I) Introduction to Quantum Field Theory (QFT)

The Standard Model (SM) is formulated as a QFT.

Special Relativity (SR) + Quantum Mechanics (QM) $\rightarrow$ QFT

1) Kinematics

- Minkowski space: 4D vector space $\mathbb{R}^{1,3}$ w/ metric

$$g_{\mu\nu} = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

- inverse metric: $g^{\alpha\nu}$: $\quad g_{\mu\alpha}\, g^{\alpha\nu} = \delta_\mu{}^\nu \qquad \mu = 0,1,2,3$

- spacetime 4-vector: $\quad x^\mu = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}, \quad \vec{x} \equiv x^i \qquad i = 1,2,3$

Set $\hbar = c = 1$: $\quad x^\mu = \begin{pmatrix} t \\ \vec{x} \end{pmatrix}$

- momentum 4-vector: $\left(p^\mu\right) = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} \equiv \begin{pmatrix} E \\ \vec{p} \end{pmatrix}$

- scalar product of 2 4-vectors a^μ, b^μ:

$$a \cdot b = g_{\mu\nu}\, a^\mu b^\nu = a^0 b^0 - \vec{a} \cdot \vec{b}$$

- indices raised/lowered with metric:

$$x_\mu = g_{\mu\nu} x^\nu \implies x_0 = x^0; \quad x_i = -x^i$$

$$a \cdot b = a_\mu b^\mu = a^0 b_0 + a^i b_i$$

- The Poincaré - Group is the group of symmetry transformations that leave the scalar product invariant, i.e.

$$x^\mu \longmapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \qquad \text{s.t.}$$

$$(x'-y')^2 = (x-y)^2$$

$\Rightarrow$ a^μ : constant dilatations

$\Lambda^\mu{}_\nu$: generator of Lorentz group

Find: $\Lambda^\mu{}_\nu \, \Lambda^\rho{}_\kappa \, g_{\mu\rho} = g_{\nu\kappa}$

$$\Leftrightarrow \quad \Lambda_\rho{}^\alpha \Lambda^\rho{}_\kappa = \delta^\alpha{}_\kappa \quad \Leftrightarrow \quad \Lambda^T \Lambda = \mathbb{1}$$

$$\Leftrightarrow \quad \boxed{\Lambda^T = \Lambda^{-1}} \quad \longrightarrow \quad \det \Lambda = \pm 1$$

- The Lorentz group is not connected. It contains 4 connected components:

$\det \Lambda = +1$ | $\det \Lambda = -1$

$\underbrace{\Lambda^0{}_0 > 0}$ & $\Lambda^0{}_0 < 0$ | $\Lambda^0{}_0 > 0$ & $\Lambda^0{}_0 < 0$

part connected to identity (PCI)

The PCI is related to other parts by

- parity: $\vec{x} \longrightarrow -\vec{x}$ $\quad \Lambda_P = \mathrm{diag}(1,-1,-1,-1)$
- time reversal : $x^0 \longrightarrow -x^0$ $\quad \Lambda_T = \mathrm{diag}(-1,1,1,1)$

· The PCI is generated by infinitesimal trafos as

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu + \dots$$

$$\Lambda^T \Lambda = 1 \Rightarrow \omega_{\mu\nu} = -\omega_{\nu\mu} \Rightarrow 6 \text{ independent generators}$$

$$\left(\Lambda^T = \Lambda^{-1} \Rightarrow (1+\omega)_\beta{}^\alpha = (1-\omega)^\alpha{}_\beta \Rightarrow \omega_\beta{}^\alpha = -\omega^\alpha{}_\beta \right)$$

A finite Lorentz trafo is given by

$$\Lambda^\rho{}_\tau = \exp\left(-\frac{i}{4} M^{\mu\nu}\omega_{\mu\nu}\right)^\rho{}_\tau$$

$$(M^{\mu\nu})^\rho{}_\tau = 2i\left(g^{\rho\mu}\delta^\nu{}_\tau - g^{\rho\nu}\delta^\mu{}_\tau\right)$$

The matrices $M^{\mu\nu}$ satisfy the Lorentz algebra

$$[M^{\mu\nu}, M^{\alpha\beta}] = 2i\left\{g^{\mu\beta}M^{\nu\alpha} + g^{\nu\alpha}M^{\mu\beta} - g^{\mu\alpha}M^{\nu\beta} - g^{\nu\beta}M^{\mu\alpha}\right\}$$

Examples:

· $\omega_{12} = -\omega_{21} = \Theta \iff \omega^1{}_2 = -\Theta$

$$\delta^\mu{}_\nu + \omega^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\Theta & 0 \\ 0 & \Theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{rotation}$$

· $\omega_{01} = -\omega_{10} = \beta \Rightarrow \begin{pmatrix} 1 & \beta & 0 & 0 \\ \beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ Lorentz boost

2) From QM to QFT

A particle is defined by its transformation behaviour under the symmetries of a physical theory, i.e. a particle forms an irreducible representation of the symmetry group.

We distinguish:

i) symmetries of spacetime
$\longleftrightarrow$ Poincaré group of $\mathbb{R}^{1,3}$
Irreps. are labelled by · mass m (rest frame)
· spin s (response to rotation)

ii) "internal" symmetries of the specific theory
$\longleftrightarrow$ global and gauge symmetries of the SM
Irreps. are labelled by "charge"

In QM a particle is described by a wavefunction $\psi(x)$.
Its properties w.r.t. i) are encoded as:

$$x \mapsto \Lambda x : \quad \psi(x) \mapsto \underbrace{R(\Lambda)}_{\substack{\text{repr. of} \\ \text{Lorentz group}}} \psi(\Lambda^{-1} x)$$

The SM comprises

Bosons $s \in \mathbb{Z}$	Fermions $s \in \mathbb{Z} + \frac{1}{2}$
· scalars φ : $s = 0$	spinors ψ_α $s = \frac{1}{2}$
· vectors A_μ : $s = 1$	

In QFT particles arise as excitations of fields.
The fields form appropriate Irreps under symmetries

2 step procedure:

- classical field theory $\Leftrightarrow$ define Lagrangians for fields of mass m, spin s, charge
- quantisation of field theory

3) The classical free scalar field

- A spin 0 particle is described by a scalar field $\varphi(x)$:

$$x \longmapsto x' = \Lambda x \qquad \varphi(x) \longmapsto \varphi'(x') = \varphi(\Lambda^{-1} x')$$
$$\varphi'(x') = \varphi(x)$$

- A neutral scalar corresponds to φ real: $\varphi(x) = \varphi^*(x)$,
  charged scalar to φ complex.

- The e.o.m. follows from mass-shell condition:

$$E^2/c^2 - \vec{p}^{\,2} = m^2 c^2$$

replace: $\vec{p} \longrightarrow -i\hbar \vec{\nabla}$

$E \longrightarrow i\hbar\, \partial_t$

$$\left(-\frac{\hbar^2}{c^2}\partial_t^2 + \hbar^2 \vec{\nabla}^2\right)\varphi(x) = m^2 c^2 \varphi(x)$$

Def.: $\partial_\mu = \dfrac{\partial}{\partial x^\mu}$ $\left(\text{s.t. } \partial'_\mu \equiv \dfrac{\partial}{\partial x'^\mu} = \Lambda_\mu{}^\nu \partial_\nu\right)$

$$\partial^2 = \partial_\mu \partial^\mu = \partial_t^2 - \vec{\nabla}^2 = \Box \qquad \text{D'Alembert op.}$$

$\hbar = 1 = c \implies$ $\boxed{(\partial^2 + m^2)\varphi(x) = 0}$ Klein-Gordon-equation

- We seek an action

$$S[\varphi] = \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

such that E.o.m. $\iff$ $\delta S[\varphi] = 0$

$$\frac{\delta S}{\delta \varphi} = 0$$

One finds: $\delta S = 0 \longleftrightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi(x)} = \frac{\partial \mathcal{L}}{\partial \varphi(x)}$ Euler-Lagrange

→ Action for free scalar (real):

$$S = \int \frac{1}{2} \left(\partial_\mu \varphi \, \partial^\mu \varphi - m^2 \varphi^2 \right) d^4 x$$

$$\Longrightarrow \quad (\partial^2 + m^2) \, \varphi(x) = 0$$

• Plane wave solution:

solution w/ definite energy $\varphi_p(x) = e^{-i p \cdot x}$

$$= e^{-iEt + i\vec{p}\cdot\vec{x}}$$

into KG: $\partial_\mu \varphi_p(x) = -i p_\mu \varphi_p(x)$

$$(-p^2 + m^2)\, \varphi_p = 0 \quad \Rightarrow \quad E^2 = \vec{p}^{\,2} + m^2$$

$$E = \pm\sqrt{\vec{p}^{\,2} + m^2}$$

Positive energy $E > 0$ ⟷ physical particle

Negative energy $E < 0$ ⟷ complex conjugate of physical particle with $E > 0$

⟶ "anti-particle"

In presence of interactions neg. energy solutions always emerge

⟶ no consistent 1-particle interpretation

• most general solution:

$$\varphi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left[f(p)\, e^{-ipx} + g(p)\, e^{ipx} \right]$$

where $E_p := +\sqrt{\vec{p}^{\,2} + m^2}$

Normalisation $\frac{1}{2E_p}$ from:

$$\int d^4p \; \delta^{(4)}(p^2+m^2) \longrightarrow \int \frac{d^3p}{2E_p}\Bigg|_{\text{on-shell}}$$

neutral field: $\varphi = \varphi^* \longrightarrow g(p) = f^*(p)$

Notation:

Oftentimes we will abbreviate

"$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}$" by $\sum_p$