3) The QED Lagrangian

We aim to find a gauge invariant Lagrangian that reproduces

$$\partial_\mu F^{\mu\nu} = j^\nu .$$

A natural guess is
$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu j^\mu \right).$$

However this is not gauge invariant under

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \chi(x) \quad \text{for arbitrary } j^\mu .$$

[Note: We need gauge invariance "off-shell", without use of e.o.m.]

Necessary: $\partial_\mu j^\mu = 0$

A candidate for j^μ is $\quad j^\mu = e\, \bar{\psi}(x) \gamma^\mu \psi(x)$

with $\psi(x)$ describing a charged fermion of charge e.

The free fermion action

$$S = \int d^4x\, \bar{\psi} \left(i \gamma^\mu \partial_\mu - m \right) \psi$$

enjoys the global symmetry

$$\psi(x) \to e^{ie\xi} \psi(x) \qquad \xi \text{ constant}$$
$$\bar{\psi}(x) \to \bar{\psi}(x)\, e^{-ie\xi}$$

The current j^μ is the conserved current associated w/ this <u>global</u> symmetry.

<u>Proposal</u>: Promote this to local gauge symmetry:

$$\psi(x) \to e^{ie\chi(x)} \psi(x) \qquad A_\mu \to A_\mu(x) + \partial_\mu \chi(x)$$

Now $\partial_\mu \psi(x) \longrightarrow \partial_\mu e^{ie\chi(x)} \psi(x) = \left(\partial_\mu + ie\left(\partial_\mu \chi(x)\right)\right) \psi(x)$

Define the covariant derivative

$$\mathcal{D}_\mu \psi(x) = (\partial_\mu - ieA_\mu)\psi(x)$$

$$\mathcal{D}_\mu \psi(x) \longrightarrow (\partial_\mu - ieA_\mu - ie\partial_\mu \chi(x))\, e^{ie\chi(x)} \psi(x)$$

$$= e^{ie\chi(x)} (\partial_\mu - ieA_\mu)\psi(x)$$

$$= e^{ie\chi(x)} \mathcal{D}_\mu \psi(x)$$

$$\Rightarrow \mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}(x)\left(i\gamma^\mu \mathcal{D}_\mu - m\right)\psi(x)$$

is gauge invariant.

In fact: $\mathcal{L}(x) = -\frac{1}{4} F^2 + \overline{\psi}(i\gamma\partial - m)\psi + \underbrace{e\,\overline{\psi}\gamma^\mu A_\mu \psi}_{A^\mu j^\mu}$

gives gauge invariant coupling of j_μ to A^μ !

<u>Interaction picture:</u>

- GP formalism for free photon - part
- $\mathcal{L}_I = j_\mu A^\mu = e\,\overline{\psi}\gamma^\mu A_\mu \psi$

leads to S-matrix

$$S = T \exp\left(ie \int d^4x\, \overline{\psi}(x) \gamma^\mu A_\mu(x) \psi(x)\right)$$

- In the Standard Model, e^+/e^- are not the only fermions of $|\text{charge}| = e$

$\longrightarrow$ 3 families of charged leptons

e^-	:	electron	$m_{e^-} = 511\ \text{keV}$
μ^-	:	muon	$m_{\mu^-} = 105.7\ \text{MeV}$
τ^-	:	tauon τ - Lepton	$m_{\tau^-} = 1.78\ \text{GeV}$

+ anti - particles

$$\Longrightarrow \mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_D + \mathcal{L}_I$$

$$\mathcal{L}_D = \bar{\psi}_e (i\gamma^\mu \partial_\mu - m_e) \psi_e + \bar{\psi}_\mu (i\gamma^\mu \partial_\mu - m_\mu)\psi_\mu + \bar{\psi}_\tau (i\gamma^\mu \partial_\mu - m_\tau)\psi_\tau$$

$$\mathcal{L}_I = e\left(\bar{\psi}_{(e)} \gamma \cdot A\, \psi_{(e)} + \bar{\psi}_{(\mu)} \gamma \cdot A\, \psi_{(\mu)} + \bar{\psi}_{(\tau)} \gamma \cdot A\, \psi_{(\tau)} \right)$$

Summarise as: $\psi = \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix}$

$$\mathcal{L}_D = \bar{\psi} (i\gamma \cdot \partial - m) \psi \quad +$$

$$\mathcal{L}_I = e\left(\bar{\psi}\, \gamma \cdot A\, \psi \right)$$

4) QED Perturbation Theory

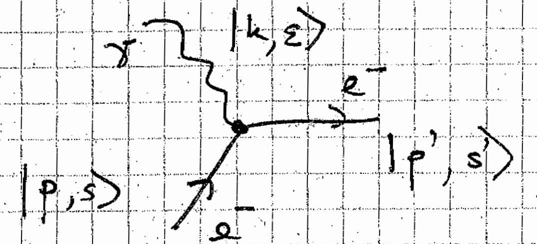
Perturbation theory in e proceeds by expansion of the S-matrix:

$$S = \mathbb{1} + S^{(1)} + S^{(2)} + \mathcal{O}(e^3)$$

$$S^{(1)} = ie \int d^4x \; \overline{\psi}(x) \gamma^\mu A_\mu \psi(x)$$

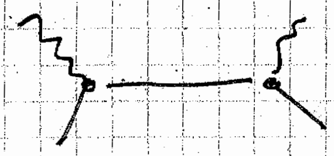
$$S^{(2)} = \frac{(ie)^2}{2!} \int d^4x \, d^4y \; T\Big(\psi(x)\, \gamma\cdot A\, \psi(x) \;\; \overline{\psi}(y)\, \gamma\cdot A\, \psi(y)\Big)$$

The elementary vertex is



- By conservation of energy & momentum not all 3 states can be physical, so $S^{(1)}$ does not contribute to physical processes.

- $S^{(2)}$ gives lowest order tree-level processes of type



etc.

Simple processes include:

a) Compton scattering: $L^{\pm} + \gamma \longrightarrow L^{\pm} + \gamma$

b) Lepton - antilepton annihilation: $L^{+} + L^{-} \longrightarrow \gamma + \gamma$

c) Lepton - lepton scattering: $L^{+} + L^{+} \longrightarrow L^{+} + L^{+}$

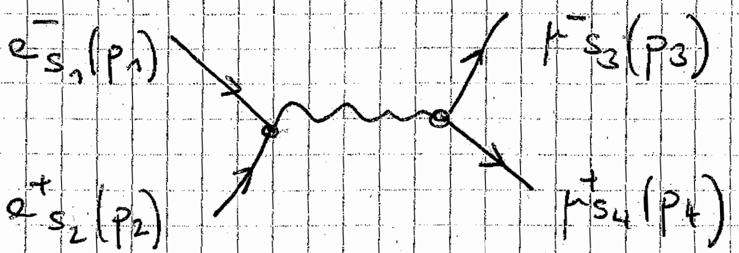
$L^{-} + L^{-} \longrightarrow L^{-} + L^{-}$

d) Lepton - antilepton scattering: $L^{+}_{i} + L^{-}_{i} \longrightarrow L^{+}_{j} + L^{-}_{j}$

We consider the process

$$e^{+} + e^{-} \longrightarrow \mu^{+} + \mu^{-}$$

at tree-level from $S^{(2)}$:



Aim: spin-averaged scattering rate

$$\omega = \sum_{\substack{p_3\\p_4}} \frac{1}{4} \sum_{\substack{s_1, s_2,\\ s_3, s_4}} (2\pi)^4 \, \delta^{(4)}(P_f - P_i) \, |T^{(2)}|^2$$

$$\underbrace{\langle \mu^{-}_{s_3}(p_3)\, \mu^{+}_{s_4}(p_4) \,|\, S^{(2)} \,|\, e^{-}_{s_1}(p_1)\, e^{+}_{s_2}(p_2) \rangle}_{=:(*)} = i\,(2\pi)^4 \, \delta^{(4)}(p_3 + p_4 - p_1 - p_2) \cdot T^{(2)}$$

$$S^{(2)} = \frac{1}{2!}(ie)^2 \int d^4x\, d^4y \; T\left[\overline{\psi}(x)\gamma\cdot A\,\psi(x)\;\overline{\psi}(y)\gamma\cdot A\,\psi(y)\right]$$

We need 2 copies of Dirac fields per initial / final state to annihilate $|i\rangle, |f\rangle$.

By Wick's theorem only contribution from:

$$T[\;] \rightarrow \;:\overline{\psi}(x)\gamma^\mu\psi(x)\;\overline{\psi}(y)\gamma^\nu\psi(y):\; i\Delta_{F\,\mu\nu}(x-y)$$

(after contracting $\overbrace{A_\mu(x)\ldots A_\nu(y)} = i\Delta_{F\,\mu\nu}(x-y)$

$$i\Delta_{F\,\mu\nu}(x-y) = (-i g_{\mu\nu})\int\frac{d^4k}{(2\pi)^4}\,\frac{e^{-ik(x-y)}}{k^2+i\varepsilon}\quad)$$

$$(*) = \frac{2}{2}(ie)^2\int d^4x\, d^4y\; i\Delta_{F\,\mu\nu}(x-y)\,\cdot$$

$$\left\langle \mu^-_{s_3}(p_3)\,\mu^+_{s_4}(p_4)\right|\,:\overline{\psi}_{(\mu)}(x)\gamma^\mu\psi_{(\mu)}(x)::\overline{\psi}_{(e)}(y)\gamma^\nu\psi_{(e)}(y):$$

$$\left| e^-_{s_1}(p_1)\,e^+_{s_2}(p_2)\right\rangle$$

from

$$\psi = \begin{pmatrix}\psi_{(e)}\\ \psi_{(\mu)}\\ \psi_{(\tau)}\end{pmatrix}$$

<u>Consider</u>: $\psi_{(e)} = \psi_{(e)}^- + \psi_{(e)}^+$

$$\psi^+_{(e)}(y) = \sum_{\tilde{p},\tilde{s}} u_{\tilde{s}}(\tilde{p})\, a^{(e)}_{\tilde{s}}(\tilde{p})\, e^{-i\tilde{p}y}$$

$$\psi^-_{(e)}(y) = \sum_{\tilde{p},\tilde{s}} v_{\tilde{s}}(\tilde{p})\, b^{+(e)}_{\tilde{s}}(\tilde{p})\, e^{i\tilde{p}y}$$

&

$$\left(\overline{\psi}_{(e)}(y)\right)^- = \sum_{\tilde{p},\tilde{s}} \overline{u}_{\tilde{s}}(\tilde{p})\, a^{+(e)}_{\tilde{s}}(\tilde{p})\, e^{i\tilde{p}y}$$

$$\left(\overline{\psi}_{(e)}(y)\right)^+ = \sum \overline{v}_{\tilde{s}}(\tilde{p})\, b^{(e)}_{\tilde{s}}(\tilde{p})\, e^{-i\tilde{p}y}$$

$$\psi^{+}_{(e)}(y)\,|e^{-}_{s_1}(p_1)\rangle = \sum_{\tilde{p},\tilde{s}} u_{\tilde{s}}(\tilde{p})\, e^{-i\tilde{p}y}\,\{a^{(e)}_{\tilde{s}}(\tilde{p}),\, a^{+(e)}_{s_1}(p_1)\}\,|0\rangle$$

$$= u_{s_1}(p_1)\, e^{-ip_1 y}\,|0\rangle$$

$$\left(\overline{\psi}_{(e)}(y)\right)^{+}|e^{+}_{s_2}(p_2)\rangle = \overline{v}_{s_2}(p_2)\, e^{-ip_2 y}\,|0\rangle$$

Similarly:

$$\langle \mu^{-}_{s_3}(p_3)|\left(\overline{\psi}_{(\mu)}\right)^{-} = \overline{u}_{s_3}(p_3)\, e^{ip_3 x}$$

$$\langle \mu^{+}_{s_4}(p_4)|\,\psi^{-}_{(\mu)} = v_{s_4}(p_4)\, e^{ip_4 x}$$

$$\Rightarrow \langle f|S^{(2)}|i\rangle = (ie)^2 \int d^4x\, d^4y\;\, i\Delta_{\mu\nu}(x-y)\; e^{i(p_3+p_4)x}\; e^{-i(p_1+p_2)y}$$

$$\times\;\; \overline{u}_{s_3}(p_3)\,\gamma^{\mu}\, v_{s_4}(p_4)\;\; \overline{v}_{s_2}(p_2)\,\gamma^{\nu}\, u_{s_1}(p_1)$$

<u>Integral:</u>

$$\int d^4x\, d^4y \int \frac{d^4k}{(2\pi)^4}\; \frac{-ig_{\mu\nu}}{k^2+i\varepsilon}\; e^{iy(k-p_1-p_2)}\, e^{ix(p_3+p_4-k)}$$

$$= \int d^4k\;\, (2\pi)^4\;\, \delta^{(4)}(k-p_1-p_2)\,\delta^{(4)}(p_3+p_4-k)\;\; \frac{-ig_{\mu\nu}}{k^2+i\varepsilon}$$

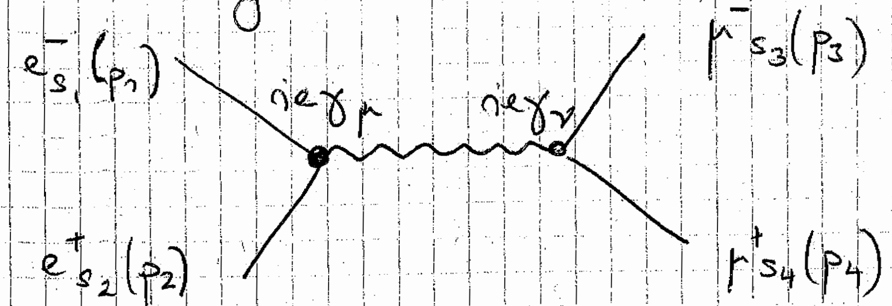
$$= (2\pi)^4\;\, \delta^{(4)}(p_1+p_2-p_3-p_4)\;\cdot\; \frac{-ig_{\mu\nu}}{s}$$

$$s = (p_1+p_2)^2 = (p_3+p_4)^2$$

$$\langle f|S^{(2)}|i\rangle = \left[i\,(2\pi)^4\,\delta^{(4)}(p_1+p_2-p_3-p_4)\right] \times$$

$$\times\;\; \overline{v}_{s_2}(p_2)\,(ie\gamma^{\mu})\,u_{s_1}(p_1)\,\cdot\left(-\frac{g_{\mu\nu}}{s}\right)\cdot\, \overline{u}_{s_3}(p_3)\,(ie\gamma^{\nu})\,v_{s_4}(p_4)$$

$\Longrightarrow$ Feynman rules

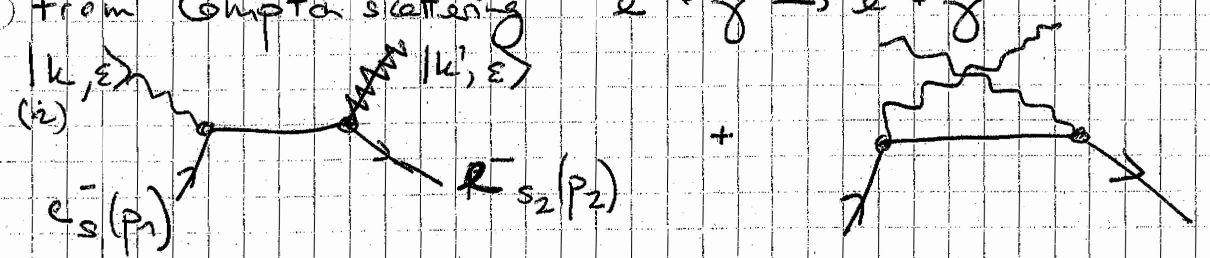


To compute the scattering amplitude T we assign

1) $ie\gamma_\mu$ for a vertex
2) $u_s(p)$ for initial state fermion
3) $\bar{u}_s(p)$ final
4) $\bar{v}_s(p)$ for initial state anti-fermion
5) $v_s(p)$ final
6) $-\frac{g^{\mu\nu}}{k^2}$ for internal photon

& impose momentum conservation at each vertex

From Compton scattering $e^- + \gamma \to e^- + \gamma$



find:

7) ε^μ for external photon line

8) $\frac{\gamma \cdot q + m}{q^2 - m^2}$ for internal electron line

Spin average

Aim: $\frac{1}{4}\sum_{\substack{s_1 s_2\\ s_3 s_4}} \left| \bar{u}_{s_3}(p_3)\gamma^{\nu} v_{s_4}(p_4) \cdot \bar{v}_{s_2}(p_2)\gamma_{\nu} u_{s_1}(p_1)\right|^2$

Use: $\left[\bar{u}_{s_3}(p_3)\gamma^{\nu} v_{s_4}(p_4)\right]^* = v^{\dagger}_{s_4}(p_4)\gamma^{\nu\dagger}\gamma^{0\dagger} u_{s_3}(p_3)$

$$= \bar{v}_{s_4}(p_4)\gamma^{\nu} u_{s_3}(p_3)$$

$\to \frac{1}{4}\sum_{s_1 s_2} \bar{u}_{s_3}(p_3)\gamma^{\nu} v_{s_4}(p_4)\bar{v}_{s_4}(p_4)\gamma^{\mu} u_{s_3}(p_3) \times$

$\sum_{s_3 s_4} \bar{v}_{s_2}(p_2)\gamma_{\nu} u_{s_1}(p_1)\bar{u}_{s_1}(p_1)\gamma_{\mu} v_{s_2}(p_2)$

Use: $\sum_s v_s(p)\bar{v}_s(p) = \gamma p - m$

$\sum_s u_s(p)\bar{u}_s(p) = \gamma p + m$

Reorganise the summation over indices of spinors to find

$\frac{1}{4} \mathrm{Tr}\left[(\gamma p_3 + m_{(\mu)})\gamma^{\nu}(\gamma p_4 - m_{(\mu)})\gamma^{\mu}\right] \times$

$\times \mathrm{Tr}\left[(\gamma p_2 - m_{(e)})\gamma_{\nu}(\gamma p_1 + m_{(e)})\gamma_{\mu}\right]$

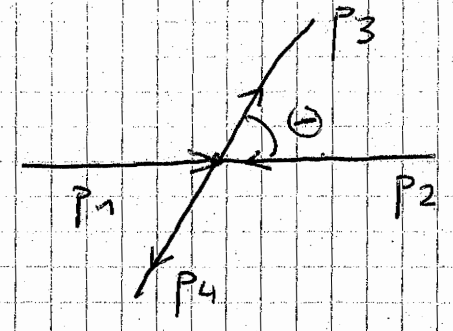
Consider high energy limit: $s \gg m_e, m_\mu$

$\frac{1}{4}\mathrm{Tr}\left[\not{p}_3\gamma^{\nu}\not{p}_4\gamma^{\mu}\right]\mathrm{Tr}\left[\not{p}_1\gamma_{\nu}\not{p}_2\gamma_{\mu}\right]$

Use identity: $\mathrm{Tr}\left[\gamma^{\alpha}\gamma^{\nu}\gamma^{\beta}\gamma^{\rho}\right] = 4\left[g^{\alpha\nu}g^{\beta\rho} + g^{\rho\alpha}g^{\nu\beta} - g^{\alpha\beta}g^{\nu\rho}\right]$

$$\longrightarrow \frac{16}{4}\left[2(p_1 p_4)(p_2 p_3) + 2(p_2 p_4)(p_1 p_3)\right] \qquad (*)$$

Scattering angle:



$$\cos\Theta = \frac{\vec{p}_1 \cdot \vec{p}_3}{|\vec{p}_1||\vec{p}_3|}$$

in high energy limit:

$$p_1 p_3 = p_2 p_4 = \frac{s}{4}\left(1 - \cos\Theta\right)$$

$$p_1 p_4 = p_2 p_3 = \frac{s}{4}\left(1 + \cos\Theta\right)$$

$$(*) \qquad s^2 \cdot \left(1 + \cos^2\Theta\right)$$

$$\omega = \int \frac{d^3 p_3}{(2\pi)^3} \frac{d^3 p_4}{(2\pi)^3} \frac{1}{2E_{p_3}} \frac{1}{2E_{p_4}} \cdot (2\pi)^4 \delta^{(4)}(p_3 + p_4 - p_1 - p_2) \cdot \frac{e^4}{s^2} \cdot \left(s^2 \left(1 + \cos^2\Theta\right)\right)$$

$$= \alpha^2 \cdot \int d^3 p_3\, d^3 p_4 \frac{1}{p_3^0\, p_4^0} \delta^{(4)}(p_1 + p_2 - p_3 - p_4)\left(1 + \cos^2\Theta\right)$$

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$$

CMS: $\vec{p}_1 + \vec{p}_2 = 0 \qquad p_1^0 + p_2^0 \simeq \sqrt{s} \qquad \left(\text{for } \frac{m^2}{s} \to 0\right)$