II Quantum Electrodynamics

1) The Electromagnetic field

The e-m. fields are encoded in the antisymmetric tensor $F_{\mu\nu}(x)$ – the field strength – as

$$E_i = F_{0i} = F^{i0}\,, \qquad B_i = -F_{jk} = -F^{jk}$$

(i, j, k cyclic)

Maxwell's equation are

$$(1)\quad \begin{rcases} \vec{\nabla}\times\vec{E} = -\dot{\vec{B}} \\ \vec{\nabla}\cdot\vec{B} = 0 \end{rcases} \text{homogeneous}$$

$$(2)\quad \begin{rcases} \vec{\nabla}\cdot\vec{E} = \rho \\ \vec{\nabla}\times\vec{B} = \dot{\vec{E}} + \vec{j} \end{rcases} \text{inhomogeneous}$$

ρ : charge density, $\vec{j}$: electric current density

In terms of $F_{\mu\nu}$ one finds:

$$(1) \iff \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$$

(Bianchi identity)

$$(2) \iff \partial_\mu F^{\mu\nu} = j^\nu \quad \text{for} \quad j^\nu = \begin{pmatrix} \rho \\ \vec{j} \end{pmatrix}$$

Note that: $\partial_\nu j^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = 0$

$\Rightarrow \dot{\rho} + \vec{\nabla}\vec{j} = 0 \quad \Leftarrow$ current conservation

Quantum mechanically, the fundamental field is not $F_{\mu\nu}$, but the <u>vector potential A_μ</u>, defined such that:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

Note that eqn. (1) is now <u>automatic</u>, hence the name Bianchi identity.

Eqn. (2) becomes: $\quad \partial^2 A^\nu - \partial^\nu \partial \cdot A = j^\nu \qquad (*)$

However, A_μ is not uniquely defined because $(*)$ is invariant under the <u>gauge transformation</u>

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \chi(x) \equiv A'_\mu(x)$$

under which $\quad F_{\mu\nu}(x) \longrightarrow F_{\mu\nu}(x) \equiv F'_{\mu\nu}(x)$

One can <u>fix the gauge</u> by an extra constraint on A_μ.
We will use <u>Lorenz gauge</u> :

$$\partial_\mu A^\mu = 0$$

[This is always possible : if $\partial_\mu A^\mu = \xi$, gauge transform A_μ by $A_\mu \to A_\mu + \partial_\mu \chi$, $\partial^2 \chi = -\xi$]

Then: $\quad \partial^2 A^\mu = j^\mu$

$$\partial_{[\mu} F_{\nu\rho]} = 0$$

Consider first the *free* e-m. field, i.e. $j^\mu = 0$.

The gauge invariant action for $F_{\mu\nu}$ is given in terms of

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) \quad \text{because}$$

$$\delta S = -\frac{1}{2}\int d^4x\, \delta F_{\mu\nu} F^{\mu\nu} = -\int d^4x\, \partial_\mu \delta A_\nu F^{\mu\nu} = \int d^4x\, \delta A_\nu \partial_\mu F^{\mu\nu}$$

$$\Rightarrow \quad \delta S = 0 \iff \partial_\mu F^{\mu\nu} = 0 \quad \checkmark$$

Note: $\int d^4x\, -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2}\int d^4x \left(\partial_\mu A_\nu \partial^\mu A^\nu - (\partial\cdot A)(\partial\cdot A)\right)$

For quantisation we will use the gauge fixed action w/ Lagrangian in Lorenz gauge: $\partial A = 0$

$$\Rightarrow \mathcal{L}(x)_{g.f.} = -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu$$

and e.o.m. $\quad \partial^2 A_\mu = 0 \quad \checkmark$

2) Quantisation of the e-m field

The canonically conjugate field is

$$\pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -\dot{A}^\mu(x).$$

We impose $\quad [A_\mu(t,\vec{x}), \pi^\nu(t,\vec{x}\,')] = i\,\delta^\nu{}_\mu\, \delta(\vec{x}-\vec{x}\,')$

From $\partial^2 A_\mu = 0$ it follows that

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a_\mu(k) e^{-ikx} + a_\mu^+(k) e^{ikx} \right)$$

where $k^\mu = (\omega, \vec{k})$, $\omega^2 - \vec{k}^2 = 0$

$\omega = +|\vec{k}|$

The canonical commutation relations are:

$$[a_\mu(k), a_\nu^\dagger(k')] = -g_{\mu\nu} (2\pi)^3 2\omega \, \delta(\vec{k} - \vec{k}')$$

$$[a_\mu(k), a_\nu(k')] = 0 = [a_\mu^\dagger(k), a_\nu^\dagger(k')]$$

Let us proceed to construct Fock space:

- vacuum $|0\rangle$: $a_\mu(k)|0\rangle = 0$
- $$H = -\int \frac{d^3k}{(2\pi)^3 2\omega} \, \omega \, a_\mu^\dagger(k) a^\mu(k)$$

Naively: $a_\mu^\dagger(k)|0\rangle$, $a_\mu^\dagger(k) a_\nu^\dagger(k)|0\rangle$, ... are eigenstates of H

2 problems:
- looks like 4 independent polarisations
- Consider $|\psi\rangle = a_0^\dagger(k)|0\rangle$

$$\langle\psi|\psi\rangle = \langle 0|a_0(k) a_0^\dagger(k)|0\rangle = -(2\pi)^3 2\omega \, \delta(0) < 0$$

$\longrightarrow$ negative norm states

Remedy: We still need to impose the gauge condition $\partial_\mu A^\mu = 0$.

We cannot impose them as an operator equation, see e.g. $[A_\mu, \Pi^\nu] = i\, \delta^\mu{}_\nu\, \delta(\;)$.

The idea of Gupta-Bleuler is to define <u>physical states</u> $|\psi\rangle$ as the subset of states s.t.

$$(**) \quad \partial . A^+(x) |\psi\rangle = 0 \iff \langle\psi| \partial . A^-(x) = 0$$

$$\left(\text{where} \quad A = A^+ + A^- , \quad A^+ = \int \frac{d^3k}{(2\pi)^3 2\omega} a(k)\, e^{-ikx} \right)$$

The $\langle\psi| \partial A(x) |\psi\rangle = \langle\psi| \partial A^+ + \partial A^- |\psi\rangle = 0 \quad \checkmark$

In modes $(**)$ reads: $\quad k^\mu a_\mu |\psi\rangle = 0$.

Consider the physical state:

$$\boxed{|k, \varepsilon\rangle = \varepsilon^\nu \, a^+_\nu(k) | 0\rangle} \qquad \varepsilon^\nu : \text{polarisation 4-vector}$$

$$0 = k^\mu a_\mu \varepsilon^\nu a^+_\nu(k) |0\rangle = [k^\mu a_\mu, \varepsilon^\nu a^+_\nu(k)] |0\rangle$$

$$= k \cdot \varepsilon \left(-(2\pi)^3 \cdot 2\omega\, \delta(0) \right) |0\rangle$$

$$\longrightarrow \quad \boxed{k^\mu \varepsilon_\mu = 0}$$

<u>2 kinds of solutions:</u>

(i) Longitudinal photons: $\quad \varepsilon_\mu = k_\mu \qquad (k^2 = 0)$

(ii) Transverse photons: $\quad \varepsilon^{(r)} = (0, \vec{\varepsilon}^{(r)}) \qquad r = 1, 2$

$\qquad \vec{k} \cdot \vec{\varepsilon}^{(r)} = 0$

It turns out that longitudinal photons are <u>unphysical</u> because they decouple from all scattering processes.

Suppose $|\psi\rangle = k^\mu a_\mu^+(k)|0\rangle \equiv a^+(k)|0\rangle$

Then $\langle\psi|\psi\rangle = k_\mu k_\nu \langle 0|\underbrace{[a^\mu(k), a^{\nu+}(k)]}_{= -g^{\mu\nu}(2\pi)^3 2\omega\cdot\delta(0)}|0\rangle \sim k^2 = 0$

and more generally $\langle\varphi|\psi\rangle = 0 \quad \forall\ |\varphi\rangle$ physical

$|\psi\rangle$ longitudinal

In fact, all expectation values involving physical operators depend only on <u>transverse states</u>.

Longitudinal states guarantee Lorentz invariance because under a general boost a transverse state picks up longitudinal components.

<u>Remark</u>:

One can reverse the logic as follows:

Given a vector field $A_\mu(x)$ the only known way to make sense of it in a quantum theory is by making A_μ a gauge field. $S = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu$ leads to neg. norm states unless a constraint like $\langle\psi|\partial\cdot A|\psi\rangle = 0$ is imposed.

In turn, this can only be justified if there is extra gauge redundancy in A_μ.

Propagator

We will need $\quad i\Delta_{F\,\mu\nu}(x-y) = \langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle$

One can show that:

$$i\Delta_{F\,\mu\nu}(x-y) = -i g_{\mu\nu} D_F(x-y)$$

$$i D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{i}{k^2+i\varepsilon}$$

Note that $\; iD_F(x-y) = i\Delta_F(x-y)\Big|_{m=0}$ for a scalar field

Remark:

- The Lagrangian action $\int d^4x \; -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)$ reads in momentum space: $S = \int d^4x \, \mathcal{L}(x)$ with

$$\mathcal{L} = -\frac{1}{2} A^\mu(-p) \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) p^2 \, A^\nu(p)$$

- The gauge fixing can be performed by adding a Lagrange multiplier term $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$:

$$\mathcal{L}_{g.f.} = -\frac{1}{2} A^\mu(-p) \left[\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) p^2 - \frac{1}{\xi} \frac{p_\mu p_\nu}{p^2} p^2 \right] A^\nu(p)$$

Propagator: $\; + i\Delta_{\mu\nu}(p) = \left(g_{\mu\nu} - (1-\xi) \frac{p_\mu p_\nu}{p^2} \right) \frac{-i}{p^2 + i\varepsilon}$

Lorentz gauge: $\xi = 1$)