f |\langle f|S|i\rangle|^2$$

Use: $|\delta^{(4)}(P_f - P_i)|^2 = \delta^{(4)}(P=0)\, \delta^{(4)}(P_f - P_i)$

$$\rightarrow \quad \mathcal{P} = (2\pi)^4\, \delta^{(4)}(0) \sum_f (2\pi)^4\, \delta^{(4)}(P_f - P_i)\, |T_{fi}|^2$$

Use: $(2\pi)^4\, \delta^{(4)}(P) = \int d^4x\, e^{-iPx}$

$$\Rightarrow (2\pi)^4\, \delta^{(4)}(0) = \mathcal{V}\cdot\mathcal{T}$$

$\mathcal{V}$: volume of space

$\mathcal{T}$: time of transition

Use: $\phi^+(x)\,|p\rangle = e^{-ipx}\,|0\rangle$

$$\Rightarrow \left(\phi^+(x)\right)^2 |p_1 p_2\rangle = \underset{\Uparrow}{2}\, e^{-ip_1 x}\, e^{-ip_2 x}\,|0\rangle$$

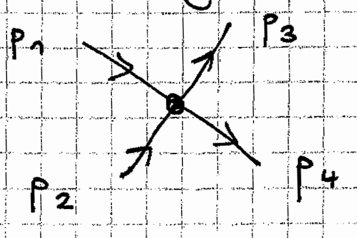
'first' $\phi^+(x)$ can act on $|p_1\rangle$ or $|p_2\rangle$
⇝ factor 2

$$\langle p_3 p_4|\left(\phi^-(x)\right)^2 = 2\, e^{ip_3 x}\, e^{ip_4 x}\,\langle 0|$$

$$\Rightarrow \langle p_3, p_4|S|p_1, p_2\rangle = (-i\lambda)\cdot\int\frac{d^4x}{4!}\binom{4}{2}\cdot 2!\,2!\cdot e^{i(p_3+p_4-p_1-p_2)x}$$

$$= i(2\pi)^4\cdot\delta^{(4)}(p_3+p_4-p_1-p_2)\,\underbrace{(-\lambda)}_{=:\,T(p_3,p_4,p_1,p_2)}$$

The result can be represented as a Feynman diagram:

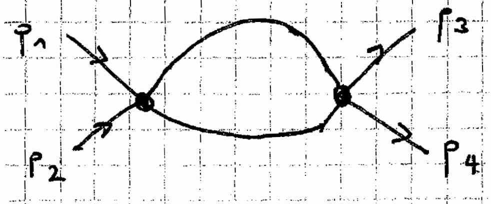


Feynman rules:

to compute T in momentum space:

- $(-\lambda)$ at each vertex
- momentum conservation at each vertex

Next order:



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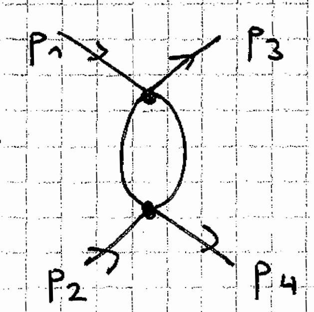

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