2) Renormalisation

$$\mathcal{L}_{QCD} = -\frac{1}{4}\,\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \sum_f \bar{q}_f\left(i\gamma^\mu D_\mu - m\right) q_f$$

* If we set $m_f = 0$ for simplicity, QCD depends on dimensionless coupling g_s in a classically scale-independent manner.

* $\alpha_s = \frac{g_s^2}{4\pi} = 0.118$ @ $\mu = M_Z$

  $\Rightarrow$ g_s not very small & higher order perturbative corrections essential

* Loop-amplitudes are UV divergent

  Interpretation: QFT only "effective" theory. At high energies new fundamental degrees of freedom become important & theory must be modified — at least by including gravity. UV divergences signal breakdown of effective theory at high energies. (Wilsonian picture)

Pragmatic remedy: Regularisation $\longleftrightarrow$ introduce cutoff to isolate singular behaviour

$$\int_0^\infty dp \longrightarrow \int_0^M dp$$

In renormalisable theory one can introduce finite # of cutoff-dependent counterterms to subtract UV divergences & then let $M \to \infty$.

But: This introduces a mass scale that does not disappear as $M \to \infty$.

$\leftrightarrow$ Classical scale invariance is anomalous

Systematics of renormalisation:

* Starting point: classical "bare" Lagrangian w/ "bare" couplings $g_0, \ldots$, "bare" masses m_0

g_0: not physical quantity by itself, diverges as $M \to \infty$

* Compute physical amplitude (perturbatively)

$F(g_0, M; p_i)$ $\quad$ p_i: external momenta, M: cutoff

* If theory renormalisable, then $\exists$ $Z(M, \mu, g)$ s.t.

$Z(M, \mu, g) \cdot F(g_0, M; p_i) = f(g, \mu; p_i)$, where

- $f(g, \mu, p_i)$ finite as $M \to \infty$
- μ arbitrary mass scale (renormalisation scale)
- g renormalised coupling, independent of external momenta, determined from measurement of $f(g, \mu; p_i)$ at scale μ & for some p_i
- $Z(M, \mu, g)$: wavefunction renormalisation

* g is related perturbatively to g_0; only other dimensionless combination is M/μ, so

perturbatively:

$$g_0\left(g, \frac{M}{\mu}\right) = g + \mathcal{O}(g^3), \quad Z\left(g, \frac{M}{\mu}\right) = 1 + \mathcal{O}(g^2)$$

Example: Consider typical QCD amplitude @ 1-loop level

$$F(g_0, M; p) = g_0^2 - 2 g_0^4 \left(a + b \ln M/p\right)$$
$$= g_0^2 - 2 b g_0^4 \left(\tilde{a} + \ln M/\mu + \ln \mu/p\right)$$

Define g via

$$g_0\left(g, M/\mu\right) = g + g^3\left(\tilde{\tilde{a}} + b \ln M/\mu\right) + \mathcal{O}(g^5)$$

$\longleftrightarrow$ absorb ∞ in g_0

Plugging g_0 into $F(g_0, M; p)$ find the renormalised amplitude

$$f(g, \mu; p) = g^2 - 2 g^4 \left(b \cdot \ln \mu/p + b\tilde{a} - \tilde{\tilde{a}}\right) + \mathcal{O}(g^6)$$

- coefficient b of divergent loop fixed
- coeff. $\tilde{\tilde{a}}$ of finite part of g_0 can be chosen arbitrarily $\longleftrightarrow$ renormalisation scheme

simplest: $\tilde{\tilde{a}} = a \rightarrow b\tilde{a} - \tilde{\tilde{a}} = 0$

Note: In this example no wavefunction renorm. was needed to obtain finite renorm. amplitude

In general: $Z = 1 + g^2\left(c \cdot \ln M/\mu + d\right) + \mathcal{O}(g^4)$

c fixed by removing divergencies

d arbitrary $\longleftrightarrow$ scheme dep.

Note: In each scheme g is inferred by measuring f
$\rightsquigarrow$ overall physics is scheme-independent!

Renormalisation group equation:

* governs scale-dependence of renormalised couplings
* $\exists$ fixed g_0, M: $F(g_0, M; p_i)$ independent of scale μ

$$\mu \cdot \frac{d}{d\mu} F(g_0, M; p_i) = 0$$

Using $f(g, \mu; p_i) = z(M, \mu; g) F(g_0, M; p_i)$

$$\Rightarrow \quad \mu \frac{d}{d\mu} f(g, \mu; p_i)\Big|_{g_0 \text{ fixed}} = \mu \frac{d}{d\mu} z(M, \mu; g)\Big|_{g_0 \text{ fixed}} \cdot \underbrace{z^{-1} \cdot f}_{F}$$

Def.: $\beta(g) = \mu \cdot \frac{d}{d\mu} \cdot g \Big|_{g_0 \text{ fixed}}$

$$\gamma(g) = -\mu \cdot \frac{d}{d\mu} z \Big|_{g_0} \cdot z^{-1}$$

$$\rightarrow \left(\mu \cdot \frac{\partial}{\partial \mu} + \beta(g) \cdot \frac{\partial}{\partial g} \right) f(g, \mu; p_i) = -\gamma(g) f(g, \mu; p_i)$$

Since f & g indep. of M $\rightarrow$ $\beta(g), \gamma(g)$ indep. of M

Back to example:

$$f(g, \mu; p) = g^2 - 2g^4 \left(b \cdot \ln \tfrac{\mu}{p} + b\tilde{a} - \tfrac{\tilde{c}}{2}\right) + O(g^6)$$

$$Z = 1 + g^2 \left(c \cdot \ln \frac{M}{\mu} + d\right) + \mathcal{O}(g^4)$$

Had: $$g_0\left(g, \frac{M}{\mu}\right) = g + g^3 \left(\tilde{a} + b \cdot \ln \frac{M}{\mu}\right) + \mathcal{O}(g^5)$$

$$0 = \mu \cdot \frac{d g_0}{d\mu} = \beta(g) + 3 g^2 \cdot \beta(g) \cdot \left(\tilde{a} + b \cdot \ln \frac{M}{\mu}\right) - g^3 \cdot b + \mathcal{O}(g^5)$$

$$\Rightarrow \beta(g) = g^3 \cdot b + \mathcal{O}(g^5)$$

$\gamma(g)$ directly from Z: $\gamma(g) = g^2 \cdot c + \mathcal{O}(g^4)$

<u>Note:</u> b, c uniquely def. by divergence

At higher order β, γ will also depend on renormalisation scheme

3) Running Coupling

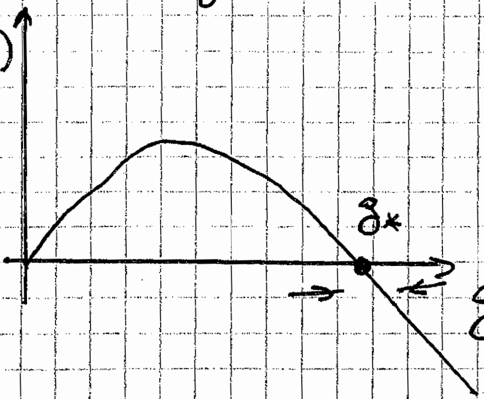
RGE: $\mu \frac{d}{d\mu} f(g,\mu;p_i) = -\gamma(g) \cdot f(g,\mu;p_i)$

f depends on μ (only) via overall scaling, but
$g = g(\mu)$ varies with scale

$$\beta(g) = \mu \frac{d}{d\mu} g \Big|_{g_0}$$

Example:

a)



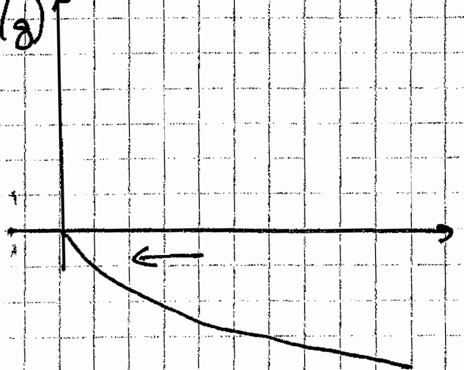
$\beta(g_*) = 0$

$\beta'(g_*) < 0$

$\rightarrow$ $g > g_* \Rightarrow \beta(g) < 0$
$g < g_* \Rightarrow \beta(g) > 0$ $\Big\}$ $g_\mu \longrightarrow g_*$ as $\mu \to \infty$

$\Rightarrow$ <u>UV fixed point</u>

b)



$\beta(0) = 0$

$\beta(g) < 0$ in some neighbourhood of $g = 0$

$g(\mu) \longrightarrow 0$ as $\mu \to \infty$ $\Longleftrightarrow$ origin is UV fixed point

$\Longleftrightarrow$ <u>asymptotic freedom</u> $\Longleftrightarrow$ perturbation theory valid at high scales

$\Longleftrightarrow$ $g(\mu)$ increases at lower scales