IV Electroweak Theory

Experimentally, we have seen that weak interactions are described by exchange of charged, massive vector bosons $W_\mu^\pm$. This requires a generalisation of of the theory of massless, abelian gauge fields as in QED.

1) Non-abelian gauge theories = Yang-Mills Theory

Recall: QED is formulated as abelian $= U(1)$ gauge theory

$$(*) \quad \psi(x) \longrightarrow e^{i\chi(x)}\psi(x) \qquad : \text{phase rotation} = U(1) \text{ action}$$

$$A_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{e}\partial_\mu \chi(x)$$

$$D_\mu = \partial_\mu + ieA_\mu(x) \Rightarrow D_\mu\psi \longrightarrow e^{i\chi(x)} D_\mu\psi(x)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -\frac{i}{e}[D_\mu, D_\nu] \quad : U(1) \text{ invariant}$$

Aim: Generalise $(*)$ to transformation under a general compact Lie group G.

Reminder:

(i) Lie group G = continuous group with elements

$$h(\underline{\Theta}) = \exp(i\Theta_a T_a) = \mathbb{1} + i\Theta_a T_a + \mathcal{O}(\Theta^2)$$

$$a = 1, \ldots, \dim G \qquad \Theta_a \in \mathbb{R}$$

(ii) T_a are called generators. They form a vector space $\mathfrak{g}$ endowed with an antisymmetric map called Lie bracket:

$$[T_a, T_b] = i f_{abc} T_c$$

f_{abc}: structure constants

subj. to the Jacobi identity:

$$[T_a,[T_b,T_c]] + [T_b,[T_c,T_a]] + [T_c,[T_a,T_b]] = 0$$

$\mathfrak{g}$ is called the <u>Lie algebra of G</u>.

A state ψ forms a unitary, irreducible representation R of G if

$$\psi_i \longrightarrow U_{ij}\psi_j = \exp\left(i\,\Theta_a T_a^{(R)}\right)_{ij}\psi_j$$

$T_a^{(R)}$: hermitian $N\times N$ matrix $\qquad T_a^+ = T_a$

$i,j = 1,\dots,N \qquad\qquad N : \dim(R)$

$$[T_a^{(R)}, T_b^{(R)}] \equiv T_a^{(R)}T_b^{(R)} - T_b^{(R)}T_a^{(R)} = i f_{abc} T_c^{(R)}$$

$$\mathrm{tr}\left(T_a^{(R)}T_b^{(R)}\right) = C(R)\,\delta_{ab}$$

<u>Ex.:</u> $G = SU(N) \qquad \dim(G) = N^2-1$

fundamental repr.: $\psi_j \quad j=1,\dots,N$ complex

$\rightarrow \dim(R) = N \qquad T_a$: $N\times N$ hermitian, $\mathrm{tr}\, T_a = 0$

$C(R) = \frac{1}{2} \qquad a = 1,\dots,N^2-1$

Aim: Construct theory invariant under gauged version

$$\psi_i \longrightarrow U_{ij}\psi_j = \exp\left(i\,T_a\,\Theta_a(x)\right)_{ij}\psi_j$$

Idea: Introduce gauge field $\underline{A}_\mu = (A_\mu)_a \qquad a = 1,\dots,\dim(G)$

(i.e. 1 gauge field per generator T_a)

<u>A_μ^a : Yang-Mills fields</u>

Logic: Want covariant derivative

$$\boxed{(D_\mu)_{ij}\psi_j = (\partial_\mu \mathbb{1} + ig\,\underline{T}\cdot\underline{A}_\mu)_{ij}\psi_j}$$

s.t. $D_\mu \psi \longrightarrow \exp(i\,T_a\Theta_a(x))D_\mu\psi$ (*)

Use this to fix gauge trafo of $\underline{A}_\mu$

Find: (*) requires that

$$\boxed{\underline{T}\cdot\underline{A}_\mu \to U\,\underline{T}\cdot\underline{A}_\mu\,U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}}$$

$U = \exp(i\,\Theta_a(x)T_a)$, g: YM coupling

$$\underline{T}\cdot\underline{A}_\mu \longrightarrow U\left(\underline{T}\,\underline{A}_\mu - \frac{1}{g}\underline{T}\cdot\partial_\mu\Theta\right)U^{-1}$$

$\Rightarrow$ field strength: $T_a(F_a)_{\mu\nu} \equiv -\frac{i}{g}[D_\mu, D_\nu]$

$$\boxed{(F_a)_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\,f_{abc}(A_b)_\mu(A_c)_\nu}$$

or: $(\underline{T}\cdot\underline{F})_{\mu\nu} = \partial_\mu(\underline{T}\,\underline{A})_\nu - \partial_\nu(\underline{T}\cdot\underline{A})_\mu + ig\left[\underline{T}\,\underline{A}_\mu, \underline{T}\,\underline{A}_\nu\right]$

Find: $\boxed{\underline{T}\cdot\underline{F}_{\mu\nu} \longrightarrow U(\underline{T}\cdot\underline{F})_{\mu\nu}U^{-1}}$

$\Rightarrow$ pure gauge part:

$$\mathcal{L}_{gauge} = \int d^4x\left(-\frac{1}{2}\right)\mathrm{Tr}\left((\underline{T}\cdot\underline{F}_{\mu\nu})(\underline{F}^{\mu\nu}\cdot\underline{T})\right)$$

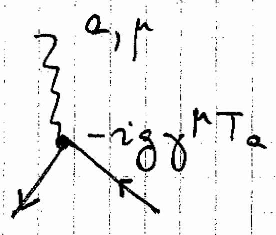
$$= \int d^4x\left(-\frac{1}{4}\right)F_{a\mu\nu}F_a^{\mu\nu}$$ is gauge invariant

by cyclicity of Tr; here $\mathrm{tr}(T_aT_b) = \frac{1}{2}\delta_{ab}$

full YM Lagrangian:

$$\boxed{\mathcal{L}_{YM} = \int d^4x\left(-\frac{1}{4}\right)(F_a)_{\mu\nu}(F^a)^{\mu\nu} + \overline{\psi}_i\left(i\gamma^\mu D^{ij}_\mu - M\delta^{ij}\right)\psi_j}$$

Interactions



i) $\mathcal{L}_{int} \ni -g\, \bar{\psi}_i\, \gamma^\mu (A_\mu)_a\, T_{a\, ij}\, \psi_j$

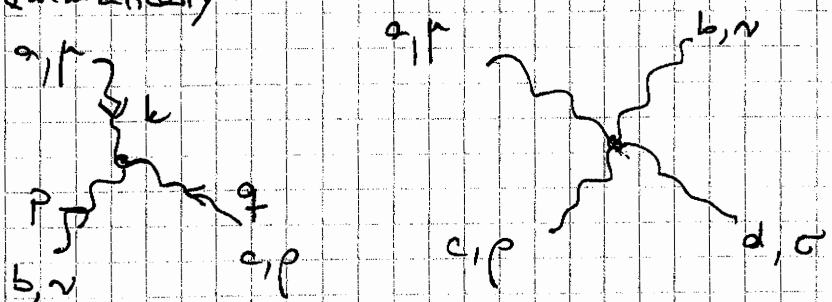
$\longrightarrow$ generalisation of el.-magn. current

ii) New: The pure YM term $-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a$ contains trilinear & quartic interactions of the gauge fields:

$$\mathcal{L}^{gauge}_{int} \ni g\, f_{abc}\, \partial_\mu A_{a\nu}\, A^\mu_b\, A^\nu_c$$
$$- \frac{1}{4} g^2\, f_{abc}\, f_{ade}\, A^\mu_b\, A^\nu_c\, A_{d\mu}\, A_{e\nu}$$

Note: For $U(1)$ theory: $f_{abc} \equiv 0 \rightarrow$ no self-interactions

Diagrammatically:



Note: This is reason behind charge of $W^\pm$ boson (see later)

Quantisation

- Due to self-interaction this is much more complicated than for abelian theory

- As for QED we need to fix the gauge

Reminder: Abelian gauge fixing / propagator:

$$S = \int d^4x \left(-\frac{1}{4}\right) F_{\mu\nu} F^{\mu\nu} = +\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4}\, A^\mu(-p) \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}\right) p^2\, A^\nu(p)$$

$$\mathcal{L} = +\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, A^\mu(-p)\, \Theta_{\mu\nu}(p)\, A^\nu(p)$$

The propagator is the inverse of $\Theta_{\mu\nu}(p)$:

$$\Theta_{\mu\nu}(p)\, \tilde{\Delta}^{\nu\rho}(p) = i\, \delta_\mu{}^\rho$$

Problem: $\Theta_{\mu\nu}(p)$ is not invertible (see tutorial)

$\to$ Add Lagrange multiplier $\int d^4x \; -\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ to implement gauge fixing

$$\Rightarrow S = +\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, A^\mu(-p)\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}\left(1+\tfrac{1}{\xi}\right)\right) p^2\, A^\nu(p)$$

$$\tilde{\Delta}_{\mu\nu}(p) = \left(g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)\frac{(-i)}{p^2+i\varepsilon}$$

Physics independent of gauge choice:

$\xi = 1$ Feynman gauge

$\xi = 0$ Landau gauge

Non-abelian: Same logic applies:

$$\to \boxed{i\Delta^{ab}_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\frac{(-i)}{k^2+i\varepsilon}\,\delta^{ab}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]}$$

- In QED the stated Feynman rules suffice to show that only transverse photons contribute to scattering.
  In YM this does not work due to self-interactions.
  But: Careful quantisation à la Faddeev–Popov introduces extra FP ghosts $\longrightarrow$ cancel contribution from long. photons

  See QFT course or Peskin–Schröder: 16.2 & 16.3.