4 Quantisation of the free scalar field

Recap on QM:

$$S = \int dt\, L(q_i, \dot{q}_i) \quad , \quad \text{canonical momentum: } \pi_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$H = \sum_i \pi_i \dot{q}_i - L$$

Quantisation: $q_i, \pi_i \Rightarrow$ operators subject to equal time commutation relations: $[q_i(t), \pi_j(t)] = i\delta_{ij}$

Field Theory:

$$q_i(t) \longrightarrow \varphi(t, \vec{x}) \equiv \varphi(x) \qquad \text{continuous label "}\vec{x}\text{"}$$

$$\pi_i = \frac{\partial L}{\partial \dot{q}_i} \longrightarrow \pi(t, \vec{x}) = \frac{\partial}{\partial(\partial_t \varphi(t, \vec{x}))} \int d^3x' \, \mathcal{L}(t, \vec{x}')$$

Klein-Gordon field:

$$\mathcal{L}(x) = \frac{1}{2} \dot{\varphi}(t, \vec{x})^2 - \frac{1}{2}\left[(\vec{\nabla} \varphi(t, \vec{x}))^2 + m^2 (\varphi(t, \vec{x}))^2 \right]$$

$$\Longrightarrow \pi(x) = \dot{\varphi}(x)$$

Quantisation: $\varphi(t, \vec{x})$ becomes operator $\Phi(t, \vec{x})$ acting on some vacuum

Equal time commutators:

$$[\Phi(t, \vec{x}), \pi(t, \vec{x}')] = i\,\delta^{(3)}(\vec{x} - \vec{x}')$$

$$[\phi(t, \vec{x}), \phi(t, \vec{x}')] = 0$$

$$[\pi(t, \vec{x}), \pi(t, \vec{x}')] = 0$$

Real fields $\varphi = \varphi^*$ correspond to hermitian operators $\Phi = \Phi^+$

In this case:

$$\Phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left[a(p)\, e^{-ipx} + a^+(p)\, e^{ipx} \right]$$

$a(p)$, $a^\dagger(p)$ are operators satisfying

$$\left.\begin{aligned} [a(p), a^\dagger(p')] &= (2\pi)^3\, 2E_p\, \delta^{(3)}(\vec{p}-\vec{p}\,') \\ [a(p), a(p')] &= 0 = [a^\dagger(\vec{p}), a^\dagger(\vec{p}\,')] \end{aligned}\right\} \text{harmonic essillator } \forall \text{ mode } p$$

Note: If $\Phi \neq \Phi^\dagger$ then $a(p)$, $b^\dagger(p)$, $b^\dagger(p) \neq (a(p))^\dagger$

Define the operator

$$P^\mu = \int d^3x' \, \pi(t,\vec{x}) \, \partial^\mu \phi(t,\vec{x}\,')$$

One can show that

$$\exp(-ia.P)\, \Phi(x)\, e^{ia.P} = \Phi(x-a),$$ identifying

P^μ as the 4-momentum operator

In modes:

$$P^\mu = \frac{1}{2} \sum_p p^\mu \left(a(p) a^\dagger(p) + a^\dagger(p) a(p) \right)$$

$$[P^\mu, a^\dagger(p)] = p^\mu a^\dagger(p)$$ by use of commutation relations

$$[P^\mu, a(p)] = -p^\mu a(p)$$

i.e. $P^\mu a^\dagger(p) = a^\dagger(p) (P^\mu + p^\mu)$

P^μ is hermitian and thus diagonalisable

Define $|k^\mu\rangle$ as eigenstate with momentum k^μ:

$$P^\mu |k^\mu\rangle = k^\mu |k^\mu\rangle$$

$$\Rightarrow P^\mu a^\dagger(p)|k^\mu\rangle = (k^\mu + p^\mu) \, |k^\mu\rangle$$

$$\rightarrow \left.\begin{matrix} a^\dagger(p) \text{ adds} \\ a(p) \text{ removes} \end{matrix}\right\} \text{1-particle excitation of momentum } p^\mu$$

Assumption:

Hamiltonian $H \equiv P^0$ bounded below

$\longleftrightarrow$ $\exists$ groundstate $|0\rangle$ of minimal energy

$\longleftrightarrow$ $a(p)|0\rangle = 0$

Then: $P^\mu \, a^+(p)\,|0\rangle = p^\mu \, a^+(p)|0\rangle$

$\rightarrow$ $a^+(p)|0\rangle$ 1-particle state w/ momentum p^μ

$$a^+(p)|0\rangle =: |p\rangle$$

Normalisation:

$$\langle p'|p\rangle = \langle 0|\, a(\vec{p}\,')\, a^+(\vec{p})|0\rangle = \langle 0|\,[\,a(\vec{p}),a^+(\vec{p}\,')]\,|0\rangle$$

$$= (2\pi)^3 \; 2E_p \; \delta^{(3)}(\vec{p}-\vec{p}\,') \underbrace{\langle 0|0\rangle}_{\equiv 1}$$

General 1-particle state:

$$|f\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \; f(\vec{p}) \; a^+(p)|0\rangle$$

N-particle states:

- 2 particle: $a^+(p_2)\,a^+(p_1)\,|0\rangle$
- 3 particle: $a^+(p_3)\,a^+(p_2)\,a^+(p_1)|0\rangle$

Bose symmetry: $a^+(p_2)\,a^+(p_1) = a^+(p_1)\,a^+(p_2)$

5) Time and normal ordering, Wicks theorem

Split $\phi(x) = \phi^+(x) + \phi^-(x)$

$$\phi^+(x) = \int_p a(p) e^{-ipx} \qquad \int_p \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}$$

$$\phi^-(x) = \int_p a^+(p) e^{ipx}$$

Note: $\phi^+(x)\,|p\rangle = \phi^+(x)\, a^+(p) = \int_{p'} [a(p'), a^+(p)]\,|0\rangle\, e^{-ip'x}$

$$= e^{-ipx}\,|0\rangle$$

$$\langle p|\phi^-(x) = \langle 0| e^{ipx}$$

Define the normal-ordered product as "creation operators to left":

$$:\phi(x)\phi(y): \;=\; \phi^-(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^+(x)\phi^+(y)$$

Note: $\langle 0| :\phi(x)\phi(y): |0\rangle = 0$

Define the time-ordered product as:

$$T\big(\phi(x)\phi(y)\big) = \Theta(x^0-y^0)\,\phi(x)\phi(y) + \Theta(y^0-x^0)\,\phi(y)\phi(x)$$

where $\Theta(x^0-y^0) = \begin{cases} 1 & \text{if } x^0-y^0>0 \\ 0 & \text{otherwise} \end{cases}$

We define: $i\Delta_F(x-y) = \langle 0|T\big(\phi(x)\phi(y)\big)|0\rangle$

$i\Delta_F(x-y)$ is called Feynman propagator.

One can show that:

$$i\Delta_F(x) = i\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ipx}}{p^2-m^2+i\varepsilon}$$

Note: $i\Delta_F(x)$ is the Green's function for the KG equ.:

$$(\partial^2+m^2)\, i\Delta_F(x) = -i\,\delta^{(4)}(x)$$

Wick's theorem

relates $: :$ to $T(\ )$:

$$T(\phi(x)\phi(y)) = :\phi(x)\phi(y): + i\Delta_F(x-y)$$

either by calculation or by noting that if

$$T(\phi(x)\phi(y)) = :\phi(x)\phi(y): + X, \text{ then}$$

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = \underbrace{\langle 0|:\phi(x)\phi(y):|0\rangle}_{=0} + \langle X\rangle$$

and X is c-number.

Generalisation:

$$T(\phi(1)\phi(2)\dots\phi(n)) = :\phi(1)\phi(2)\dots\phi(n):$$

$$+\sum_{j<k} :\phi(1)\dots\hat{\phi}(j)\dots\hat{\phi}(k)\dots\phi(n):\ i\Delta_F(j-k)$$

$+ \ \dots$ 2 contractions

$+ \ \dots$