Insert: Interpretation of renorm. scale

$$f(g,\mu,p) = g^2 - 2g^4\left(b \cdot \ln \mu/p + b\bar{c} - \hat{a}\right) + \mathcal{O}(g^6)$$

A priori: μ arbitrary energy scale

However: If $\mu/p \gg 1$ or $\mu/p \ll 1$ the
$\left|\ln \mu/p\right| \gg 1$ & perturbation theory becomes less exact because higher order corrections become more important

Natural to identify $\mu \sim$ energy scale of scattering process

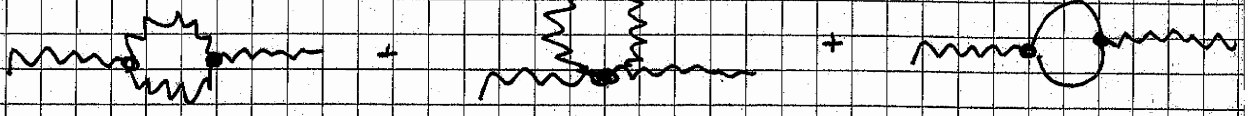
Then $g(\mu)$ describes effective coupling at this scale.

i.e. if $\mu \cong \mathcal{O}(p)$, the $g(\mu)$ defines a coupling suitable for perturbation theory.

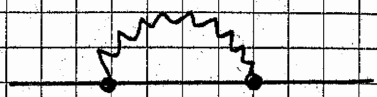
Consider non-ab. YM theory coupled to fermions (e.g. QCD).

At 1-loop have the following UV divergent amplitudes

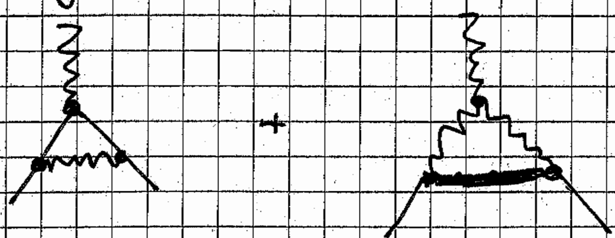
* gluon propagator (self-energy of gluon)



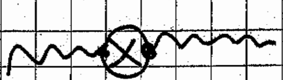
(plus ghosts)

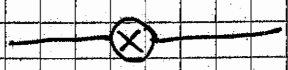
* fermion propagator 

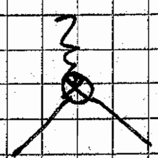
* gluon-fermion vertex:



The associated divergence is canceled by appropriate counter-terms corresponding to Feynman diagram

*  $-i\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\delta^{ab}\ \delta_3$

*  $i\,\not{p}\cdot\delta_2$

*  $igt^a\gamma^\mu\,\delta_1$

δ_i involve $\ln\left(M/\mu\right)$

M : cutoff

μ : scale

Fact: β-function determined by δ_i via

$$\beta(g) = g \cdot \mu \cdot \frac{\partial}{\partial\mu}\left(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3\right)$$

Computation: See e.g. Peskin-Schröder, 16.5.

Asymptotic freedom can be read off from β-function:

as before: $$\beta(g) = b \cdot g^3 + \mathcal{O}(g^5)$$

$\Longrightarrow$ AF requires $b < 0$

__Fact:__ For non-Abelian gauge theory with simple gauge group G ($\longleftrightarrow$ single coupling g)

$$\beta(g) = -\beta_0 \frac{g^3}{16\pi^2} + \mathcal{O}(g^5) \quad @ \text{ 1-loop}$$

where β_0 depends on matter content as follows:

* Consider matter in repr. R:

$$D_\mu R = \partial_\mu R + i g A_{\mu a} (T_R)_a \cdot R$$

$$[(T_R)_a, (T_R)_b] = i f_{abc} (T_R)_c$$

$$\mathrm{tr}\,(T_R^a T_R^b) = C_R \cdot \delta^{ab}$$

* For theory with Weyl spinor ψ_{R_a} & scalar fields Φ_{R_i} get

$$\boxed{\beta_0 = \underbrace{\frac{11}{3} C_{adj.}}_{\text{gauge fields in adjoint repr.}} - \frac{2}{3} \sum_a C_{R_a} - \frac{1}{6} \sum_i C_{R_i}}$$

Weyl spinors   real scalars

Note: For adjoint repr.:

$$(T_{adj.\,a})_{bc} = -i f_{abc} \Longrightarrow$$

$$\mathrm{tr}\,(T_{adj.\,a} T_{adj.\,b}) = - f_{abd} f_{bdc} = f_{acd} f_{bcd}$$

$$\rightarrow \quad \alpha_s(\mu^2) = \frac{4\pi}{\beta_0} \frac{1}{\ln\left(\mu^2/\mu_0^2\right) + \frac{4\pi}{\beta_0 \alpha_s(\mu_0^2)}}$$

$\rightarrow$ for $\beta_0 > 0$: $\alpha_s \rightarrow 0$ as $\mu^2 \rightarrow \infty$

and α_s increases as μ^2 decreases

$$\alpha_s(\mu^2) = \frac{4\pi}{\beta_0} \frac{1}{\ln(\mu^2) - \underbrace{\left(\ln(\mu_0^2) - \frac{4\pi}{\beta_0 \alpha_s(\mu_0^2)}\right)}_{=: \ln(\Lambda_{QCD}^2)}}$$

$$\boxed{\alpha_s(\mu^2) = \frac{4\pi}{\beta_0} \frac{1}{\ln\left(\mu^2/\Lambda_{QCD}^2\right)}}$$

From def. of Λ_{QCD} : $\alpha_s(\Lambda_{QCD}^2) \rightarrow \infty$

$\Longrightarrow$ basic QCD energy scale!

The appearance of a fundamental scale in a classically scale invariant theory due to quantum effects is called <u>dimensional transmutation</u>.

Experimentally: $\Lambda_{QCD} \sim 0.3$ GeV

Note: $\frac{4\pi}{\beta_0} \sim 1.5 \Longrightarrow$ pert. theory meaningless when $\ln\left(\mu^2/\Lambda_{QCD}^2\right) \sim 0.5$

Compare: masses of hadrons ~ 1 GeV

$\rightarrow$ 2 "phases" of QCD

$\mu^2 \gg \Lambda_{QCD}^2$: pert. regime, quark-gluon phase

& $f_{acd} f_{bcd} = C_{adj} \cdot \delta_{ab}$ $\quad C_{adj} = N$ for $SU(N)$

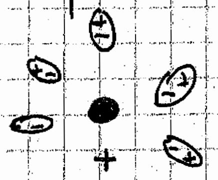
QCD with n_f flavours:

$$\beta_0 = \frac{11}{3} \cdot 3 - \frac{2}{3} \cdot \left(\frac{1}{2} + \frac{1}{2}\right) \cdot n_f$$

$$\boxed{\beta_0 = 11 - \frac{2}{3} n_f} \qquad \text{QCD}: \beta_0 = 7 > 0$$

Physical interpretation:

* charged matter tends to screen a source comparable to dipole screening of el. magn. field:



$\longleftrightarrow$ pair production of dipoles

$\Longrightarrow$ screening increases for large distance $=$ small μ

$\Longrightarrow$ weakens g as $\mu \to 0$

present also for abelian theory

* for non-abelian gauge theory:
  in addition anti-screening from production of gluons $\longrightarrow$ opposite sign due to opposite statistics
  Nice argument see Peskin-Schröder, 16.7.

Evolution of $\alpha_s = \frac{g_s^2}{4\pi}$

$$\frac{d\alpha_s(\mu^2)}{d\ln(\mu^2)} = -\frac{\beta_0}{4\pi}\alpha_s^2(\mu^2) + \mathcal{O}(\alpha_s^3), \quad \beta_0 > 0$$

Integration: $\displaystyle -\int_{\mu_0^2}^{\mu^2} d\ln\tilde{\mu}^2 = \frac{4\pi}{\beta_0}\int_{\alpha_s(\mu_0^2)}^{\alpha_s(\mu^2)} \frac{d\tilde{\alpha}_s}{\tilde{\alpha}_s^2}$

e.g. @ $\mu = M_Z$: $\alpha_s(M_Z^2) \simeq 0.118$

$\mu^2 \gtrsim \Lambda^2_{QCD}$: coupling so strong that no coloured objects isolated existent

$\rightsquigarrow$ confinement

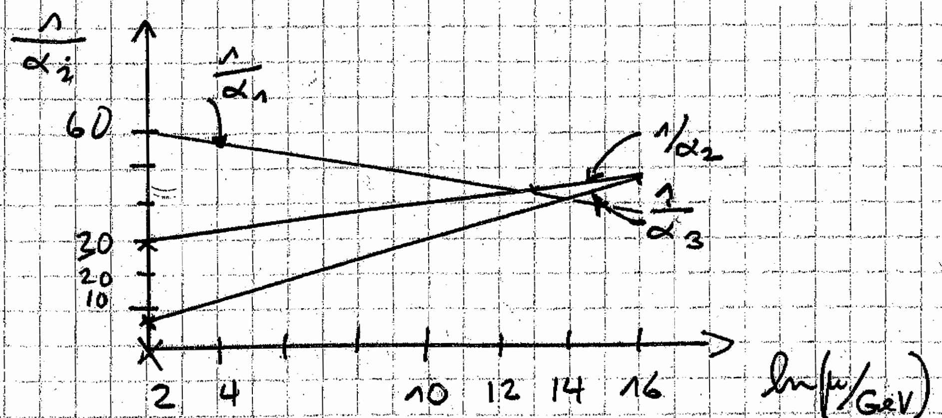
Running of other couplings in SM:

Define $g_s \equiv g_3$, $g \equiv g_2$ (weak SU(2))

$g_1 \equiv \sqrt{\frac{5}{3}} \cdot g'$ g': Y-coupling

$$\Rightarrow \beta_{g_i} = -\frac{1}{16\pi^2} (\beta_0)_{g_i} \cdot g_i^3$$

$$(\beta_0)_{g_i} = \left(-\frac{41}{10}, \frac{19}{6}, 7\right)$$



Note: For $U(1)_Y$ $C_a \longrightarrow \frac{3}{5} Y_i^2$

Summary

- SM based on $SU(3) \times SU(2) \times U(1)_Y$
- successful effective theory based on at least 19 (26) input parameters:

EW : g, g', λ_H, v_H

Yukawas : 3 (Leptons) + 6 (Quarks)

CKM 3 angles + 1 phase

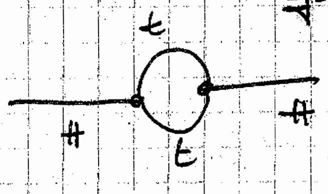
QCD : g_s, Θ-angle

Neutrinos : 3 masses

PMNS : 3 angles + 1 phase (Dirac) + 2 phases (Majorana)

- Open theoretical questions: Why all that mess?
  - $\rightsquigarrow$ why $SU(3) \times SU(2) \times U(1)$
  - $\rightsquigarrow$ values of coupling constants?
  - $\rightsquigarrow$ hierarchies of family masses
  - $\rightsquigarrow$ origin of family replication?

Biggest question: stability of EW scale w.r.t. quantum corrections



Expect: $m_H \sim M_{Planck}$

$\Longrightarrow$ LHC will uncover it